Principalization Algorithm via Class Group Structure

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Abstract.

For an algebraic number field K with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ or $(9,3)$, the structure of the 3-class groups $\mathrm{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_i, $1 \le i \le 4$, of K is calculated with the aid of presentations for the metabelian Galois group $\mathrm{G}_3^2(K) = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field $\mathrm{F}_3^2(K)$ of K. In the case of a quadratic base field $K = \mathbb{Q}(\sqrt{D})$ it is shown that the structure of the 3-class groups of the four S_3-fields $N_1,\ldots,N_4$ determines the type of principalization of the 3-class group of K in $N_1,\ldots,N_4$. This provides an alternative to the classical principalization algorithm by Scholz and Taussky. The new algorithm, which is easily automatizable and executes very quickly, is implemented in PARI/GP and applied to all 4 596, resp. 1 146, quadratic fields with discriminant $-10^6 < D < 10^7$ and 3-class group of type $(3,3)$, resp. $(9,3)$, to obtain extensive statistics of their principalization types and the distribution of their second 3-class groups $\mathrm{G}_3^2(K)$ on the coclass graphs $\mathcal{G}(3,r)$, $1 \le r \le 6$, in the sense of Eick, Leedham-Green, and Newman.

References.

[1] D. C. Mayer, Transfers of metabelian p-groups, *Monatsh. Math.* (2010), DOI 10.1007/s00605-010-0277-x.
[2] D. C. Mayer, The second p-class group of a number field, *Int. J. Number Theory* (2010).
[3] D. C. Mayer, Principalisation algorithm via class group structure, *J. Th. Nombres Bordeaux.*
[4] D. C. Mayer, *The distribution of second p-class groups on coclass graphs* (27th Journées Arithmétiques, Vilnius, Lithuania, 2011).

Introduction

The principal ideal theorem, which has been conjectured by Hilbert in 1898, states that each ideal of a number field K becomes principal when it is extended to the Hilbert class field $\mathrm{F}^1(K)$ of K, that is the maximal abelian unramified extension field of K. Inspired by the Artin-Furtwängler proof of the principal ideal theorem, Scholz and Taussky investigated the principalization in intermediate fields $K < N < \mathrm{F}_3^1(K)$ between a base field K with 3-class group of type $(3,3)$ or $(9,3)$ and its Hilbert 3-class field $\mathrm{F}_3^1(K)$. They developed an algorithm for computing the principalization of K in its four unramified cyclic cubic extension fields $N_1,\ldots,N_4$ for the case of a complex quadratic base field K. This algorithm is probabilistic, since it decides whether an ideal $\mathfrak{a}$ of K becomes principal in N_i, for some $1 \le i \le 4$, by testing local cubic residue characters of a principal ideal cube $(\alpha) = \mathfrak{a}^3$, associated with the ideal $\mathfrak{a}$, and of a fundamental unit ε_i of the non-Galois cubic subfield L_i of the complex S_3-field N_i with respect to a series of rational test primes $(p_\ell)_{\ell\ge 1}$ and terminating when a critical test prime occurs. An upper bound for the minimal critical test prime p_{ℓ_0} cannot be given effectively. It can only be estimated by means of Chebotarëv's density theorem, thus causing uncertainty.

An entirely different approach to the principalization problem will be presented in this lecture. It is based on a purely group theoretical connection between the structure of the abelianizations $M_i/\gamma_2(M_i)$ of the four maximal normal subgroups M_i of an arbitrary metabelian 3-group G with $G/\gamma_2(G)$ of type $(3,3)$ or $(9,3)$ and the kernels $\ker(\mathrm{T}_i)$ of the transfers $\mathrm{T}_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$, $1 \le i \le 4$. By the Artin reciprocity law of class field theory, a corresponding number theoretical connection is established between the structure of the 3-class groups $\mathrm{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_i of an arbitrary algebraic number field K with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ or $(9,3)$ and the principalization kernels $\ker(\mathrm{j}_{N_i|K})$ of the class extension homomorphisms $\mathrm{j}_{N_i|K} : \mathrm{Cl}_3(K) \longrightarrow \mathrm{Cl}_3(N_i)$, $1 \le i \le 4$, applying the group theoretical statements to the second 3-class group $\mathrm{G}_3^2(K) = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of K, that is the Galois group of the second Hilbert 3-class field $\mathrm{F}_3^2(K) = \mathrm{F}_3^1(\mathrm{F}_3^1(K))$ of K.

1. Computational Performance of the Algorithm

The history of determining principalization types of quadratic fields is shown in the following two tables 1 and 2.

