FINITE 3-GROUPS WITH TRANSFER KERNEL TYPE F

DANIEL C. MAYER AND M. F. NEWMAN

Abstract. For finite metabelian 3-groups G with abelianization G/G' of type $(3,3)$, coclass $r = \mathrm{cc}(G) \in \{4,6\}$, class $c = \mathrm{cl}(G) = r+1$, and transfer kernel type F, we determine the smallest non-trivial members of the cover, that is the set $\mathrm{cov}(G)$ of all finite 3-groups H whose second derived quotient H/H'' is isomorphic to G. We provide evidence of arithmetical realizations of these groups by second 3-class groups $G = \mathrm{G}_3^2 K = \mathrm{Gal}(\mathrm{F}_3^2 K/K)$, respectively 3-class tower groups $H = \mathrm{G}_3^\infty K = \mathrm{Gal}(\mathrm{F}_3^\infty K/K)$, of quadratic fields $K = \mathbb{Q}(\sqrt{d})$.

1. Introduction

For a finite 3-group G, let $(\gamma_j G)_{j\geq 1}$ denote the lower central series. In several recent presentations and papers [24, 25, 26, 11, 27, 28, 29, 30], we succeeded in determining the *cover* $\mathrm{cov}(G) = \{H \mid H/H'' \simeq G\}$ of all metabelian 3-groups G with class-1 quotient $G/\gamma_2 G \simeq C_3 \times C_3$ and *transfer kernel type* (TKT) E or c [21]. These groups share the fixed coclass $\mathrm{cc}(G) = 2$, and the common class-2 quotient $G/\gamma_3 G \simeq \langle 27, 3\rangle$, in the notation of the SmallGroups Library [3, 4]. Their class-3 quotient $G/\gamma_4 G$ is given by either $\langle 243, \mathbf{6}\rangle$ for type c.18, $\varkappa(G) \sim (0313)$, E.6, $\varkappa(G) \sim (1313)$, and E.14, $\varkappa(G) \sim (2313)$, or $\langle 243, \mathbf{8}\rangle$ for type c.21, $\varkappa(G) \sim (0231)$, E.8, $\varkappa(G) \sim (1231)$, and E.9, $\varkappa(G) \sim (2231)$ [19, Tbl., pp. 79–80], [21, § 3.3, Tbl. 6–7, pp. 492–494]. The cover of metabelian groups with type E or c is finite with cardinality proportional to the nilpotency class. Since the derived length of the members is bounded by 3, an algebraic number field with capitulation type E or c must have a 3-class tower with at most three stages [11, 29, 30].

In the present article, we determine the smallest members H with $\mathrm{dl}(H) \geq 3$ of the cover $\mathrm{cov}(G)$ of metabelian 3-groups G with abelianization $G/G' \simeq (3,3)$ and TKT F [21]. These groups may have any elevated coclass $r := \mathrm{cc}(G) \geq 3$, and thus share the common class-3 quotient $G/\gamma_4 G \simeq \langle 243, \mathbf{3}\rangle$.

Since our main intention is to shed light on the 3-class tower of quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with capitulation type F (§ 2), we focus on metabelian 3-groups G with even coclass $r = \mathrm{cc}(G) \in \{4,6\}$ and odd class $c := \mathrm{cl}(G) = r+1$ which admit an automorphism $\sigma \in \mathrm{Aut}(G)$ acting as inversion $\sigma : x \mapsto x^{-1}$ on the abelianization G/G'. Such groups are called σ-groups.

The groups G arise as *sporadic vertices of coclass graphs* $\mathcal{G}(3,r)$, outside of coclass trees (§ 3). Members of *periodic infinite sequences on coclass trees* $\mathcal{T}^r \subset \mathcal{G}(3,r)$ [12, 13] will be investigated in a subsequent paper.

2. First step: Gathering number theoretic information

2.1. **History of transfer kernel type** F. *Complex* quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 K$ of type $(3,3)$ and transfer kernel type (TKT) F have been detected by Brink in 1984 [9]. The absolute values of their fundamental discriminants d set in with 27 156, outside of the ranges investigated by Scholz and Taussky in 1934 [36], and by Heider and Schmithals in 1982 [16]. However, the computational results in Brink's Thesis [9, Appendix A, pp. 96–113] were unknown to us until we got a copy via ProQuest in 2006. Their actual extent is not mentioned explicitly in

Date: October 23, 2016.

2000 *Mathematics Subject Classification.* Primary 20D15, 20F12, 20F14; secondary 11R37, 11R29, 11R16.

Key words and phrases. Finite 3-groups of coclass 4 or 6, transfer kernel type F, polycyclic pc-presentations, central series, lattice of normal subgroups, cover, relation rank, nuclear rank, descendant trees; 3-class towers, 3-capitulation type, second 3-class group, quadratic fields.

Research of the first author supported by the Austrian Science Fund (FWF): P 26008-N25.

the official paper [10] by Brink and his academic advisor Gold. Therefore, we previously believed to have the priority in discovering the discriminant $d = -27\,156$ of a field K with type F.11 in 1989 [19, Tbl., p. 84], and the discriminants $d = -31\,908$, $-67\,480$, $-124\,363$ of fields K with types F.12, F.13, F.7 in 2003 [20, Tbl. 3, p. 497], all of them with second 3-class groups G_3^2K of coclass 4. In 2006, it turned out that our claim must be restricted to $d = -124\,363$, which after nearly 20 years eventually provided the first example for type F.7, called the unique undiscovered type by Brink [9, § 7.2, p. 91].

It required further 10 years until we had the courage to study the 3-class tower of number fields with transfer kernel type F, based on abelian type invariants of second order, as developed in [28].

As opposed to coclass 4, we can definitely claim priority in discovering the discriminant $d = -423\,640$ of a *complex* quadratic field $K = \mathbb{Q}(\sqrt{d})$ with type F.12 in 2010 [20, Tbl. 3, p. 497], and the discriminants $d = -1\,677\,768$, $-2\,383\,059$, $-4\,838\,891$ of fields K with types F.7, F.13, F.11 in 2016, all of them with second 3-class groups G_3^2K of coclass 6.

Similarly, we were the first who found the discriminant $d = 8\,321\,505$ of a *real* quadratic field $K = \mathbb{Q}(\sqrt{d})$ with type F.13 in 2010 [20, Tbl. 4, p. 498], and the discriminants $d = 10\,165\,597$, $22\,937\,941$, $66\,615\,244$ of fields K with types F.7, F.12, F.11 in 2016 [32, Tbl. 4, p. 1291], all of which possess second 3-class groups G_3^2K of coclass 4.

TABLE 1. Abelian type invariants $\tau^{(2)}K$ of 2nd order for $K = \mathbb{Q}(\sqrt{d})$ real with $\mathrm{cc}(\mathrm{G}_3^2K) = 4$

Type	$\tau^{(2)}K = [1^2; (32; 2^31, T_1), (32; 2^31, T_2), (1^3; 2^31, T_3), (1^3; 2^31, T_4)]$			
d	T_1	T_2	T_3	T_4
F.7				
10 165 597	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(21^2)^3, (1^3)^9$
49 425 848	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(\mathbf{21^3})^{\mathbf{3}}, (\mathbf{1^4})^{\mathbf{9}}$
85 309 765	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
F.11				
66 615 244	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
75 246 413	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
76 575 261	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
F.12				
22 937 941	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
32 466 649	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
64 177 681	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
69 716 760	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(\mathbf{1^4})^{\mathbf{12}}$
95 283 149	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
97 052 709	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(21^2)^3, (1^3)^9$
F.13				
8 321 505	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
17 373 109	$(31^2)^3$	$(31^3)^3$	$(21^2)^3, (1^3)^9$	$(21^2)^3, (1^3)^9$
51 376 888	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(\mathbf{1^4})^{\mathbf{12}}$
72 034 376	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(21^2)^3, (1^3)^9$
93 285 944	$(31^2)^3$	$(\mathbf{31^2})^{\mathbf{3}}$	$(\mathbf{21^2})^{\mathbf{12}}$	$(21^2)^3, (1^3)^9$

