POWER-COMMUTATOR PRESENTATIONS FOR INFINITE SEQUENCES OF 3-GROUPS

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ABSTRACT. For certain infinite sequences of 3-groups G with derived length $2 \le \mathrm{dl}(G) \le 3$, and either with $G/G' \simeq (3,3)$, coclass $\mathrm{cc}(G) \ge 1$, or with $G/G' \simeq (9,3)$, coclass $\mathrm{cc}(G) \ge 3$, parametrized pc-presentations are given in dependence on the class $c = \mathrm{cl}(G)$.

1. PRELIMINARIES

For the detailed description of p-groups G with abelianization G/G' of type (p,p) or (p^2,p), we need two advanced invariants which coincide for any p-group G and for its metabelianization G/G'' [18, Thm. 1.1, p. 402].

Definition 1.1. Let G be a p-group of generator rank $d(G) = d$, put $m = \frac{p^d-1}{p-1}$, let $H_1, \ldots, H_m$ be the maximal (normal) subgroups (of index p) in G, and denote by

$$T_i : G/G' \to H_i/H_i',\ gG' \mapsto T_i(gG')$$

the *Artin transfer* from G to H_i, for $1 \le i \le m$, [17, (4), p. 470].

The family $\varkappa(G) = (\ker(T_i))_{1 \le i \le m}$ is called the *transfer kernel type*, TKT, of G, and the family $\tau(G) = (H_i/H_i')_{1 \le i \le m}$ is called the *transfer target type*, TTT, of G.

In this article, we shall be concerned with 3-groups G of particular TKTs, resp. pTKTs (punctured), with $d = 2$, $m = 4$, specified at the beginning of each section. By the symbol $\sim$ we indicate that some of the TKTs are equivalent, since they generate the same orbit under action of the symmetric group S_4 [17].

Definition 1.2. An infinite sequence of p-groups $(G_j)_{j \ge 0}$ is called a *periodic sequence* or *coclass family* if its members

1. share a common coclass $\mathrm{cc}(G_j) = r \ge 1$,
2. are descendants of a common ancestor M_0, which is the root of a coclass tree $\mathcal{T}_r(M_0)$,
3. share a common TKT $\varkappa(G_j)$,
4. share $m - 1$ common components of the TTT $\tau(G_j)$, whereas the single remaining component depends on the nilpotency class,
5. share a common parametrized pc-presentation $G_j = \langle x_1, \ldots, x_d \mid \mathcal{R}_{c_j}(x_1, \ldots, x_d)\rangle$ having the nilpotency class $c_j = \mathrm{cl}(G_j)$ as the only parameter.

Remark 1.1. The mainline of the coclass tree $\mathcal{T}_r(M_0)$ consists of one or more periodic sequences $(M_j)_{j \ge 0}$, whose parametrized pc-presentations approach the pro-p presentation of their projective limit $L = \varprojlim_{j \ge 0} M_j$ when the class $c_j = \mathrm{cl}(M_j)$ tends to infinity.

In the sequel we characterize 3-groups by their identifer in the SmallGroups library [4] and their descendants of order bigger than 3^7 by the notation used in the ANUPQ package [14] of GAP [13] and MAGMA [15].

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2. 3-groups G with $G/G' \simeq (3,3)$

For coclass $\mathrm{cc}(G)=1$ and abelianization $G/G' \simeq (3,3)$ we need

- all three cases of TKTs in section a, that is,
  a.1, $\varkappa=(0,0,0,0)$,
  a.2, $\varkappa=(1,0,0,0)$, and
  a.3, $\varkappa=(2,0,0,0)\sim(3,0,0,0)\sim(4,0,0,0)$.

2.1. **Periodic 3-groups on coclass tree $\mathcal{T}_1(\langle 9,2\rangle)$.** As a familiar entrance, we begin by showing that certain 3-groups of class at least 5 on the coclass tree $\mathcal{T}_1(\langle 9,2\rangle)$ belong to $4+3=7$ periodic coclass sequences with period length 2.