Table 1. History of investigating quadratic fields of type $(3, 3)$

History		complex		real	
authors	year	range	number	range	number
Scholz, Taussky	1934	$-10\,000 < D$	2		
Heider, Schmithals	1982	$-20\,000 < D$	13	$D < 1 \cdot 10^5$	5
Brink	1984	$-96\,000 < D$	66		
Mayer	1989	$-30\,000 < D$	35		
Mayer	1991			$D < 2 \cdot 10^5$	16
Mayer	2010	$-10^6 < D$	2 020	$D < 10^7$	2 576

Table 2. History of investigating quadratic fields of type $(9, 3)$

History		complex		real	
authors	year	range	number	range	number
Scholz, Taussky	1934	$-10\,000 < D$	2		
Heider, Schmithals	1982	$-20\,000 < D$	7		
Mayer	1989	$-30\,000 < D$	9		
Mayer	2011	$-10^6 < D$	875	$D < 10^7$	271

2. Theoretical Foundations of the Algorithm

2.1. Little and big two-stage towers of 3-class fields.

K an algebraic number field,
$\mathrm{Cl}_3(K)$ its 3-class group of type $(3,3)$ or $(9,3)$,

$N_1,\ldots,N_4$ the four unramified cyclic cubic extensions of K,
$0 \le \varepsilon \le 4$ the counter $\#\{1 \le i \le 4 \mid \mathrm{rank}_3(\mathrm{Cl}_3(N_i)) \ge 3\}$,
$\tilde{N}_4 = \prod_{i=1}^4 N_i$ the Frattini extension of K,

$\mathrm{F}_3^1(K)$ the first Hilbert 3-class field of K,
$\mathrm{F}_3^1(N_i)$ the first Hilbert 3-class field of N_i, $1 \le i \le 4$,
$\mathrm{F}_3^2(K)$ the second Hilbert 3-class field of K.

Definition 2.1. For $1 \le i \le 4$,
the $\Gamma_i = \mathrm{Gal}(\mathrm{F}_3^1(N_i)|K)$ denote the Galois groups of the four *little two-stage towers* of K, $K < \mathrm{F}_3^1(K) \le \mathrm{F}_3^1(N_i)$,

$G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ denotes the Galois group of the *big two-stage tower* of K, $K < \mathrm{F}_3^1(K) \le \mathrm{F}_3^2(K)$.
$G = \mathrm{G}_3^2(K)$ is called the *second* 3-*class group* of K.

Further notation.

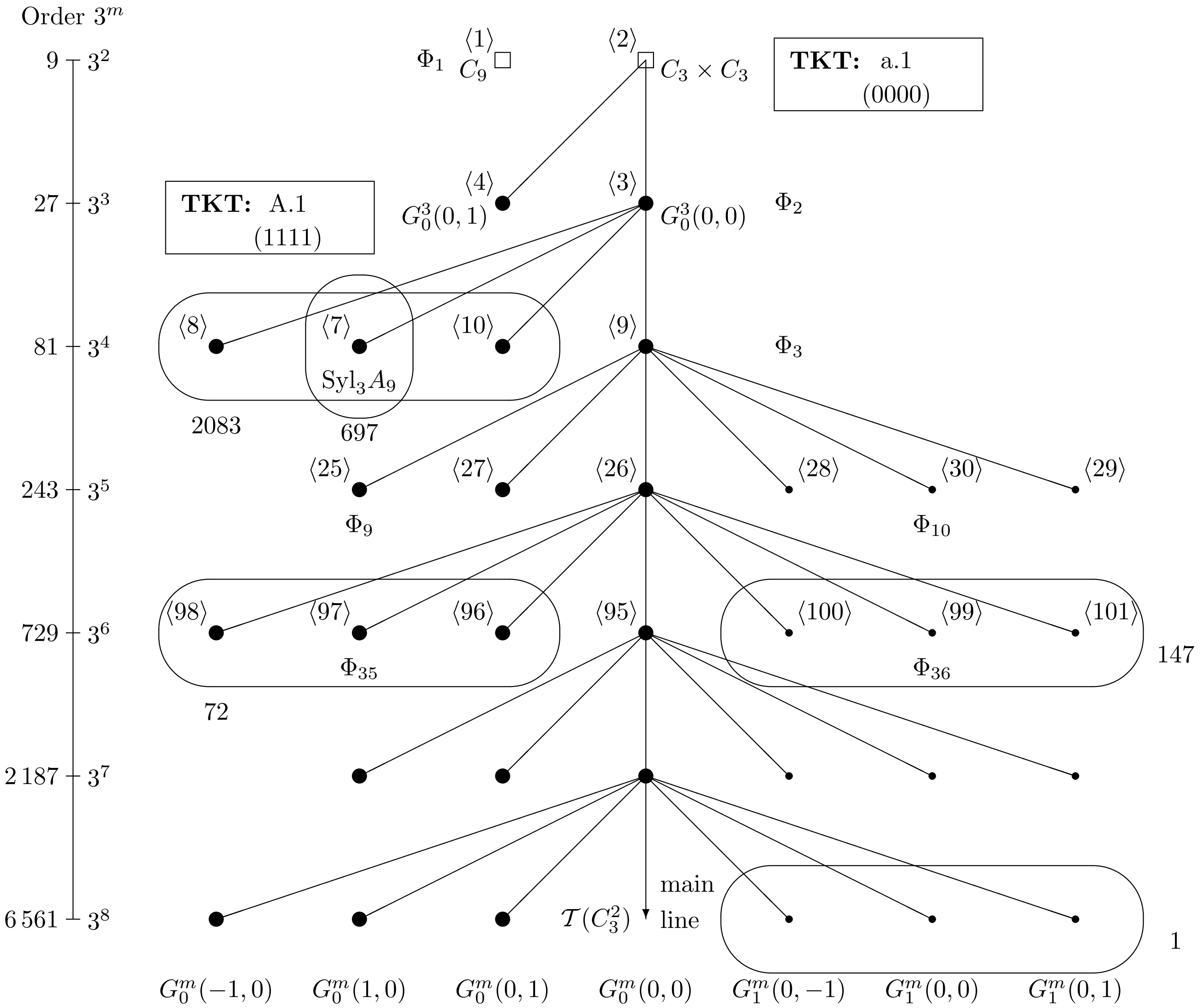
$\gamma_j(G)$, $j \geq 1$, the members of the lower central series of G,
$\chi_j(G)$, $j \geq 2$, the two-step centralizers of $\gamma_j(G)/\gamma_{j+2}(G)$,
$\mathrm{T}_i : G/G' \to M_i/M_i'$ the transfer from G to the maximal subgroup M_i, $1 \leq i \leq 4$,
$\tau = (\mathrm{str}(M_i/M_i'))_{1\leq i\leq 4}$ the *transfer target type* (TTT) of G, where $M_i/M_i' \simeq \mathrm{Cl}_3(N_i)$,
$\varkappa = (\ker(\mathrm{T}_i))_{1\leq i\leq 4}$ the *transfer kernel type* (TKT) of G, briefly $\varkappa = (\varkappa(i))_{1\leq i\leq 4}$, where $\ker(\mathrm{T}_i) = \tilde{M}_{\varkappa(i)}/G'$.