2.2. **Artin patterns for coclass 4.** In Table 1, resp. 2, we present arithmetic information about iterated index-p abelianization data (IPADs), $\tau^{(2)}K = \left[\mathrm{Cl}_3K; \left(\mathrm{Cl}_3L_i; (\mathrm{Cl}_3M)_{M\in\mathrm{Lyr}_1L_i}\right)_{1\le i\le 4}\right]$, of second order for *real*, resp. *complex*, quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 K$ of type $1^2 \widehat{=} (3,3)$, TKT F, and a second 3-class group $\mathrm{G}_3^2 K$ of coclass 4, which occur in the range $0 < d < 10^8$, resp. $-5\cdot 10^5 < d < 0$, of fundamental discriminants. With exception of $d = 8\,321\,505$ [20, Tbl. 4, p. 498], the positive discriminants were discovered and investigated in March 2016 and published in [32, Tbl. 4, p. 1291]. The negative discriminants were taken from the lower half range

TABLE 2. Abelian type invariants $\tau^{(2)}K$ of 2nd order for $K = \mathbb{Q}(\sqrt{d})$ complex with $\mathrm{cc}(\mathrm{G}_3^2 K) = 4$

Type	$\tau^{(2)}K = [1^2; (32; 2^31, T_1), (32; 2^31, T_2), (1^3; 2^31, T_3), (1^3; 2^31, T_4)]$			
$-d$	T_1	T_2	T_3	T_4
F.7				
124 363	$\mathbf{(421^2)^3}$	$\mathbf{(321^2)^3}$	$\mathbf{(2^31)^3, (1^6)^3, (2^21^2)^6}$	$\mathbf{(2^21^3)^3, (21^4)^3, (2^21^2)^6}$
225 299	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
260 515	$\mathbf{(3^221)^3}$	$\mathbf{(3^221)^3}$	$\mathbf{(32^21)^3, (2^31)^3, (1^6)^3, (2^21^2)^3}$	$\mathbf{(32^21)^3, (1^6)^3, (21^4)^3, (2^21^2)^3}$
343 380	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
423 476	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
486 264	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
F.11				
27 156	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
241 160	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
394 999	$(\mathbf{31}^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(1^5)^3, (21^3)^3, (1^4)^6}$
477 192	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
484 804	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
F.12				
31 908	$(31^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
135 587	$(31^3)^3$	$(31^3)^3$	$\mathbf{(2^21^2)^3, (21^3)^6}, (2^21)^3$	$\mathbf{(1^5)^3, (21^3)^3, (1^4)^6}$
160 403	$\mathbf{(321^2)^3}$	$\mathbf{(321^2)^3}$	$\mathbf{(2^4)^3, (21^4)^6, (2^21^2)^3}$	$\mathbf{(2^21^2)^3, (1^5)^6, (21^3)^3}$
184 132	$(\mathbf{41}^3)^3$	$(31^3)^3$	$\mathbf{(2^21^2)^3, (21^3)^6}, (2^21)^3$	$\mathbf{(2^21^2)^3, (21^3)^3, (1^4)^6}$
189 959	$(31^3)^3$	$(31^3)^3$	$\mathbf{(2^21^2)^3, (21^3)^6}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
291 220	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
454 631	$(31^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
499 159	$(31^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^3, (1^5)^3, (1^4)^6}$
F.13				
67 480	$(41^3)^3$	$(31^3)^3$	$\mathbf{(321^2)^3, (21^3)^6}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
104 627	$(41^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(1^5)^3, (21^3)^3, (1^4)^6}$
167 064	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
224 580	$\mathbf{(321^2)^3}$	$\mathbf{(321^2)^3}$	$\mathbf{(21^4)^3, (2^21^2)^3, (1^5)^6}$	$\mathbf{(2^21^2)^3, (1^5)^3, (21^3)^6}$
287 155	$(41^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
296 407	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
317 747	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
344 667	$(41^3)^3$	$(31^3)^3$	$\mathbf{(2^21^2)^3, (21^3)^6}, (2^21)^3$	$\mathbf{(2^21^2)^3, (21^3)^3, (1^4)^6}$
401 603	$(41^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
426 891	$(41^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^6, (1^4)^6}$
487 727	$(\mathbf{31}^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^9}, (2^21)^3$	$\mathbf{(21^3)^3, (1^5)^3, (1^4)^6}$

of [20, Tbl. 3, p. 497], but they were separated into the four TKTs in June 2016. The IPAD of first order of such a field has the form $\tau^{(1)}K = \left[\mathrm{Cl}_3K; (\mathrm{Cl}_3L_i)_{1\le i\le 4}\right] = \left[1^2; \left(32, 32, 1^3, 1^3\right)\right]$, according to [23, Thm. 4.5, pp. 444–445, and Tbl. 6.10, p. 455]. We point out that we use logarithmic type invariants throughout this article, e.g., $32 \widehat{=} (27, 9)$ and $1^3 \widehat{=} (3, 3, 3)$. Since the Hilbert 3-class field of the fields K under investigation has 3-class group $\mathrm{Cl}_3\mathrm{F}_3^1K \simeq 2^31 \widehat{=} (9, 9, 9, 3)$, the iterated IPAD of second order of K has the shape $\tau^{(2)}K = \left[1^2; (32; 2^31, T_1), (32; 2^31, T_2), (1^3; 2^31, T_3), (1^3; 2^31, T_4)\right]$, where the families T_1, T_2, resp. T_3, T_4, consist of 3, resp. 12, remaining components. Exceptional entries are printed in **boldface** font.

Definition 2.1. The 3-class tower of the *real* quadratic field K with type F and G_3^2K of coclass 4 resides in the *tower ground state*, if the iterated IPAD of second order of K is given by

$$\tau^{(2)}K = [1^2; (32; 2^31, (31^2)^3), (32; 2^31, (31^3)^3), (1^3; 2^31, (21^2)^3, (1^3)^9)^2]. \tag{2.1}$$

The 3-class tower of the *complex* quadratic field K with type F and G_3^2K of coclass 4 resides in the *tower ground state*, if the iterated IPAD of second order of K is given by

$$\tau^{(2)}K = [1^2; (32; 2^31, T_1), (32; 2^31, (31^3)^3), (1^3; 2^31, (2^21)^3, (21^2)^9)^2], \tag{2.2}$$

where $T_1 = (41^3)^3$ if K is of type F.11 or F.13, and $T_1 = (31^3)^3$ if K is of type F.7 or F.12.

TABLE 3. Abelian type invariants $\tau^{(2)}K$ of 2nd order for $K = \mathbb{Q}(\sqrt{d})$ complex with $\mathrm{cc}(\mathrm{G}_3^2K) = 6$