Theorem 2.1. *For each integer $c\ge 5$, there are 4 metabelian descendants G of $\langle 9,2\rangle$, having nilpotency class $\mathrm{cl}(G)=c$, coclass $\mathrm{cc}(G)=1$, and order $|G|=3^{c+1}$, with two generators x,y and parametrized pc-presentation*

$$\begin{aligned} G \;=\; &\langle\ x,y,s_2,s_3,s_4,\ldots,s_c\ \mid \\ &s_2=[y,x],\ s_j=[s_{j-1},x] \text{ for } 3\le j\le c,\\ &s_j^3=s_{j+2}^2 s_{j+3} \text{ for } 2\le j\le c-3,\ s_{c-2}^3=s_c^2,\\ &R(x)=1,\ R(y)=1\ \rangle,\end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by

$$R(x) \;=\; \begin{cases} x^3 & \text{for } G \text{ of TKT a.1 or a.3,}\\ x^3 s_c^{-1} & \text{for } G \text{ of TKT a.2,}\end{cases} \tag{1}$$

$$R(y) \;=\; \begin{cases} y^3 s_3^{-2} s_4^{-1} & \text{for } G \text{ of TKT a.1 or a.2,}\\ y^3 s_3^{-2} s_4^{-1} s_c^{-1} & \text{or}\\ y^3 s_3^{-2} s_4^{-1} s_c^{-2} & \text{for } G \text{ of TKT a.3.}\end{cases} \tag{2}$$

For odd class $c\ge 5$ the 4 groups are pairwise non-isomorphic σ-groups.
For even class $c\ge 6$, the pair of groups sharing the same TKT (a.3) is isomorphic, and thus only 3 groups are pairwise non-isomorphic, and only the mainline group with TKT a.1 is a σ-group.

Remark 2.1. The presentations in Theorem 2.1 are similar to but not identical with Blackburn's well-known presentations [5]. See also [17, § 2, pp. 469–470].

Remark 2.2. Pro-3 presentations for the projective limit L of the mainline of coclass tree $\mathcal{T}_1(\langle 9,2\rangle)$ are given in two different forms by Eick and Feichtenschlager [9, § 9.2, p. 11] resp. [12, App. A, pp. 90–91]:

(1) either for L as an extension of $\gamma_3(L)\simeq\mathbb{Z}_3^2$ by $L/\gamma_3(L)$,

$$\begin{aligned} L \;=\; &\langle\ x,y,s_2,t_1,t_2\ \mid\ [y,x]=s_2,\ [s_2,x]=t_1,\ [t_1,x]=t_2,\ [t_2,x]=t_1^{-4}t_2^{-2},\\ &x^3=1,\ y^3=t_1^2t_2,\ s_2^3=t_1^{-3}t_2^{-1}\ \rangle,\end{aligned}$$

(2) or for L as an extension of $\gamma_4(L)\simeq\mathbb{Z}_3^2$ by $L/\gamma_4(L)$,

$$\begin{aligned} L \;=\; &\langle\ x,y,s_2,s_3,t_1,t_2\ \mid\ [y,x]=s_2,\ [s_2,x]=s_3,\ [s_3,x]=t_2,\ [t_1,x]=t_2^3,\ [t_2,x]=t_1^{-2}t_2^{-2},\\ &x^3=1,\ y^3=s_3^2t_2,\ s_2^3=t_1^{-1}t_2^{-1},\ s_3^3=t_1\ \rangle.\end{aligned}$$

For coclass $\mathrm{cc}(G) \geq 2$ and abelianization $G/G' \simeq (3,3)$ we need

- all four cases of TKTs in section E, that is,
  E.6, $\varkappa = (1,1,2,2)$,
  E.14, $\varkappa = (3,1,2,2) \sim (4,1,2,2)$,
  E.8, $\varkappa = (1,1,3,4)$, and
  E.9, $\varkappa = (3,1,3,4) \sim (4,1,3,4)$,
- both TKTs in section c, that is,
  c.18, $\varkappa = (0,1,2,2)$, and
  c.21, $\varkappa = (0,1,3,4)$,
- TKT H.4, $\varkappa = (2,1,2,2)$,
- TKT G.16, $\varkappa = (2,1,3,4)$.