2.2. **Nearly homocyclic 3-class groups of 3-rank two.**

Definition 2.2. For an integer $n \geq 2$, denote by $\mathrm{A}(3,n)$ the *nearly homocyclic abelian* 3-*group* of order 3^n, i.e., the abelian group of type $(3^{q+r}, 3^q)$, where $n = 2q + r$ with integers $q \geq 1$ and $0 \leq r < 2$.

2.2.1. *Second* 3*-class groups* G *of coclass.* $\mathrm{cc}(G) = 1$

FIGURE 1. Root $C_3 \times C_3$ and branches $\mathcal{B}(j)$, $2 \le j \le 7$, on the coclass graph $\mathcal{G}(3,1)$



TKT:	a.3	a.3	a.2	a.1	a.1	a.1	a.1
	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

Proof: N.Blackburn, 1958, [Bl] On a special class of p-groups.

Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group containing an abelian maximal subgroup,
small full circle: metabelian group without abelian maximal subgroups.

Theorem 2.1. *(Transfer target type, TTT, for $G \in \mathcal{G}(3,1)$)*
The structure of the 3*-class groups* $\mathrm{Cl}_3(N_i)$, $1 \le i \le 4$,
for $\mathrm{cc}(G) = 1$, $|G| = 3^m$, $\mathrm{cl}(G) = m-1$, $m \ge 3$,
is given by

$$\mathrm{Cl}_3(N_1) \simeq \begin{cases} \mathrm{A}(3, m-1), & \textit{if } [\chi_2(G), \gamma_2(G)] = 1,\ m \ge 5, \\ \mathrm{A}(3, m-2), & \textit{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G),\ m \ge 6, \end{cases}$$