Type	$\tau^{(2)}K = [1^2; (43; 3^32, T_1), (43; 3^32, T_2), (1^3; 3^32, T_3), (1^3; 3^32, T_4)]$			
$-d$	T_1	T_2	T_3	T_4
F.7				
1 677 768	$(521^2)^3$	$(421^2)^3$	$\boldsymbol{(321^2)^3}, \boldsymbol{(21^3)^6}, (2^21)^3$	$\boldsymbol{(21^3)^6}, \boldsymbol{(1^4)^6}$
5 053 191	$(421^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
8 723 023	$(421^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
F.11				
4 838 891	$(521^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
5 427 023	$(421^2)^3$	$(421^2)^3$	$\boldsymbol{(21^3)^9}, \boldsymbol{(2^21)^3}$	$\boldsymbol{(1^5)^3}, \boldsymbol{(21^3)^3}, \boldsymbol{(1^4)^6}$.
8 493 815	$(521^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
F.12				
423 640	$(421^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
8 751 215	$(521^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
F.13				
2 383 059	$\boldsymbol{(42^21)^3}$	$\boldsymbol{(42^21)^3}$	$\boldsymbol{(21^4)^3}, \boldsymbol{(2^21^2)^3}, \boldsymbol{(1^5)^6}$	$\boldsymbol{(21^4)^3}, \boldsymbol{(2^31)^6}, \boldsymbol{(2^21^2)^3}$
5 444 651	$(421^2)^3$	$(421^2)^3$	$\boldsymbol{(2^21^2)^3}, \boldsymbol{(21^3)^6}, \boldsymbol{(2^21)^3}$	$\boldsymbol{(2^21^2)^3}, \boldsymbol{(21^3)^3}, \boldsymbol{(1^4)^6}$
5 606 283	$(421^2)^3$	$(421^2)^3$	$\boldsymbol{(21^3)^6}, \boldsymbol{(1^4)^6}$	$\boldsymbol{(21^3)^9}, \boldsymbol{(2^21)^3}$
5 765 812	$\boldsymbol{(52^21)^3}$	$\boldsymbol{(42^21)^3}$	$\boldsymbol{(21^4)^3}, \boldsymbol{(1^5)^6}, \boldsymbol{(2^21^2)^3}$	$\boldsymbol{(21^4)^3}, \boldsymbol{(2^21^2)^9}$
6 863 219	$(521^2)^3$	$(421^2)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$
8 963 839	$(421^2)^3$	$(421^2)^3$	$\boldsymbol{(2^21^2)^3}, \boldsymbol{(21^3)^6}, (2^21)^3$	$\boldsymbol{(1^5)^3}, \boldsymbol{(21^3)^3}, \boldsymbol{(1^4)^6}$

2.3. **Artin patterns for coclass 6.** In Table 3 we summarize the iterated IPADs of second order $\tau^{(2)}K = \left[\mathrm{Cl}_3K; \left(\mathrm{Cl}_3L_i; (\mathrm{Cl}_3M)_{M\in\mathrm{Lyr}_1L_i}\right)_{1\le i\le 4}\right]$ of the few complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3K \simeq 1^2$, TKT F, and G_3^2K of coclass 6, which occur in the range

$-10^7 < d < 0$ of fundamental discriminants. With exception of $d = -423\,640$ [20, Tbl. 3, p. 497], these discriminants were discovered and investigated in June 2016. The IPAD of first order of such a field has the form $\tau^{(1)}K = \left[\mathrm{Cl}_3 K; (\mathrm{Cl}_3 L_i)_{1\le i\le 4}\right] = \left[1^2; \left(43, 43, 1^3, 1^3\right)\right]$, according to [23, Thm. 4.5, pp. 444–445, and Tbl. 6.11, p. 455]. Since the Hilbert 3-class field of these fields K has 3-class group $\mathrm{Cl}_3\mathrm{F}_3^1 K \simeq 3^3 2$, the iterated IPAD of second order of K has the form $\tau^{(2)}K = \left[1^2; (43; 3^3 2, T_1), (43; 3^3 2, T_2), (1^3; 3^3 2, T_3), (1^3; 3^3 2, T_4)\right]$, where the families T_1, T_2, resp. T_3, T_4, consist of 3, resp. 12, remaining components. As before, exceptional entries are printed in **boldface** font.

Definition 2.2. The 3-class tower of the *complex* quadratic field K with type F and $\mathrm{G}_3^2 K$ of coclass 6 resides in the *tower ground state*, if the iterated IPAD of second order of K is given by

$$\tau^{(2)}K = [1^2; (43; 3^3 2, T_1), (43; 3^3 2, (421^2)^3), (1^3; 3^3 2, (2^2 1)^3, (21^2)^9)^2], \tag{2.3}$$

where $T_1 = (521^2)^3$ if K is of type F.11 or F.13, and $T_1 = (421^2)^3$ if K is of type F.7 or F.12.

3. Second step: Searching for suitable metabelian 3-groups

3.1. **Nebelung's infinite main trunk.** 3-groups G with coclass $\mathrm{cc}(G) = 1$ were investigated by N. Blackburn [5] in 1958. All of these CF-groups have abelianization $G/G' \simeq (3, 3)$ and abelian commutator subgroup G'. Twenty years later, J. A. Ascione wrote her Thesis [1, 2] about two-generated 3-groups with coclass $\mathrm{cc}(G) = 2$, which split into CF-groups with $G/G' \simeq (9, 3)$ and non-CF groups with $G/G' \simeq (3, 3)$. The latter arise from immediate descendants of step size $s = 2$ of Blackburn's group $G_0^3(0, 0) = \langle 27, 3\rangle$. Ascione recognized that many groups under her investigation can be arranged in periodic branches of infinite *coclass trees* [27], as they were called after rigorous proofs of their structure were developed by M. du Sautoy [12] and independently by B. Eick and C. Leedham-Green [13].

Further ten years later, B. Nebelung [33] succeeded in determining parametrized presentations $G = G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$ for all *metabelian* 3-groups G with $G/G' \simeq (3, 3)$, in particular for the non-CF groups with *elevated coclass* $\mathrm{cc}(G) \ge 3$, which were unknown previously. Her crucial idea was to show the existence of an infinite *main trunk* $(P_{2j+1})_{j\ge 1}$ (Figure 1) consisting of metabelian 3-groups such that all desired groups with fixed coclass $r = j+1$ arise from the vertex P_{2j+1}, more precisely, from an immediate descendant of step size $s = 2$ of P_{2j+1} (which causes the non-CF property). In contrast to the mainline of a coclass tree, where each successor is an immediate descendant of step size $s = 1$ of its predecessor, the vertex P_{2j+3} is an immediate descendant of step size $s = 2$ of the vertex P_{2j+1}, for each $j \ge 1$. Thus, the main trunk was the first example of *periodic bifurcations* in a descendant tree [27]. (Cfr. [21, p. 485], [22, Thm. 3.15, pp. 440–441]).

Theorem 3.1. *(The main trunk; Nebelung,* 1989, [33, p. 192]*)*

(1) *In the descendant tree $\mathcal{T}(R)$ of the abelian root $R := C_3 \times C_3 = \langle 9, 2\rangle$, there exists a unique infinite path of (reverse) directed edges $(P_{2j+1} \leftarrow P_{2j+3})_{j\ge 1}$ such that, for each fixed coclass $r = j + 1 \ge 2$, every metabelian* 3*-group G with $G/G' \simeq (3, 3)$ and $\mathrm{cc}(G) = r$ is a proper descendant of P_{2j+1}.*

(2) *The trailing vertex P_3 is exactly the extra special Blackburn group $G_0^3(0, 0) = \langle 27, 3\rangle$ with exceptional transfer kernel type (TKT)* a.1, *$\varkappa = (0000)$.*

(3) *All the other vertices P_{2j+1} with $j \ge 2$ share the common TKT* b.10, *$\varkappa = (0043)$, possess nilpotency class $c = j + 1$, coclass $r = j$, logarithmic order $c + r = 2j + 1$, abelian commutator subgroup of type $D := \mathrm{A}(3, c - 1) \times \mathrm{A}(3, r - 1)$, IPAD of first order $\tau^{(1)} = \left[1^2; \mathrm{A}(3, c), \mathrm{A}(3, r + 1), 1^3, 1^3\right]$, where $r + 1 = c$, and iterated IPAD of second order $\tau^{(2)} = \left[1^2; (\mathrm{A}(3, c); D, \mathrm{B}(3, c - 1) \times C(3))^2, (1^3; D, (1^3)^{12})^2\right]$, where*

$$\mathrm{B}(3, c-1) := \begin{cases} C(3^t) \times C(3^{t-1}) & \text{if } c = 2t \text{ is even,} \\ C(3^t) \times C(3^{t-2}) & \text{if } c = 2t - 1 \text{ is odd.} \end{cases}$$