2.2. **Periodic 3-groups on coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$.** Now we show that certain 3-groups of class at least 5 on the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$ belong to $6 + 4 = 10$ periodic coclass sequences with period length 2.

Theorem 2.2. *For each integer $c \geq 5$, there are 6 metabelian descendants G of $\langle 243, 6\rangle$, having nilpotency class $\mathrm{cl}(G) = c$, coclass $\mathrm{cc}(G) = 2$, and order $|G| = 3^{c+2}$, with two generators x, y and parametrized pc-presentation*

$$\begin{aligned} G \quad = \quad & \langle\ x, y, s_2, t_3, s_3, s_4, \ldots, s_c\ \mid \\ & s_2 = [y,x],\ t_3 = [s_2, y],\ s_j = [s_{j-1}, x] \text{ for } 3 \leq j \leq c, \\ & s_j^3 = s_{j+2}^2 s_{j+3} \text{ for } 2 \leq j \leq c-3,\ s_{c-2}^3 = s_c^2,\ t_3^3 = 1, \\ & R(x) = 1,\ R(y) = 1\ \rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by

$$R(x) \quad = \quad \begin{cases} x^3 & \text{for } G \text{ of TKT c.18 or H.4,} \\ x^3 s_c^{-1} & \text{for } G \text{ of TKT E.6 or E.14,} \end{cases} \tag{3}$$

$$R(y) \quad = \quad \begin{cases} y^3 s_3^{-2} s_4^{-1} & \text{for } G \text{ of TKT c.18 or E.6,} \\ y^3 s_3^{-2} s_4^{-1} s_c^{-1} & \text{or} \\ y^3 s_3^{-2} s_4^{-1} s_c^{-2} & \text{for } G \text{ of TKT H.4 or E.14.} \end{cases} \tag{4}$$

For odd class $c \geq 5$ the 6 groups are pairwise non-isomorphic σ-groups.
*For even class $c \geq 6$, the two pairs of groups sharing the same TKT (*H.4 *and* E.14*) are isomorphic, and thus only 4 groups are pairwise non-isomorphic, and only the mainline group with TKT* c.18 *is a σ-group.*

Proof. G is a metabelian 3-group with abelian commutator subgroup $G' = \langle s_2, t_3, s_3, s_4, \ldots, s_c\rangle$. Due to the nilpotency relation $[s_c, x] = 1$, G is of class $\mathrm{cl}(G) = c$. Since t_3 is not contained in the subgroup $\langle s_3, s_4, \ldots, s_c\rangle$ and the centre of G is given by $\zeta_1(G) = \langle t_3, s_c\rangle$, the lower central series of G has a single bicyclic factor $\gamma_3 = \langle t_3, s_3, \gamma_4(G)\rangle$ and G is of coclass $\mathrm{cc}(G) = 2$, and thus of order $|G| = 3^{c+2}$.

The four maximal subgroups of G are given by $H_1 = \langle y, G'\rangle$, $H_2 = \langle x, G'\rangle$, $H_3 = \langle xy, G'\rangle$, $H_4 = \langle xy^2, G'\rangle$.

For the investigation of the transfers

$$T_i : G/G' \to H_i/H_i',\ gG' \mapsto \begin{cases} g^3 H_i' & \text{if } g \in G \setminus H_i, \\ g^{S_3(h)} H_i' & \text{if } g \in H_i, \end{cases}$$

where $S_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$ for some $h \in G \setminus H_i$, generators of the derived subgroups H_i', $1 \le i \le 4$, must be determined. Since G' is a normal subgroup of index 3 in each maximal subgroup, we obtain $H_i' = [G', H_i] = (G')^{g_i - 1}$ when $H_i = \langle g_i, G'\rangle$, and some commutator calculus yields

$$\begin{aligned} H_1' &= \langle t_3\rangle, \\ H_2' &= \langle s_3, s_4, \ldots, s_c\rangle, \\ H_3' &= \langle s_3 t_3, s_4, \ldots, s_c\rangle, \\ H_4' &= \langle s_3 t_3^2, s_4, \ldots, s_c\rangle. \end{aligned}$$

Now we can calculate the transfer kernels. For this purpose we represent the elements $g \in G$ in the form $g \equiv x^j y^\ell \mod G'$ and solve the congruence $T_i(gG') \equiv 1 \mod H_i'$.