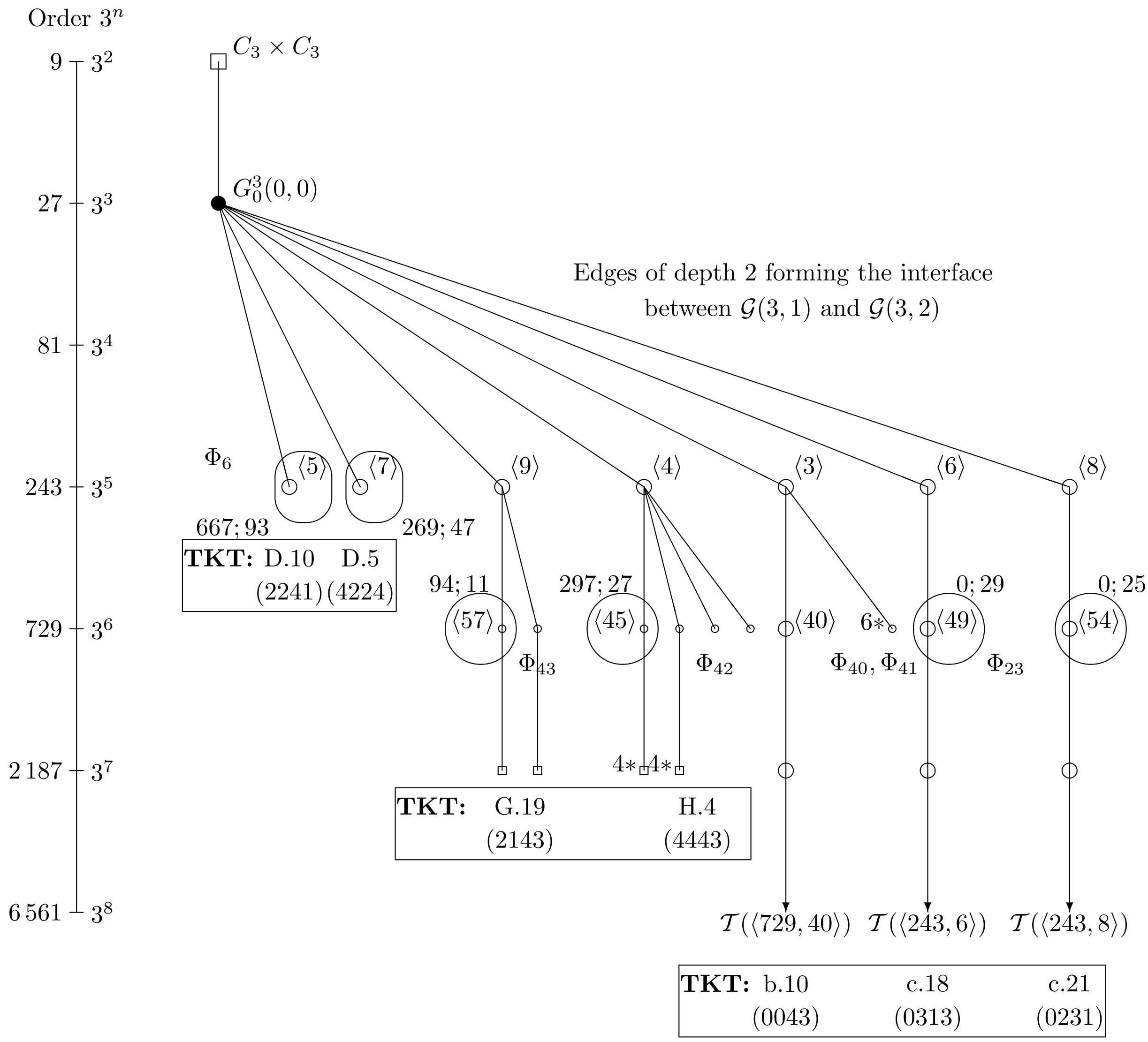
$$\mathrm{Cl}_3(N_i) \simeq \mathrm{A}(3,2) \textit{ for } 2 \le i \le 4,\ \textit{if } m \ge 4.$$

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.3.1.

Remark 2.1. Our Thm.2.1 covers all stem groups in Φ_9, Φ_{35}, Φ_{36}, and in higher isoclinism families.
It remains to investigate two groups in Φ_2, four groups in Φ_3, and three groups in Φ_{10}.

2.2.2. *Second* 3-*class groups* G, $\mathrm{cc}(G) \geq 2$, $G/G' \simeq (3,3)$

Figure 2. Sporadic groups and roots of coclass trees on the coclass graph $\mathcal{G}(3,2)$



Proof: B.Nebelung, 1989, [Ne] Klassifikation metabelscher 3-Gruppen.

Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group containing an abelian maximal subgroup,
big contour circle: metabelian group with centre of type $(3,3)$,
small contour circle: metabelian group with centre of type (3),
small contour square: non-metabelian group.

Theorem 2.2. *(Partial TTT $(\tau_1(G), \tau_2(G))$ for $\mathrm{cc}(G) \geq 2$)* *The structure of the 3-class groups $\mathrm{Cl}_3(N_i)$, $1 \leq i \leq 4$, for $\mathrm{cc}(G) \geq 2$, $|G| = 3^n$, $\mathrm{cl}(G) = m-1$, $G/G' \simeq (3,3)$, and invariant $e = n - m + 2 \geq 3$, where $4 \leq m < n \leq 2m-3$, is given by*

$$\mathrm{Cl}_3(N_1) \simeq \begin{cases} \mathrm{A}(3, m-1), & \textit{if } [\chi_s(G), \gamma_e(G)] = 1,\ m \geq 5\,, \\ \mathrm{A}(3, m-2), & \textit{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G),\ m \geq 6\,, \end{cases}$$

$$\mathrm{Cl}_3(N_2) \simeq \mathrm{A}(3, e) \textit{ for } e \geq 4\,,$$

$$\mathrm{Cl}_3(N_i) \simeq \mathrm{A}(3,3) \textit{ for } 3 \leq i \leq 4,\ \textit{if } \Gamma_i \not\simeq G_0^4(1,0) \simeq \mathrm{Syl}_3 A_9\,.$$

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.3.2.

Remark 2.2. Our Thm.2.2 covers $\mathrm{Cl}_3(N_i)$, $1 \leq i \leq 2$, for all stem groups in Φ_{23} and in higher isoclinism families.
It remains to investigate seven groups in Φ_6, three groups in Φ_{40}, three groups in Φ_{41}, four groups in Φ_{42}, and two groups in Φ_{43} completely,
$\mathrm{Cl}_3(N_2)$, for $\mathrm{cc}(G) = 2$,
and $\mathrm{Cl}_3(N_i)$, $3 \leq i \leq 4$, for $\mathrm{cc}(G) \geq 2$.