(4) *For $j \geq 4$, periodicity of length 2 sets in, P_{2j+1} has nuclear rank $\nu = 2$, p-multiplicator rank $\mu = 6$, and immediate descendant numbers (including non-metabelian groups)*

$$(N_1/C_1, N_2/C_2) = \begin{cases} (21/1, 151/21) & \text{if } j \geq 4 \text{ is even,} \\ (30/1, 295/37) & \text{if } j \geq 5 \text{ is odd.} \end{cases}$$

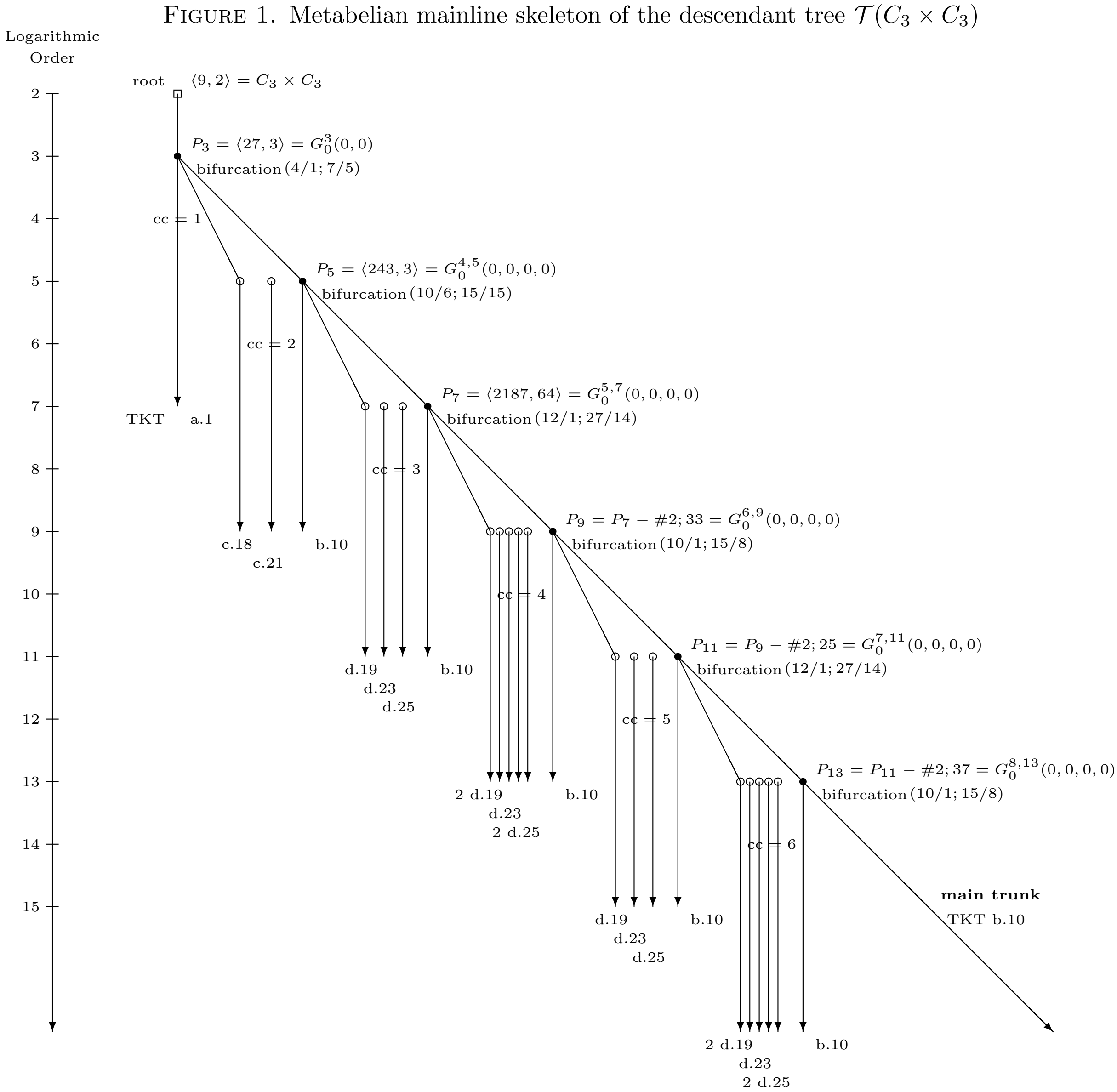
Restricted to metabelian groups, the immediate descendant numbers are

$$(\tilde{N}_1/\tilde{C}_1, \tilde{N}_2/\tilde{C}_2) = \begin{cases} (10/1, 15/8) & \text{if } j \geq 4 \text{ is even,} \\ (12/1, 27/14) & \text{if } j \geq 3 \text{ is odd.} \end{cases}$$

All immediate descendants are σ-groups, if $j \geq 1$ is odd, but only $(3/3, 1/1)$, if $j = 2$, and $(3/1, 1/1)$, if $j \geq 4$ is even.

Corollary 3.1. *(All coclass trees with metabelian mainlines; Nebelung,* [33, § 5.2, pp. 181–195]*)*

FIGURE 1. Metabelian mainline skeleton of the descendant tree $\mathcal{T}(C_3 \times C_3)$



The coclass trees of 3*-groups* G *with* $G/G' \simeq (3,3)$*, whose mainlines consist of metabelian vertices, possess the following remarkable periodicity of length* 2*, drawn impressively in Figure* 1*.*

(1) *For even* $j \geq 2$*, the vertex* P_{2j+1} *with subscript* $2j+1 \geq 5$ *of the main trunk has exactly* 4 *immediate descendants of step size* $s = 2$ *giving rise to coclass trees* $\mathcal{T}^{j+1} \subset \mathcal{G}(3, j+1)$ *whose mainline vertices are metabelian* 3*-groups* G *with odd* $\mathrm{cc}(G) = j+1$ *and fixed TKT, either* d.19*,* $\varkappa = (4043)$*, or* d.23*,* $\varkappa = (1043)$*, or* d.25*,* $\varkappa = (2043)$*, or* b.10*,* $\varkappa = (0043)$*, the latter with root* P_{2j+3}*.*

(2) *For odd* $j \geq 3$*, the vertex* P_{2j+1} *with subscript* $2j+1 \geq 7$ *of the main trunk has exactly* 6 *immediate descendants of step size* $s = 2$ *giving rise to coclass trees* $\mathcal{T}^{j+1} \subset \mathcal{G}(3, j+1)$ *whose mainline vertices are metabelian* 3*-groups* G *with even* $\mathrm{cc}(G) = j+1$ *and fixed TKT, either* d.19*,* $\varkappa = (4043)$*, twice, or* d.23*,* $\varkappa = (1043)$*, or* d.25*,* $\varkappa = (2043)$*, twice, or* b.10*,* $\varkappa = (0043)$*, the latter with root* P_{2j+3}*.*

(3) *The unique pre-periodic exception is the vertex* P_3 *of the main trunk, which has exactly* 3 *immediate descendants of step size* $s = 2$ *giving rise to coclass trees* $\mathcal{T}^2 \subset \mathcal{G}(3,2)$ *whose mainline vertices are metabelian* 3*-groups* G *with even* $\mathrm{cc}(G) = 2$ *and fixed TKT, either* c.18*,* $\varkappa = (0313)$*, or* c.21*,* $\varkappa = (0231)$*, or* b.10*,* $\varkappa = (0043)$*, the latter with root* P_5*.*

3.2. **Sporadic vertices outside of coclass trees.** Now we begin our search for finite *metabelian* σ-groups of minimal order with type F. According to Theorem 3.1, they belong to the sporadic part of the coclass graph $\mathcal{G}(3,4)$, because groups with type F and coclass 3 are not σ-groups.