First we derive expressions for the transfer images of the generators x, y. Since $x, y \notin H_i$ and $s_4, s_c \in H_i'$ for $i = 3, 4$, we have $T_i(xG') \equiv x^3 \equiv 1 \mod H_i'$ and $T_i(yG') \equiv y^3 \equiv s_3^2 \mod H_i'$ for $i = 3, 4$.

Further, since $y \notin H_2$ and $s_3, s_4, s_c \in H_2'$, we have $T_2(yG') \equiv y^3 \equiv 1 \mod H_2'$, and since $x \notin H_1$, we have $T_1(xG') \equiv x^3 \mod H_1'$.

However, for the action of trace elements as symbolic exponents we need [17, eqn. (6), p. 486].

Since $x \in H_2$ and $s_c \in H_2'$, we have $T_2(x) \equiv x^{S_3(y)} \equiv x^3[x,y]^3[[x,y],y] \equiv 1 \cdot s_2^{-3}[s_2^{-1}, y] \equiv s_4^{-2} s_5^{-1}[y, s_2^{-1}]^{-1} \equiv [y, s_2]^{s_2^{-1}} \equiv t_3^{-1} \mod H_2'$.

Since $y \in H_1$ and since we can prove by induction that $s_3^3 s_4^3 s_5 = 1$, we have $T_1(y) \equiv y^{S_3(x)} \equiv y^3[y,x]^3[[y,x],x] \equiv s_3^2 s_4 s_c^e s_2^3 s_3 \equiv s_3^2 s_4 s_c^e s_4^2 s_5 s_3 \equiv s_3^3 s_4^3 s_5 s_c^e \equiv s_c^e \mod H_1'$ for some exponent $0 \le e \le 2$.

Consequently, we obtain the following expressions for the transfer images:

$$\begin{aligned} T_1(x^j y^\ell G') &\equiv \begin{cases} s_c^{e\ell} \mod H_1' & \text{if } x^3 = 1,\ y^3 = s_3^2 s_4 s_c^e, \\ s_c^{j+e\ell} \mod H_1' & \text{if } x^3 = s_c,\ y^3 = s_3^2 s_4 s_c^e, \end{cases} \\ T_2(x^j y^\ell G') &\equiv t_3^{-j} \mod H_2', \\ T_i(x^j y^\ell G') &\equiv s_3^{2\ell} \mod H_i' \text{ for } i \in \{3, 4\}. \end{aligned}$$

□

2.3. **Periodic 3-groups on coclass tree $\mathcal{T}_2(\langle 243,8\rangle)$.** Similarly to the previous section, we now show that certain 3-groups of class at least 6 on the coclass tree $\mathcal{T}_2(\langle 243,8\rangle)$ belong to $6+4=10$ periodic coclass sequences with period length 2.

Theorem 2.3. *For each integer $c\ge 6$, there are 6 metabelian descendants G of $\langle 243,8\rangle$, having nilpotency class $\mathrm{cl}(G)=c$, coclass $\mathrm{cc}(G)=2$, and order $|G|=3^{c+2}$, with two generators x,y and parametrized pc-presentation*

$$\begin{aligned} G \;=\; \langle\; & x,y,t_2,s_3,t_3,t_4,\ldots,t_c \;\mid \\ & t_2=[y,x],\; s_3=[t_2,x],\; t_j=[t_{j-1},y] \text{ for } 3\le j\le c,\\ & t_j^3=t_{j+2}^2t_{j+3} \text{ for } 2\le j\le c-3,\; t_{c-2}^3=t_c^2,\; s_3^3=1,\\ & R(y)=1,\; R(x)=1\;\rangle, \end{aligned}$$

where the relators $R(y)$ and $R(x)$ are given by

$$R(y) \;=\; \begin{cases} y^3s_3^{-1} & \text{for } G \text{ of TKT c.21 or G.16},\\ y^3s_3^{-1}t_c^{-1} & \text{for } G \text{ of TKT E.8 or E.9}, \end{cases} \tag{5}$$

$$R(x) \;=\; \begin{cases} x^3t_3^{-1}t_4^{-2}t_5^{-1} & \text{for } G \text{ of TKT c.21 or E.8},\\ x^3t_3^{-1}t_4^{-2}t_5^{-1}t_c^{-1} & \text{or}\\ x^3t_3^{-1}t_4^{-2}t_5^{-1}t_c^{-2} & \text{for } G \text{ of TKT G.16 or E.9}. \end{cases} \tag{6}$$