2.3. **Searching for 3-class groups of 3-rank three.**

2.3.1. *Second 3-class groups G of coclass* $\mathrm{cc}(G) = 1$.

Theorem 2.3. *(TTT of stem groups G in $\Phi_2, \Phi_3, \Phi_{10}$)*
The structure of the 3-class groups $\mathrm{Cl}_3(N_i)$, $1 \le i \le 4$,
for $\mathrm{cc}(G) = 1$, $|G| = 3^m$, $3 \le \mathrm{cl}(G) + 1 = m \le 5$,
ist given by

$$\mathrm{Cl}_3(N_1) \simeq \begin{cases} \mathrm{A}(3,2), & \textit{if } m = 3, \\ \mathrm{A}(3,3), & \textit{if } m = 4,\ G \not\simeq G_0^4(1,0) \simeq \mathrm{Syl}_3 A_9, \\ C_3 \times C_3 \times C_3, & \textit{if } m = 4,\ G \simeq G_0^4(1,0) \simeq \mathrm{Syl}_3 A_9, \\ \mathrm{A}(3,3), & \textit{if } k = 1,\ m = 5, \end{cases}$$

$$\mathrm{Cl}_3(N_i) \simeq \begin{cases} \mathrm{A}(3,2), & \textit{if } G \simeq G_0^3(0,0), \\ C_9, & \textit{if } G \simeq G_0^3(0,1), \end{cases} \quad \textit{for } 2 \le i \le 4,\ \textit{if } m = 3,$$

where $\mathrm{Gal}(\mathrm{F}_3^2(K)|N_1) = \langle y, \gamma_2(G)\rangle$ *with* $y^3 = 1$, *for* $m = 3$.

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.4.1.

Corollary 2.3.1. *(TTT $\tau(G)$ of stem groups G in $\Phi_2, \Phi_3, \Phi_{10}$) The following table gives the structure of the 3-class groups $\mathrm{Cl}_3(N_i)$, $1 \le i \le 4$, for the 3-groups $G \in \mathcal{G}(3,1)$ of small nilpotency class $1 \le \mathrm{cl}(G) = m-1 \le 4$ in dependence on the principalisation or transfer kernel type, TKT, $\varkappa$. ε counts 3-class groups of type $(3,3,3)$.*

m	k	TKT	$\varkappa$	$\mathrm{Cl}_3(\mathrm{F}_3^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε
2	0	a.1	(0000)	1	(3)	(3)	(3)	(3)	0
3	0	a.1	(0000)	(3)	(3,3)	(3,3)	(3,3)	(3,3)	0
3	0	A.1	(1111)	(3)	(3,3)	(9)	(9)	(9)	0
4	0	a.1	(0000)	(3,3)	(9,3)	(3,3)	(3,3)	(3,3)	0
4	0	a.2	(1000)	(3,3)	(9,3)	(3,3)	(3,3)	(3,3)	0
4	0	a.3	(2000)	(3,3)	(9,3)	(3,3)	(3,3)	(3,3)	0
4	0	a.3*	(2000)	(3,3)	(3,3,3)	(3,3)	(3,3)	(3,3)	1
5	1	a.1	(0000)	(9,3)	(9,3)	(3,3)	(3,3)	(3,3)	0

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Cor.4.1.1.

Example 2.1. The smallest discriminant with TKT a.3*, where $\varepsilon = 1$, is $D = 142\,097$.

2.3.2. *Second* 3*-class groups* G, $\mathrm{cc}(G) = 2$, $G/G' \simeq (3,3)$.

Theorem 2.4. *(TTT $\tau(G)$ of stem groups G in Φ_6)*
If $|G| = 3^5$, $\mathrm{cl}(G) = 3$, $G/G' \simeq (3,3)$*, i.e.,*
G is one of the 7 top vertices on $\mathcal{G}(3,2)$ with bicyclic centre, then the structure of the 3-class groups of $\mathrm{F}_3^1(K)$ and $N_1,\ldots,N_4$ is given by

TKT	$\varkappa$	$\mathrm{Cl}_3(\mathrm{F}_3^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε
D.10	(2241)	(3,3,3)	(9,3)	(9,3)	(3,3,3)	(9,3)	1
D.5	(4224)	(3,3,3)	(3,3,3)	(9,3)	(3,3,3)	(9,3)	2
G.19	(2143)	(3,3,3)	(9,3)	(9,3)	(9,3)	(9,3)	0
H.4	(4443)	(3,3,3)	(3,3,3)	(3,3,3)	(9,3)	(3,3,3)	3
b.10	(0043)	(3,3,3)	(9,3)	(9,3)	(3,3,3)	(3,3,3)	2
c.18	(0313)	(3,3,3)	(9,3)	(9,3)	(3,3,3)	(9,3)	1
c.21	(0231)	(3,3,3)	(9,3)	(9,3)	(9,3)	(9,3)	0

in dependence on the transfer kernel type, TKT (principalisation type), $\varkappa$ of K.
Here, ε denotes the number of 3-class groups $\mathrm{Cl}_3(N_i)$ of type $(3,3,3)$.