Theorem 3.2. *There exist precisely* 13 *metabelian* 3*-groups* G *of order* $|G| = 3^9$*, class* $\mathrm{cl}(G) = 5$*, coclass* $\mathrm{cc}(G) = 4$*, and relation rank* $d_2G = 4$*, having transfer kernel types (TKTs) in section* F*. They are immediate descendants of step size* $s = 2$ *of the parent group* $P_7 = \langle 2187, 64 \rangle$ *in the SmallGroups library* [3, 4]*, that is, their last lower central* $\gamma_5 G$ *is of type* $(3,3)$ *and* $P_7 \simeq G/\gamma_5 G$ *is their common class-*4 *quotient. In the notation of the ANUPQ package* [14] *of GAP* [15] *and MAGMA* [18]*, they are given by* $G = P_7 - \#2; m$ *with*

$$\begin{cases} m \in \{36, 38\} & \textit{for TKT } \mathrm{F}.11,\ \varkappa(G) = (1143), \\ m \in \{41, 47, 50, 52\} & \textit{for TKT } \mathrm{F}.13,\ \varkappa(G) = (3143), \\ m \in \{43, 46, 51, 53\} & \textit{for TKT } \mathrm{F}.12,\ \varkappa(G) = (1343), \\ m \in \{55, 56, 58\} & \textit{for TKT } \mathrm{F}.7,\ \varkappa(G) = (3443). \end{cases}$$

Proof. We use the p-group generation algorithm [34, 35, 17] as implemented in the computational algebra system Magma [6, 7, 18] to construct these 13 groups. We start with $P :=$`SmallGroup`$(2187, 64)$, $c :=$`NilpotencyClass`(P), call the Magma function $D :=$`descendants`$(P, c{+}1 :$`step sizes`$:= [2])$, and test all members of the list D for a suitable TKT in section F, making use of our own implementation of the Artin transfer homomorphisms and σ-automorphism checking. □

Remark 3.1. A different proof of Theorem 4.1 is possible by using results of Nebelung [33], which contain parametrized presentations $G_\rho^{6,9}(\alpha,\beta,\gamma,\delta)$ of the groups with type F, $\rho = 0$, index of nilpotency $\mathrm{cl}(G) + 1 = 6$, and logarithmic order $\mathrm{lo}(G) = 9$. The quartet $(\alpha,\beta,\gamma,\delta)$ is given by $(1,1,0,0)$ for $\ell = 36$, $(1,-1,0,0)$ for $\ell = 38$, $(1,1,-1,0)$ for $\ell = 41$, $(-1,-1,1,0)$ for $\ell = 47$, $(1,-1,-1,0)$ for $\ell = 50$, $(-1,1,1,0)$ for $\ell = 52$, $(1,1,0,-1)$ for $\ell = 43$, $(-1,-1,0,1)$ for $\ell = 46$, $(-1,1,0,-1)$ for $\ell = 51$, $(1,-1,0,1)$ for $\ell = 53$, $(1,1,-1,1)$ for $\ell = 55$, $(1,-1,-1,-1)$ for $\ell = 56$, and $(-1,1,1,1)$ for $\ell = 58$.

Figure 2 shows the complete normal lattice of the groups G in Theorems 3.3 and 3.2. The lattice consists of diamonds of type $(3,3)$. Omitting two of the four cyclic subgroups, we draw each diamond as a square standing on one of its vertices. The members $\gamma_j G$, $1 \leq j \leq \mathrm{cl}(G)+1$, of the lower central series are indicated by tiny full discs. Except for the mandatory bottleneck $\gamma_2 G/\gamma_3 G$, all factors $\gamma_j G/\gamma_{j+1} G$ are bicyclic. Thus we call G a BF-group (as opposed to a CF-group [33]). For such groups, the upper central series $\zeta_j G$, $0 \leq j \leq \mathrm{cl}(G)$, is just the reverse lower central series. To enable a comparison, we emphasize that the smallest metabelian Schur σ-groups of order 3^5 with type D, i.e., the two groups $\langle 243, 5|7 \rangle$, have a similar but more simple normal structure [20, 23, 22].

The following finite metabelian σ-groups of bigger order and type F belong to the sporadic part of the coclass graph $\mathcal{G}(3,6)$, again in view of Theorem 3.1.

Theorem 3.3. *There exist precisely* 13 *metabelian* 3*-groups* G *of order* $|G| = 3^{13}$*, class* $\mathrm{cl}(G) = 7$*, coclass* $\mathrm{cc}(G) = 6$*, and relation rank* $d_2G = 4$*, having transfer kernel types (TKTs) in section* F*. They are immediate descendants of step size* $s = 2$ *of the parent group* $P_{11} = P_7 - \#2; 33 - \#2; 25$ *in the notation of the ANUPQ package* [14] *of GAP* [15] *and MAGMA* [18]*, that is, their last lower central* $\gamma_7 G$ *is of type* $(3,3)$ *and* $P_{11} \simeq G/\gamma_7 G$ *is their common class-*6 *quotient. They are given by* $G = P_{11} - \#2; m$ *with*

$$\begin{cases} m \in \{40, 42\} & \textit{for TKT } \mathrm{F.11},\ \varkappa(G) = (1143), \\ m \in \{45, 51, 54, 56\} & \textit{for TKT } \mathrm{F.13},\ \varkappa(G) = (3143), \\ m \in \{47, 50, 55, 57\} & \textit{for TKT } \mathrm{F.12},\ \varkappa(G) = (1343), \\ m \in \{59, 60, 62\} & \textit{for TKT } \mathrm{F.7},\ \varkappa(G) = (3443). \end{cases}$$

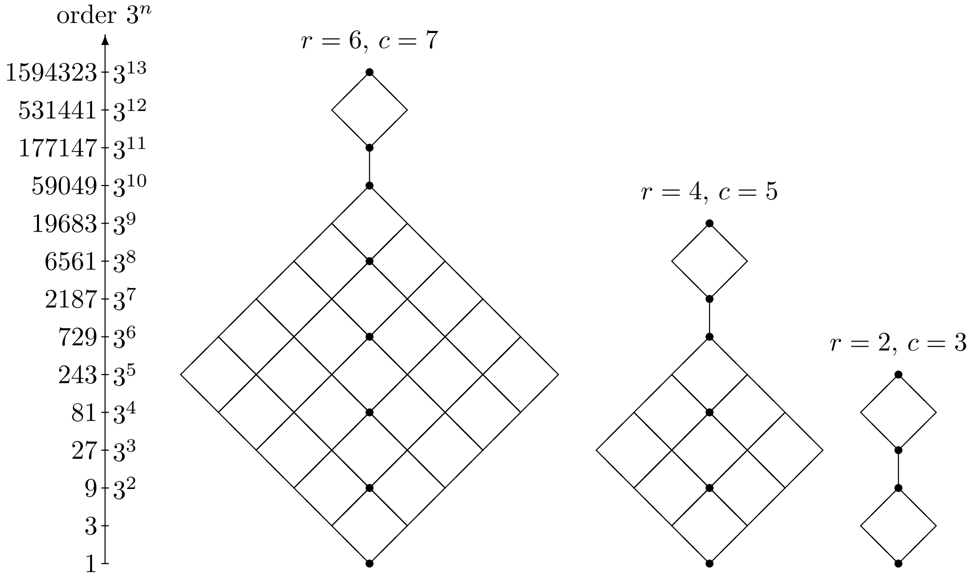
Remark 3.2. The group $\langle 2187, 64\rangle - \#2; 33$ is a sibling of the 13 groups in Theorem 3.2 and the grandparent of the 13 groups in Theorem 3.3.