For odd class $c\ge 7$ the 6 groups are pairwise non-isomorphic σ-groups.
For even class $c\ge 6$, the two pairs of groups sharing the same TKT (G.16 and E.9) are isomorphic, and thus only 4 groups are pairwise non-isomorphic, and only the mainline group with TKT c.21 is a σ-group.

Remark 2.3. Eick, Leedham-Green, Newman, and O'Brien [11] have determined the projective limit $L=\varprojlim_{j\ge 0}M_j$ of the metabelian mainline $(M_j)_{j\ge 0}$ of the coclass tree $\mathcal{T}_2(M_0)$ with root $M_0=\langle 243,6\rangle$, resp. $\langle 243,8\rangle$. It is given by the pro-3 presentation

$$\begin{aligned} L=\langle\; t,a,z \;\mid\; & a^3=z^f,\; [t,t^a]=z,\; tt^at^{a^2}=z^2,\\ & z^3=1,\; [z,a]=1,\; [z,t]=1,\;\rangle, \end{aligned}$$

where $f=0$, resp. $f=1$. The centre of L is the cyclic group $\zeta_1(L)=\langle z\rangle$ of order 3.

The mainline vertices of $\mathcal{T}_2(M_0)$ are the σ-groups

$$\begin{aligned} M_{2\ell} &\simeq L/\langle t^{3^{\ell+2}}\rangle \\ &\qquad \text{of order } 3^{2\ell+5} \text{ and odd class } 2\ell+3,\\ M_{2\ell+1} &\simeq L/\langle t^{3^{\ell+2}},t^{3^{\ell+1}}(t^a)^{-3^{\ell+1}}\rangle \\ &\qquad \text{of order } 3^{2\ell+6} \text{ and even class } 2\ell+4, \end{aligned}$$

for $\ell\ge 0$.

Proof. G is a metabelian 3-group with abelian commutator subgroup $G' = \langle t_2, s_3, t_3, t_4, \ldots, t_c\rangle$. Due to the nilpotency relation $[t_c, x] = 1$, G is of class $\mathrm{cl}(G) = c$. Since s_3 is not contained in the subgroup $\langle t_3, t_4, \ldots, t_c\rangle$ and the centre of G is given by $\zeta_1(G) = \langle s_3, t_c\rangle$, the lower central series of G has a single bicyclic factor $\gamma_3 = \langle s_3, t_3, \gamma_4(G)\rangle$ and G is of coclass $\mathrm{cc}(G) = 2$, and thus of order $|G| = 3^{c+2}$.

The four maximal subgroups of G are given by $H_1 = \langle x, G'\rangle$, $H_2 = \langle y, G'\rangle$, $H_3 = \langle yx, G'\rangle$, $H_4 = \langle yx^2, G'\rangle$.

For the investigation of the transfers

$$T_i : \ G/G' \to H_i/H_i',\ gG' \mapsto \begin{cases} g^3 H_i' & \text{if } g \in G \setminus H_i, \\ g^{S_3(h)} H_i' & \text{if } g \in H_i, \end{cases}$$

where $S_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$ for some $h \in G \setminus H_i$, generators of the derived subgroups H_i', $1 \le i \le 4$, must be determined. Since G' is a normal subgroup of index 3 in each maximal subgroup, we obtain $H_i' = [G', H_i] = (G')^{g_i - 1}$ when $H_i = \langle g_i, G'\rangle$, and some commutator calculus yields

$$\begin{aligned} H_1' &= \langle s_3\rangle, \\ H_2' &= \langle t_3, t_4, \ldots, t_c\rangle, \\ H_3' &= \langle t_3 s_3, t_4, \ldots, t_c\rangle, \\ H_4' &= \langle t_3 s_3^2, t_4, \ldots, t_c\rangle. \end{aligned}$$

Now we can calculate the transfer kernels. For this purpose we represent the elements $g \in G$ in the form $g \equiv y^j x^\ell \mod G'$ and solve the congruence $T_i(gG') \equiv 1 \mod H_i'$.