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.4.2.

Example 2.2. The first discriminant with TKT D.10, where $\varepsilon = 1$, is $D = -4\,027$.

Theorem 2.5. *(TTT of stem groups G in $\Phi_{40}, \Phi_{41}, \Phi_{42}, \Phi_{43}$)*
If $|G| = 3^6$, $\mathrm{cl}(G) = 4$, $G/G' \simeq (3,3)$,
and $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, *i.e.,*
G is one of the 12 *vertices of level* 2 *on coclass graph* $\mathcal{G}(3,2)$ *with cyclic centre,*
then the structure of the 3*-class groups of* $\mathrm{F}_3^1(K)$ *and* $N_1, \ldots, N_4$ *is given by*

TKT	$\varkappa$	ρ	$\mathrm{Cl}_3(\mathrm{F}_3^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε
G.19	(2143)	1	$(3,3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
H.4	(4443)	1	$(9,3,3)$	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	3
b.10	(0043)	-1	$(3,3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2
b.10	(0043)	1	$(9,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2

in dependence on the TKT $\varkappa$ of K and on the relational exponent ρ of G.
Again, ε denotes the number of 3*-class groups* $\mathrm{Cl}_3(N_i)$ *of type* $(3,3,3)$.

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.4.3.

Example 2.3. The first discriminant with TKT H.4, where $\varepsilon = 3$, is $D = -3\,896$.

Theorem 2.6. *(ε as a tree invariant)*
All metabelian groups G on the coclass tree $\mathcal{T}(\langle 729, 40\rangle)$, resp. $\mathcal{T}(\langle 243, 6\rangle)$, $\mathcal{T}(\langle 243, 8\rangle)$, of coclass graph $\mathcal{G}(3, 2)$ are characterized by the value $\varepsilon = 2$, resp. $\varepsilon = 1$, $\varepsilon = 0$.

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Cor.4.4.1.

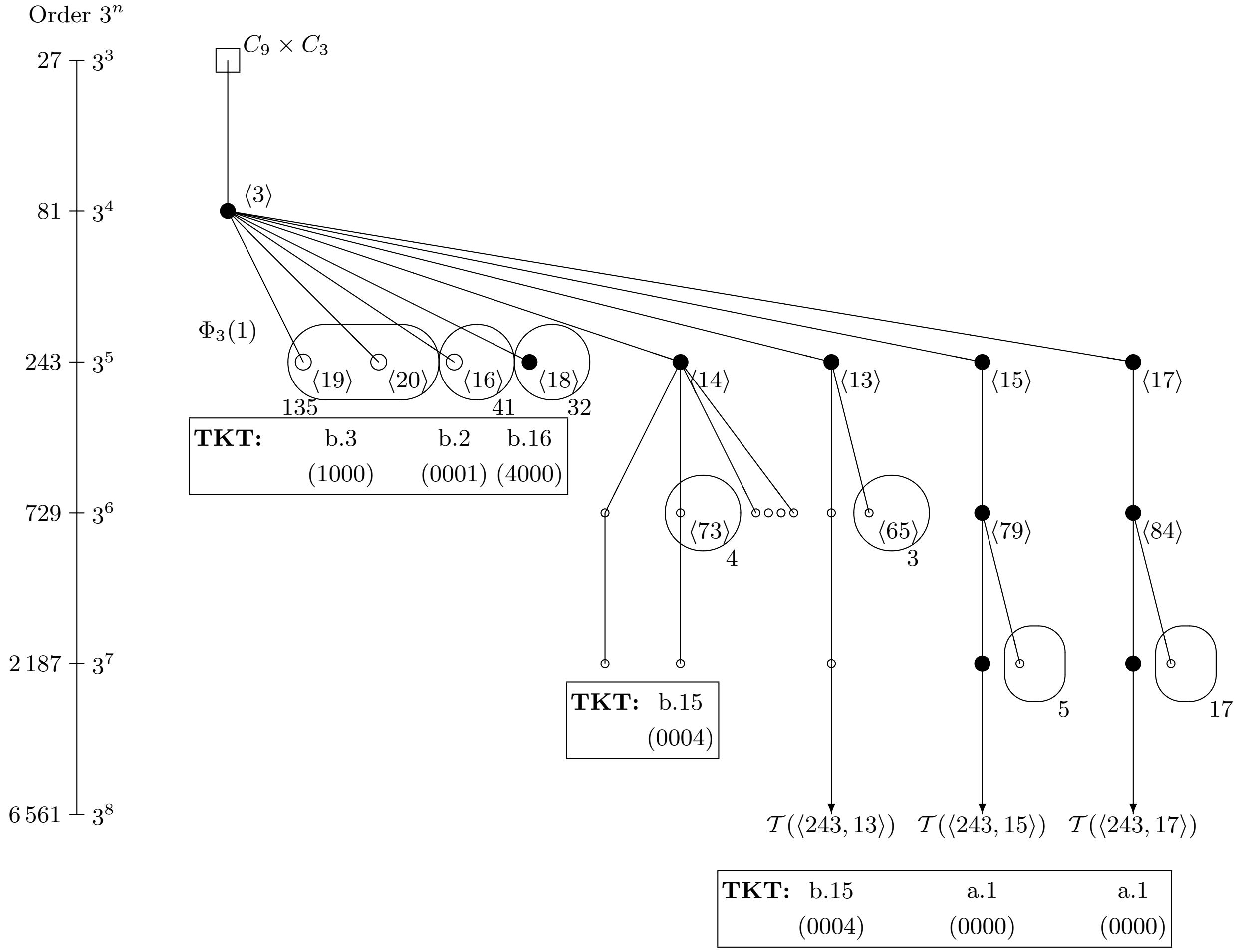
2.3.3. *Second* 3*-class groups* G, $\mathrm{cc}(G) \geq 3$, $G/G' \simeq (3,3)$.