Proof. Again, we use the p-group generation algorithm [34, 35, 17] as implemented in the computational algebra system Magma [6, 7, 18] to construct these 13 groups. We start with $P = \langle 2187, 64\rangle - \#2; 33 - \#2; 25$, given by its compact presentation s, i.e. $P :=$`PCGroup`(s), $c :=$`NilpotencyClass`(P), call the Magma function $D :=$`descendants`$(P, c+1$ `:step sizes:=` $[2])$, and test all members of the list D for a suitable TKT in section F, making use of our own implementation of the Artin transfer homomorphisms and σ-automorphism checking. $\square$

Remark 3.3. Again, Theorem 3.3 can be proved with the aid of Nebelung's Thesis [33], which gives parametrized presentations $G^{8,13}_{\rho}(\alpha, \beta, \gamma, \delta)$ of the groups with type F, $\rho = 0$, index of nilpotency $\mathrm{cl}(G) + 1 = 8$, and logarithmic order $\mathrm{lo}(G) = 13$. The quartet $(\alpha, \beta, \gamma, \delta)$ is given by $(1,1,0,0)$ for $\ell = 40$, $(1,-1,0,0)$ for $\ell = 42$, $(1,1,-1,0)$ for $\ell = 45$, $(-1,-1,1,0)$ for $\ell = 51$, $(1,-1,-1,0)$ for $\ell = 54$, and $(-1,1,1,0)$ for $\ell = 56$.

The *metabelian* σ-groups $G = P_7 - \#2; m$ of coclass 4 in Theorem 3.2 are the unique contestants for the second 3-class group $\mathrm{G}^2_3 K$ of (complex and real) quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with IPAD $\tau^{(1)} K = \left[1^2; (32)^2, (1^3)^2\right]$ and 3-capitulation type F. Since their relation rank is uniformly given by $d_2 G = 4$, the Shafarevich Theorem [37] discourages them as 3-class tower groups $\mathrm{G}^\infty_3 K$ of

FIGURE 2. 3-groups of orders 3^{13}, 3^9 with TKT F, and of order 3^5 with TKT D.



quadratic fields K, both, complex and real. All of them share the common iterated IPAD of second order

$$\tau^{(2)}G = \left[1^2; (32; 2^3 1, (31^2)^3)^2, (1^3; 2^3 1, (1^3)^{12})^2\right]. \tag{3.1}$$

The *metabelian* σ-groups $G = P_{11} - \#2; m$ of coclass 6 in Theorem 3.3 are the unique contestants for the second 3-class group $\mathrm{G}_3^2 K$ of (complex and real) quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with IPAD $\tau^{(1)}K = \left[1^2; (43)^2, (1^3)^2\right]$ and 3-capitulation type F. Since their relation rank is uniformly given by $d_2 G = 4$, the Shafarevich Theorem [37] discourages them as 3-class tower groups $\mathrm{G}_3^\infty K$ of quadratic fields K, both, complex and real. All of them share the common iterated IPAD of second order

$$\tau^{(2)}G = \left[1^2; (43; 3^3 2, (421)^3)^2, (1^3; 3^3 2, (1^3)^{12})^2\right]. \tag{3.2}$$

4. Third step: Constructing the smallest members of the cover

4.1. **Cover with relation rank 3 for real fields.** We begin with the smallest *non-metabelian* σ-groups H which have relation rank $d_2 H = 3$ and are candidates for 3-class tower groups of *real* quadratic fields K, according to Shafarevich [37].

Theorem 4.1. *The non-trivial members H of minimal order $|H| = 3^{10}$, class $\mathrm{cl}(H) = 5$, coclass $\mathrm{cc}(H) = 5$, and derived length $\mathrm{dl}(H) = 3$, of the cover $\mathrm{cov}(G) = \{H \mid H/H'' \simeq G\}$ of the 13 groups $G = P_7 - \#2; m$ with type* F *in Theorem* 3.2 *are* 96 *immediate descendants of step size* 3 *of the parent group $P_7 = \langle 2187, 64\rangle$, that is, their last lower central $\gamma_5 H$ is of type $(3,3,3)$ and $P_7 \simeq H/\gamma_5 H$ is their common class-4 quotient. They are of the form $H = P_7 - \#3; \ell$ with identifiers ℓ given in Table* 4*, where terminal groups with $d_2 H = 3$ and capable groups with $d_2 H = 4$ are distinguished.*

Table 4. Cover groups of order 3^{10} of 3-groups of order 3^9 with TKT F

m	Terminal for $\ell =$	Capable for $\ell =$	Total count
F.11			
36	$140, 141$	$239, 254, 260, 310, 313, 316$	8
38	$143, 144$	$240, 255, 261, 268, 271, 274$	8
F.13			
41	$148, 149$	$281, 296, 302, 312, 315, 318$	8
47	$158, 159$	$269, 272, 275, 324, 339, 345$	8
50	$162, 171$	$242, 248, 263, 325, 331, 346$	8
52	$164, 166$	$243, 249, 264, 283, 289, 304$	8
F.12			
43	$151, 152$	$270, 273, 276, 282, 297, 303$	8
46	$156, 157$	$311, 314, 317, 323, 338, 344$	8
51	$163, 176$	$245, 251, 257, 328, 334, 340$	8
53	$165, 177$	$246, 252, 258, 286, 292, 298$	8
F.7			
55	$168, 178$	$287, 293, 299, 330, 336, 342$	8
56	169	$285, 291, 306$	4
58	172	$326, 332, 347$	4

Proof. We use the p-group generation algorithm [34, 35, 17] as implemented in the computational algebra system Magma [6, 7, 18] to construct these 96 groups. We start with $P :=$`SmallGroup`$(2187, 64)$, $c :=$`NilpotencyClass`(P), call the Magma function $D :=$`descendants`$(P, c+1$:`step sizes`$:= [3])$, and test all members of the list D for a suitable TKT in section F, making use of our own implementation of the Artin transfer homomorphisms. Finally we check the 96 second derived quotients H/H'' against the 13 groups G of Theorem 3.2 for isomorphism $H/H'' \simeq G$, stopping at the first isomorphism encountered. The two groups G with identifiers 56, 58 of TKT F.7 turn out to be exceptional, since they are associated with four non-metabelian groups H only, instead of eight. □

4.1.1. *Cover groups H with* $\mathrm{lo}(H) = 10$. The 24 *terminal non-metabelian* σ-groups $H = P_7 - \#3; \ell$ of coclass 5 in Theorem 4.1 and Table 4 have the maximal relation rank $d_2 H = 3$ permitted for 3-class tower groups $\mathrm{G}_3^\infty K$ of *real* quadratic fields K, but too big for complex quadratic fields [37]. Since all of them share the common iterated IPAD of second order

$$(4.1) \qquad \tau^{(2)} H = \left[1^2; (32; 2^3 1, (31^2)^3), (32; 2^3 1, (31^3)^3), (1^3; 2^3 1, (21^2)^3, (1^3)^9)^2\right],$$

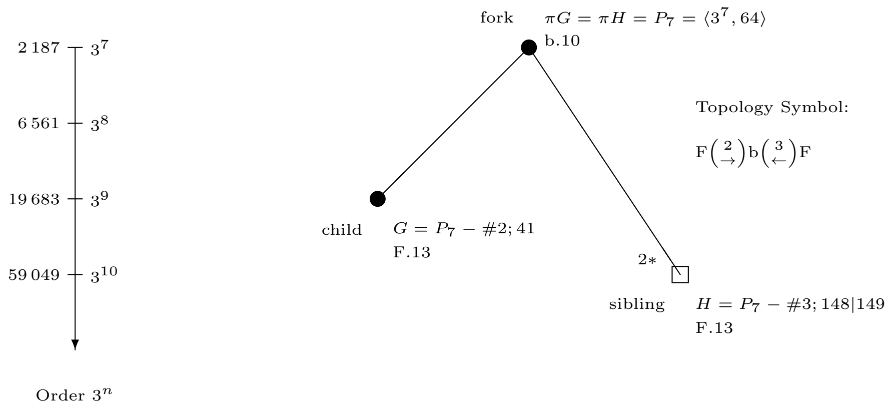
the following hypothesis is compatible with data available currently in Table 1.