First we derive expressions for the transfer images of the generators x, y. Since $x, y \notin H_i$ and $t_4, t_5, t_c \in H_i'$ for $i = 3, 4$, we have $T_i(yG') \equiv y^3 \equiv s_3 \mod H_i'$ and $T_i(xG') \equiv x^3 \equiv t_3^2 \mod H_i'$ for $i = 3, 4$.

Further, since $x \notin H_2$ and $t_3, t_4, t_5, t_c \in H_2'$, we have $T_2(xG') \equiv x^3 \equiv 1 \mod H_2'$, and since $y \notin H_1$ and $s_3 \in H_1'$, we have $T_1(yG') \equiv y^3 \equiv t_c^\varepsilon \mod H_1'$ with suitable $0 \le \varepsilon \le 1$.

However, for the action of trace elements as symbolic exponents we need [17, eqn. (6), p. 486].

Since $y \in H_2$ and $t_4, t_5, t_c \in H_2'$, we have $T_2(y) \equiv y^{S_3(x)} \equiv y^3[y,x]^3[[y,x],x] \equiv s_3 t_c^\varepsilon \cdot t_2^3 s_3 \equiv s_3^2 t_4^2 t_5 t_c^\varepsilon \equiv s_3^2 \mod H_2'$.

Since $x \in H_1$ and since we can prove by induction that $s_3^3 s_4^3 s_5 = 1$, we have $T_1(x) \equiv x^{S_3(y)} \equiv x^3[x,y]^3[[x,y],y] \equiv t_3 t_4^2 t_5 t_c^e t_2^{-3} t_3^{-1} \equiv t_3 t_4^2 t_5 t_c^e t_4^{-2} t_5^{-1} t_3^{-1} \equiv t_c^e \mod H_1'$ for some exponent $0 \le e \le 2$.

Consequently, we obtain the following expressions for the transfer images:

$$\begin{aligned} T_1(y^j x^\ell G') &\equiv \begin{cases} t_c^{e\ell} \mod H_1' & \text{if } y^3 = s_3,\ x^3 = t_3 t_4^2 t_5 t_c^e, \\ t_c^{j+e\ell} \mod H_1' & \text{if } y^3 = s_3 t_c,\ x^3 = t_3 t_4^2 t_5 t_c^e, \end{cases} \\ T_2(y^j x^\ell G') &\equiv s_3^{2j} \mod H_2', \\ T_i(y^j x^\ell G') &\equiv s_3^j t_3^\ell \mod H_i' \text{ for } i \in \{3, 4\}. \end{aligned}$$

□

2.4. **Periodic 3-groups on coclass tree $\mathcal{T}_3(\langle 729,49\rangle-\#2;1)$.** The following result shows that certain 3-groups of class at least 6 on the entirely non-metabelian coclass tree $\mathcal{T}_3(\langle 729,49\rangle-\#2;1)$, belong to $6+4=10$ periodic coclass sequences with period length 2.

Theorem 2.4. *For each integer $c\ge 6$, there are 6 descendants G of $\langle 729,49\rangle-\#2;1$, having nilpotency class $\mathrm{cl}(G)=c$, coclass $\mathrm{cc}(G)=3$, order $|G|=3^{c+3}$, and derived length $\mathrm{dl}(G)=3$, with two generators x,y and parametrized pc-presentation*