Theorem 2.7. *(TTT* $\tau(G)$ *for groups* G *with* $\mathrm{cc}(G) \geq 3$*)* *If* $|G| = 3^n$, $n \geq 7$, $G/G' \simeq (3,3)$, $\mathrm{cc}(G) \geq 3$, $m \leq n-2$, $e \geq 4$, *then* $\mathrm{Cl}_3(N_1)$, $\mathrm{Cl}_3(N_2)$ *are nearly homocyclic and* $\mathrm{Cl}_3(N_3)$, $\mathrm{Cl}_3(N_4)$ *are of type* $(3,3,3)$, *independently from the transfer kernel type* $\varkappa$. *In particular,* $\varepsilon = 2$.

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.4.5.

2.3.4. *Second* 3*-class groups* G, $\mathrm{cc}(G)=2$, $G/G'\simeq(9,3)$

FIGURE 3. Sporadic groups and roots of coclass trees on the coclass graph $\mathcal{G}(3,2)$



Proof: J.A.Ascione, 1979, [As] On 3-groups of second maximal class.

Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group with centre of type $(3,3)$,
big contour circle: metabelian group with centre of type (9),
small contour circle: metabelian group with centre of type (3).

Theorem 2.8. *(TTT $\tau(G)$ of branch 1 groups G in Φ_3)*
If $|G| = 3^5$, $\mathrm{cl}(G) = 3$, $G/G' \simeq (9,3)$, i.e.,
G is one of the 8 CF-groups of level 2 on coclass graph $\mathcal{G}(3,2)$,
then the structure of the 3-class groups of $N_1,\dots,N_4$ and the Frattini extension $\tilde{N}_4$ is given by

TKT	$\varkappa$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε	$\mathrm{Cl}_3(\tilde{N}_4)$
b.2	(0001)	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3,3)$	1	$(9,3)$
b.3	(1000)	$(27,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	1	$(9,3)$
b.3	(1000)	$(27,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	1	$(9,3)$
b.16	(4000)	$(9,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	2	$(3,3,3)$
b.15	(0004)	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3,3)$	1	$(3,3,3)$
b.15	(0004)	$(9,3)$	$(9,3)$	$(9,3)$	$(3,3,3,3)$	1	$(3,3,3)$
a.1	(0000)	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3,3)$	1	$(3,3,3)$
a.1	(0000)	$(9,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	2	$(3,3,3)$