Conjecture 4.1. (Tower ground state) The real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d \in \{66\,615\,244, 76\,575\,261\}$ of type F.11, resp. $d = 22\,937\,941$ of type F.12, resp. $d \in \{8\,321\,505, 17\,373\,109\}$ of type F.13, have 3-class field towers of exact length $\ell_3 K = 3$ with group $\mathrm{G}_3^\infty K \simeq P_7 - \#3; \ell$, where $\ell \in \{140, 141, 143, 144\}$, for type F.11, resp. $\ell \in \{151, 152, 156, 157, 163, 176, 165, 177\}$, for type F.12, resp. $\ell \in \{148, 149, 158, 159, 162, 171, 164, 166\}$, for type F.13.

For all types F.11, F.12, F.13, the tower group $H = \mathrm{G}_3^\infty K$ has $\mathrm{lo}(H) = 10$, $\mathrm{cl}(H) = 5$, $\mathrm{cc}(H) = 5$, $\zeta_1 H = (3,3,3)$, $\gamma_2^2 H = (3)$, and $\#\mathrm{Aut}(H) = 2 \cdot 3^{14}$.

Remark 4.1. Figure 3 shows one of the possible tree topologies for the real quadratic field $K = \mathbb{Q}(\sqrt{8\,321\,505})$ of type F.13, expressing the mutual location of $G = \mathrm{G}_3^2 K$ and $H = \mathrm{G}_3^3 K = \mathrm{G}_3^\infty K$, connected by the fork $\pi G = \pi H = P_7$ of type b.10.

FIGURE 3. Possible sibling topology of $K = \mathbb{Q}(\sqrt{8\,321\,505})$



4.1.2. *Cover groups H with* $\mathrm{lo}(H) = 12$. Each of the *capable non-metabelian* σ-groups $D = P_7 - \#3; \ell$ of coclass 5 in Theorem 4.1 and Table 4 with iterated IPAD of second order

$$(4.2) \quad \tau^{(2)} D = \left[1^2; (32; 2^3 1, (31^2)^3), (32; 2^3 1, (\mathbf{31^2})^{\mathbf{3}}), (1^3; 2^3 1, (\mathbf{21^2})^{\mathbf{12}}), (1^3; 2^3 1, (21^2)^3, (1^3)^9)\right],$$

has nuclear rank $\nu = 1$ and p-multiplicator rank $\mu = 4$. There are *two possible scenarios* for the immediate descendant numbers:

either the first scenario: $(N_1/C_1) = (8/5)$, there is only a single capable σ-child $P_7 - \#3; \ell - \#1; k$ with $(\nu, \mu) = (1, 4)$, $(N_1/C_1) = (3/0)$, and all three terminal grandchildren $H = P_7 - \#3; \ell - \#1; k - \#1; j$ with $1 \le j \le 3$ have $d_2 H = 3$,
or the second scenario: $(N_1/C_1) \in \{(8/8), (8/5)\}$, there is also only a single capable σ-child $P_7 - \#3; \ell - \#1; k$ with $(\nu, \mu) = (1, 4)$, $(N_1/C_1) = (1/0)$, and the terminal grandchild $H = P_7 - \#3; \ell - \#1; k - \#1; j$ with $j = 1$ has $d_2 H = 3$.
This is the maximal relation rank permitted for 3-class tower groups $\mathrm{G}_3^\infty K$ of *real* quadratic fields K [37]. Therefore, we suggest the following hypothesis, based on Table 1.

Conjecture 4.2. (Excited tower state) The real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d = 10\,165\,597$ of type F.7, resp. $d = 72\,034\,376$ of type F.13, have 3-class field towers of exact length $\ell_3 K = 3$ with group $\mathrm{G}_3^\infty K \simeq P_7 - \#3; \ell - \#1; k - \#1; j$, where

$$(\ell, k) \in \{(293, 7), (299, 5), (336, 5), (342, 5), (291, 5), (306, 7), (332, 5), (347, 5)\},$$

$1 \le j \le 3$, resp.

$$(\ell, k) \in \{(272, 7), (289, 8), (296, 5), (302, 8), (304, 5), (315, 8), (331, 5), (339, 4), (345, 4), (346, 5)\},$$

$1 \le j \le 3$, or

$$(\ell, k) \in \{(248, 8), (249, 8), (263, 7), (264, 7), (275, 4), (318, 5)\},$$

$j = 1$.

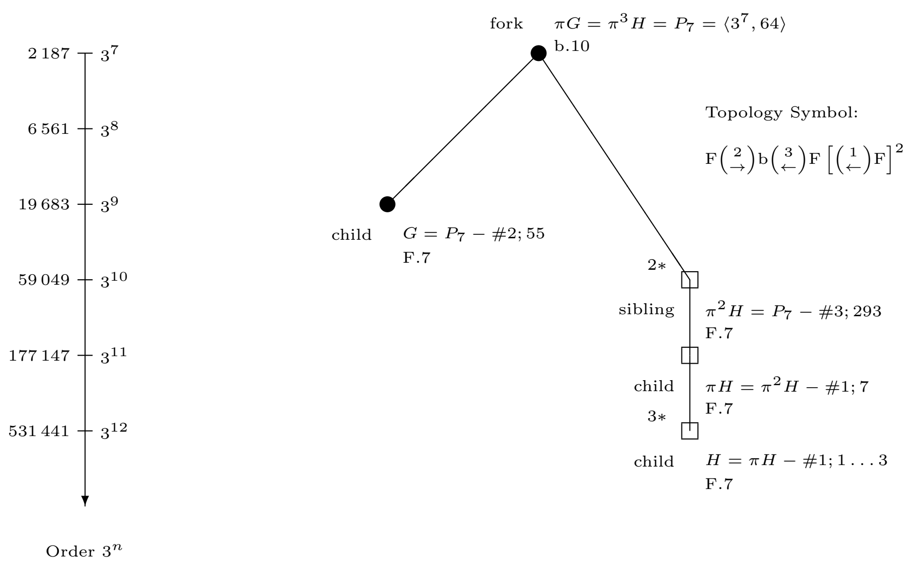
For type F.7, we always have the first scenario, and the tower group $H = \mathrm{G}_3^\infty K$ has $\mathrm{lo}(H) = 12$, $\mathrm{cl}(H) = 7$, $\mathrm{cc}(H) = 5$, $\zeta_1 H = (3, 3)$, $\gamma_2^2 H = (3, 3, 3)$, and $\#\mathrm{Aut}(H) = 2 \cdot 3^{17}$.

For type F.13, first scenario, the tower group H has $\mathrm{lo}(H) = 12$, $\mathrm{cl}(H) = 7$, $\mathrm{cc}(H) = 5$, $\zeta_1 H = (3, 3)$, $\gamma_2^2 H = (9, 3)$ or $\gamma_2^2 H = (3, 3, 3)$, and $\#\mathrm{Aut}(H) = 2 \cdot 3^{17}$.

For type F.13, second scenario, the tower group H has $\mathrm{lo}(H) = 12$, $\mathrm{cl}(H) = 7$, $\mathrm{cc}(H) = 5$, $\zeta_1 H = (9)$, $\gamma_2^2 H = (9, 3)$, and $\#\mathrm{Aut}(H) = 2 \cdot 3^{16}$.

Remark 4.2. Figure 4 shows one of the possible tree topologies for the real quadratic field $K = \mathbb{Q}(\sqrt{10\,165\,597})$ of type F.7, expressing the mutual location of $G = \mathrm{G}_3^2 K$ and $H = \mathrm{G}_3^3 K = \mathrm{G}_3^\infty K$, connected by the fork $\pi G = \pi^3 H = P_7$ of type b.10.

FIGURE 4. Possible fork topology of $K = \mathbb{Q}(\sqrt{10\,165\,597})$



Remark 4.3. (**Open problems No. 1**)
The Conjectures 4.1 and 4.2 would be theorems, when we succeeded in proving that there are neither 3-groups H of type F with $\mathrm{lo}(H) \geq 11$ satisfying Formula (4.1) nor with $\mathrm{lo}(H) \geq 13$ satisfying Formula (4.2). Due to the partial order of IPADs in descendant trees [31], this is only a *finite* (but possibly rather extensive) task, provided there does not occur a total stabilization.