$$\begin{aligned} G \quad = \quad & \langle\ x,y,s_2,t_3,s_3,s_4,\ldots,s_c,u_5\ \mid \\ & s_2=[y,x],\ t_3=[s_2,y],\ s_j=[s_{j-1},x] \text{ for } 3\le j\le c, \\ & u_5=[s_3,y]=[s_4,y],\ [s_3,s_2]=u_5^2,\ t_3^3=u_5^2, \\ & s_2^3=s_4^2s_5u_5,\ s_j^3=s_{j+2}^2s_{j+3} \text{ for } 3\le j\le c-3,\ s_{c-2}^3=s_c^2, \\ & R(x)=1,\ R(y)=1\ \rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by equations (3) and (4).
For odd class $c\ge 7$ the 6 groups are pairwise non-isomorphic σ-groups.
*For even class $c\ge 6$, the two pairs of groups sharing the same TKT (*H.4 *and* E.14*) are isomorphic, and thus only 4 groups are pairwise non-isomorphic, and only the mainline group with TKT* c.18 *is a σ-group.*

2.5. **Periodic 3-groups on coclass tree $\mathcal{T}_3(\langle 729,54\rangle-\#2;3)$.** The following result shows that certain 3-groups of class at least 6 on the entirely non-metabelian coclass tree $\mathcal{T}_3(\langle 729,54\rangle-\#2;3)$, belong to $6+4=10$ periodic coclass sequences with period length 2.

Theorem 2.5. *For each integer $c\ge 6$, there are 6 descendants G of $\langle 729,54\rangle-\#2;3$, having nilpotency class $\mathrm{cl}(G)=c$, coclass $\mathrm{cc}(G)=3$, order $|G|=3^{c+3}$, and derived length $\mathrm{dl}(G)=3$, with two generators x,y and parametrized pc-presentation*

$$\begin{aligned} G \quad = \quad & \langle\ x,y,t_2,s_3,t_3,t_4,\ldots,t_c,u_5\ \mid \\ & t_2=[y,x],\ s_3=[t_2,x],\ t_j=[t_{j-1},y] \text{ for } 3\le j\le c, \\ & u_5=[t_3,x]=[t_4,x],\ [t_3,t_2]=u_5,\ s_3^3=u_5^2, \\ & t_2^3=t_4^2t_5u_5,\ t_j^3=t_{j+2}^2t_{j+3} \text{ for } 3\le j\le c-3,\ t_{c-2}^3=t_c^2, \\ & R(y)=1,\ R(x)=1\ \rangle, \end{aligned}$$

where the relators $R(y)$ and $R(x)$ are given by equations (5) and (6).
For odd class $c\ge 7$ the 6 groups are pairwise non-isomorphic σ-groups.
*For even class $c\ge 6$, the two pairs of groups sharing the same TKT (*G.16 *and* E.9*) are isomorphic, and thus only 4 groups are pairwise non-isomorphic, and only the mainline group with TKT* c.21 *is a σ-group.*

3. 3-groups G with $G/G' \simeq (9,3)$

For coclass $\mathrm{cc}(G) \ge 3$ and abelianization $G/G' \simeq (9,3)$ we need

- pTKT C.4, $\varkappa = (3,1,1;3)$,
- two pTKTs in section D,
  D.5, $\varkappa = (2,1,1;3)$, and
  D.10, $\varkappa = (4,1,1;3)$,
- pTKT d.10, $\varkappa = (0,1,1;3)$,
- pTKT B.2, $\varkappa = (1,1,1;3)$.

3.1. **Periodic 3-groups on coclass tree $\mathcal{T}_3(\langle 729, 13\rangle)$.** Similarly as in the previous sections, we show that certain 3-groups of class at least 8 on the coclass tree $\mathcal{T}_3(\langle 729, 13\rangle)$ belong to $9+5=14$ periodic coclass sequences with period length 2.