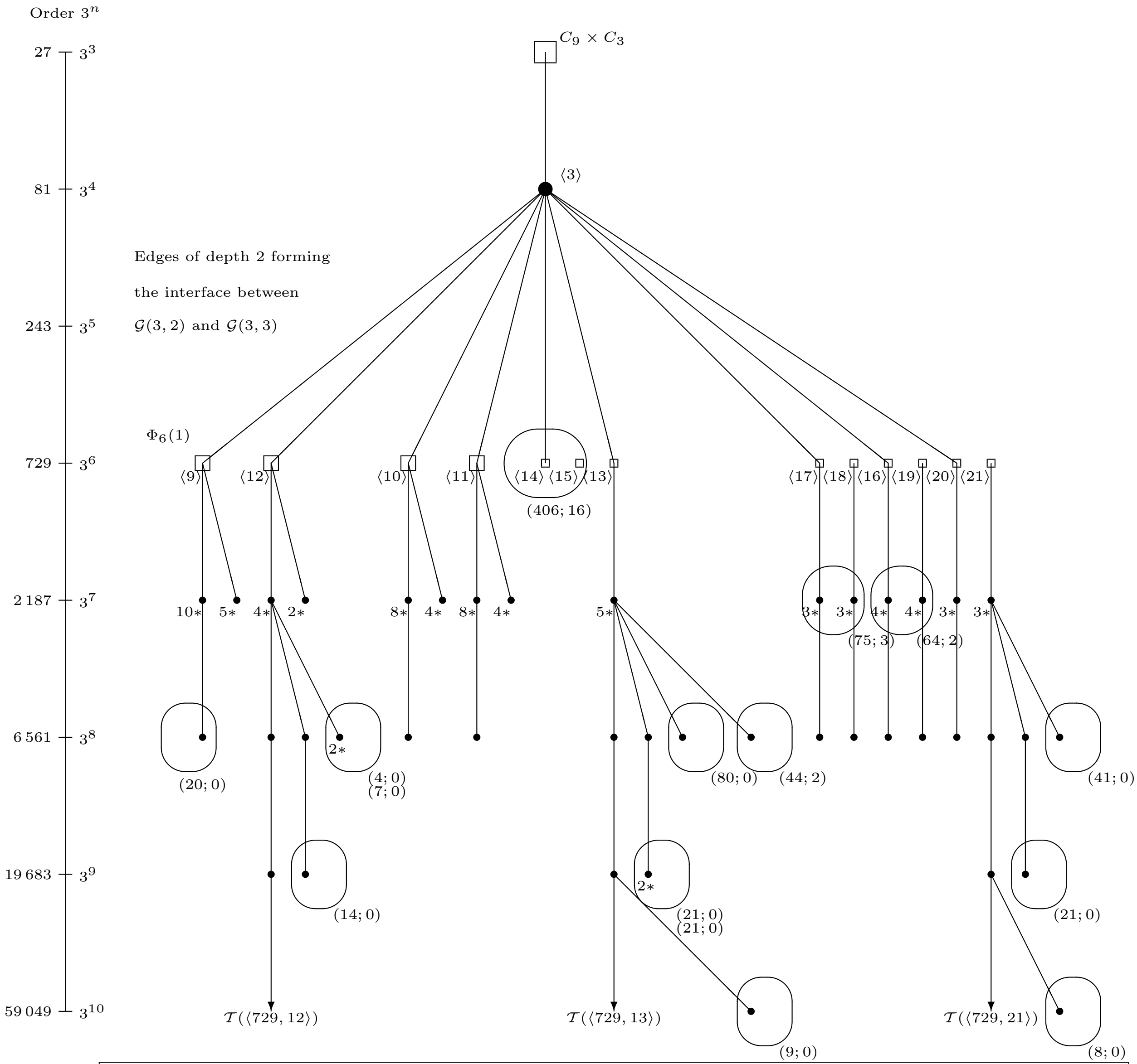
in dependence on the punctured TKT $\varkappa$ of K.
Here, ε denotes the number of 3-class groups $\mathrm{Cl}_3(N_i)$ of rank greater than 2.

Proof: D.C.Mayer, July 2011, [6] Metabelian 3-groups with abelianisation of type $(9,3)$, Thm.6.1.

Example 2.4. The smallest discriminant with punctured TKT b.3, where $\varepsilon = 1$, is $D = 635\,909$.

2.3.5. *Second* 3*-class groups* G, $\mathrm{cc}(G) = 3$, $G/G' \simeq (9,3)$

FIGURE 4. Sporadic groups and roots of coclass trees on the coclass graph $\mathcal{G}(3,3)$



TKT:	A.20	b.15	A.20	B.18	c.27	b.31	D.11	d.10	B.2	D.10	C.4	E.12	B.7	e.14	E.12	D.6
	(4444)	(0004)	(4444)	(1444)	(0440)	(0444)	(4232)	(0112)	(1112)	(1142)	(1122)	(1234)	(1114)	(1230)	(1234)	(1231)

Proof: E.A.O'Brien, 2011, private communication.

Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group with centre of type $(3, 3)$,
medium contour square: metabelian group with centre of type $(3, 3, 3)$,
small contour square: metabelian group with centre of type $(9, 3)$,
small full circle: metabelian group of order at least 2187.

Theorem 2.9. *(TTT $\tau(G)$ of branch 1 groups G in Φ_6)*
If $|G| = 3^6$, $\mathrm{cl}(G) = 3$, $G/G' \simeq (9,3)$, i.e.,
G is one of the 13 top vertices on coclass graph $\mathcal{G}(3,3)$,
then the structure of the 3-class groups of $N_1, \ldots, N_4$ and
the Frattini extension $\tilde{N}_4$ is given by

TKT	$\varkappa$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε	$\mathrm{Cl}_3(\tilde{N}_4)$
D.11	(4232)	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$
D.11	(4322)	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$
B.7	(1114)	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1	$(9,3,3)$
B.7	(1114)	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1	$(9,3,3)$
E.12	(1234)	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$
E.12	(1324)	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$
d.10	(0112)	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$
e.14	(1320)	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$
e.14	(1230)	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$
A.20	(4444)	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4	$(3,3,3,3)$
b.15	(0004)	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4	$(3,3,3,3)$
c.27	(0440)	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	4	$(3,3,3,3)$
b.31	(0444)	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	4	$(3,3,3,3)$

in dependence on the punctured TKT $\varkappa$ of K.
Again, ε denotes the number of 3-class groups $\mathrm{Cl}_3(N_i)$ of rank greater than 2.

Proof: D.C.Mayer, July 2011, [6] Metabelian 3-groups with abelianisation of type $(9,3)$, Thm.7.1.

Example 2.5. The first discriminant with punctured TKT D.11, where $\varepsilon = 2$, is $D = -3\,299$.

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