Theorem 3.1. *For each integer $c \ge 8$, there are 9 metabelian descendants G of $\langle 729, 13\rangle$, having nilpotency class $\mathrm{cl}(G) = c$, coclass $\mathrm{cc}(G) = 3$, and order $|G| = 3^{c+3}$, with two generators x, y and parametrized pc-presentation*

$$
\begin{aligned}
G \quad = \quad & \langle\ x, y, \tau, t_2, s_3, t_3, t_4, \ldots, t_c \ \mid \\
& \tau = x^3,\ t_2 = [y,x],\ s_3 = [t_2, x],\ t_j = [t_{j-1}, y] \text{ for } 3 \le j \le c, \\
& t_j^3 = t_{j+2}^2 t_{j+3} \text{ for } 2 \le j \le c-3,\ t_{c-2}^3 = t_c^2,\ s_3^3 = 1, \\
& [\tau, y] = t_4 t_5^2 t_6,\ R(y) = 1,\ R(\tau) = 1\ \rangle,
\end{aligned}
$$

where the relators $R(y)$ and $R(\tau)$ are given by

$$
R(y) \quad = \quad \begin{cases} y^3 & \text{for } G \text{ of pTKT d.10 or B.2(1) or B.2(2)}, \\ y^3 t_c^{-1} & \text{for } G \text{ of pTKT C.4(1) or D.5(2) or D.10(1)}, \\ y^3 t_c^{-2} & \text{for } G \text{ of pTKT C.4(2) or D.5(1) or D.10(2)}, \end{cases} \tag{7}
$$

$$
R(\tau) \quad = \quad \begin{cases} \tau^3 s_3^{-1} t_5^{-2} t_6^{-2} t_7^{-1} & \text{for } G \text{ of pTKT d.10 or D.10}, \\ \tau^3 s_3^{-1} t_5^{-2} t_6^{-2} t_7^{-1} t_c^{-1} & \text{for } G \text{ of pTKT B.2(1) or C.4(1) or D.5(1)}, \\ \tau^3 s_3^{-1} t_5^{-2} t_6^{-2} t_7^{-1} t_c^{-2} & \text{for } G \text{ of pTKT B.2(2) or C.4(2) or D.5(2)}. \end{cases} \tag{8}
$$

For odd class $c \ge 9$ the 9 groups are pairwise non-isomorphic σ-groups.
For even class $c \ge 8$, the four pairs of groups sharing the same pTKT (B.2, C.4, D.5 and D.10) are isomorphic, and thus only 5 groups are pairwise non-isomorphic, and only the mainline group with pTKT d.10 is a σ-group.

3.2. **Periodic 3-groups on coclass tree $\mathcal{T}_4(\langle 2187, 168\rangle - \#2; 7)$.** The following result shows that certain 3-groups of class at least 8 on the entirely non-metabelian coclass tree $\mathcal{T}_4(\langle 2187, 168\rangle - \#2; 7)$, belong to $9 + 5 = 14$ periodic coclass sequences with period length 2.

Theorem 3.2. *For each integer $c \geq 8$, there are 9 descendants G of $\langle 2187, 168\rangle - \#2; 7$, having nilpotency class $\mathrm{cl}(G) = c$, coclass $\mathrm{cc}(G) = 4$, order $|G| = 3^{c+4}$, and derived length $\mathrm{dl}(G) = 3$, with two generators x, y and parametrized pc-presentation*

$$\begin{aligned} G \;=\; & \langle\ x, y, \tau, t_2, s_3, t_3, t_4, \ldots, t_c, u_5 \ \mid \\ & \tau = x^3,\ t_2 = [y, x],\ s_3 = [t_2, x],\ t_j = [t_{j-1}, y] \text{ for } 3 \leq j \leq c, \\ & u_5 = [t_3, x] = [t_4, x],\ [\tau, t_2] = [t_3, t_2] = u_5,\ s_3^3 = u_5^2, \\ & t_2^3 = t_4^2 t_5 u_5,\ t_j^3 = t_{j+2}^2 t_{j+3} \text{ for } 3 \leq j \leq c-3,\ t_{c-2}^3 = t_c^2, \\ & [\tau, y] = t_4 t_5^2 t_6,\ R(y) = 1,\ R(\tau) = 1\ \rangle, \end{aligned}$$

where the relators $R(y)$ and $R(\tau)$ are given by equations (7) and (8).
For odd class $c \geq 9$ the 9 groups are pairwise non-isomorphic σ-groups.
For even class $c \geq 8$, the four pairs of groups sharing the same pTKT (B.2, C.4, D.5 and D.10) are isomorphic, and thus only 5 groups are pairwise non-isomorphic, and only the mainline group with pTKT d.10 is a σ-group.

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