4.2. **Cover with relation rank 2 for complex fields.** Next we search for the smallest *non-metabelian* σ-groups H with relation rank $d_2H = 2$ (the so-called *Schur* σ-groups), which are candidates for 3-class tower groups of *complex* quadratic fields K [37] with $G = \mathrm{G}_3^2K$ of coclass $\mathrm{cc}(G) = 4$. Since this process is of considerable complexity, we prefer a splitting into the TKTs F.7, F.11, F.12, and F.13.

TABLE 5. Abelian quotient invariants of second order, $\tau^{(2)}D$, for $D = P_7 - \#4; \ell$

Identifier	Cat.	$\tau^{(2)}D = [1^2; (32; 2^31, T_1), (32; 2^31, T_2), (1^3; 2^31, T_3), (1^3; 2^31, T_4)]$			
$\ell =$		T_1	T_2	T_3	T_4
$23, 24, 26$	1	$(31^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^3, (1^4)^9}$	$\mathbf{(21^3)^3, (1^4)^9}$
$42, 44, 50, 54,$ $68, 72, 78, 80$	2	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$\mathbf{(21^3)^3, (1^4)^9}$
$121, \mathbf{123}, \mathbf{128}, 131, 142,$ $145, \mathbf{165}, 169, 174, 196$	3	$(31^3)^3$	$(31^3)^3$	$(2^21)^3, (21^2)^9$	$(2^21)^3, (21^2)^9$

4.2.1. *Type* F.7. Table 5 classifies the $21 = 3 + 8 + 10$ immediate descendants D of step size 4 of $P_7 := \langle 2187, 64\rangle$ with type F.7 into *three categories*, according to the IPAD of second order.

For the *first* category, all abelian quotients of subgroups of index 9 possess 3-rank 4. For the *second* category, T_3 consists of twelve abelian quotients with 3-rank 3. For the *third* category, T_3 and T_4 both consist of twelve abelian quotients with 3-rank 3. Note that the abelianization of the commutator subgroup, (2^31), which occurs in all four components of the IPAD of second order, has 3-rank 4.

Among the 10 members D of category three, 7 give rise to batches of 27, resp. 18, Schur σ-groups H each. Their identifiers in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4; \ell - \#2; k - \#4; j - \#1; i - \#2; 1, \tag{4.3}$$

where ℓ is one of the counters different from **123**, **128** and **165** in category three of Table 5, $1 \leq k \leq 41$, resp. $1 \leq k \leq 21$, has a unique value in dependence on ℓ (the unique σ-group among the immediate descendants of step size 2), j completely runs through the range $1 \leq j \leq 27$, resp. $1 \leq j \leq 18$, and $1 \leq i \leq 5$ is a unique value in dependence on j.

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 20$, class $\mathrm{cl}(H) = 9$, coclass $\mathrm{cc}(H) = 11$, derived length $\mathrm{dl}(H) = 3$, and IPAD of second order,

$$\tau^{(2)}H = [1^2; (32; 2^31, (31^3)^3)^2, (1^3; 2^31, (2^21)^3, (21^2)^9)^2]. \tag{4.4}$$

Their automorphism group is of uniform order $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$. However, Table 7 shows that the centre, $\zeta_1 H$, the second derived subgroup, $\gamma_2^2 H$, and the number t of possible values for j occur in three variants of Table 6.
Table 7 also gives the number m of the metabelianization $H/H'' \simeq G = P_7 - \#2; m$ from Theorem 3.2, in dependence on ℓ.

Table 6. Variants of $\zeta_1 H$, $\gamma_2^2 H$, and the number t

Variant	IV	V	VI
$\zeta_1 H$	2^2	21^2	2^2
$\gamma_2^2 H$	$3^2 21^3$	$32^3 1^2$	$3^2 21^3$
t	27	27	18

Table 7. Association of the values m, k, and the variant to each value of ℓ

ℓ	121	131	142	145	169	174	196
m	55	55	56	58	56	58	55
k	14	13	19	21	14	13	31
var.	V	V	VI	VI	V	V	IV

Table 8. Abelian quotient invariants of second order, $\tau^{(2)}D$, for $D = P_7 - \#4; \ell$

Identifier	Cat.	$\tau^{(2)}D = [1^2; (32; 2^3 1, T_1), (32; 2^3 1, T_2), (1^3; 2^3 1, T_3), (1^3; 2^3 1, T_4)]$			
$\ell =$		T_1	T_2	T_3	T_4
$4, 6$	1	$(31^3)^3$	$(31^3)^3$	$(\mathbf{21^3})^{\mathbf{3}}, (\mathbf{1^4})^{\mathbf{9}}$	$(\mathbf{21^3})^{\mathbf{3}}, (\mathbf{1^4})^{\mathbf{9}}$
$37, 46, 64, 73,$ $86, 87, 89, 90$	2	$(31^3)^3$	$(31^3)^3$	$(2^2 1)^3, (21^2)^9$	$(\mathbf{21^3})^{\mathbf{3}}, (\mathbf{1^4})^{\mathbf{9}}$
$119, 127, 139, 144,$ $164, 172, 180, 182$	3	$(31^3)^3$	$(31^3)^3$	$(2^2 1)^3, (21^2)^9$	$(2^2 1)^3, (21^2)^9$

4.2.2. *Type* F.11. Table 8 classifies the $18 = 2 + 8 + 8$ immediate descendants D of step size 4 of $P_7 := \langle 2187, 64 \rangle$ with type F.11 into three categories, according to the IPAD of second order.

All 8 members D of category three give rise to batches of 27, resp. 81, Schur σ-groups H each. Their identifiers in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4; \ell - \#2; k - \#4; j - \#1; i - \#2; 1, \tag{4.5}$$

where ℓ is one of the counters in category three of Table 8, $1 \le k \le 41$ has a unique value in dependence on ℓ (the unique σ-group among the immediate descendants of step size 2), j completely runs through the range $1 \le j \le 27$, resp. $1 \le j \le 81$, and $1 \le i \le 5$ is a unique value in dependence on j.

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 20$, class $\mathrm{cl}(H) = 9$, coclass $\mathrm{cc}(H) = 11$, derived length $\mathrm{dl}(H) = 3$, and IPAD of second order,

$$\tau^{(2)}H = [1^2; (32; 2^3 1, (41^3)^3), (32; 2^3 1, (31^3)^3), (1^3; 2^3 1, (2^2 1)^3, (21^2)^9)^2]. \tag{4.6}$$

Their centre, $\zeta_1 H$, is of uniform type (32). However, Table 10 shows that the order $\#\mathrm{Aut}(H)$ of the automorphism group, the second derived subgroup, $\gamma_2^2 H$, and the number t of possible values for j occur in two variants of Table 9.
Table 10 also gives the number m of the metabelianization $H/H'' \simeq G = P_7 - \#2; m$ from Theorem 3.2, in dependence on ℓ.

4.2.3. *Type* F.12. Table 11 classifies the $36 = 4 + 16 + 16$ immediate descendants D of step size 4 of $P_7 := \langle 2187, 64 \rangle$ with type F.12 into three categories, according to the IPAD of second order.

TABLE 9. Variants of $\#\mathrm{Aut}(H)$, $\gamma_2^2 H$, and the number t

Variant	1	2
$\#\mathrm{Aut}(H)$	$2\cdot 3^{26}$	$2\cdot 3^{25}$
$\gamma_2^2 H$	$32^3 1^2$	$3^2 21^3$
t	81	27

TABLE 10. Association of the values m, k, and the variant to each value of ℓ

ℓ	119	127	139	144	164	172	180	182
m	38	36	38	36	38	36	36	38
k	41	32	41	41	41	32	32	31
var.	1	1	2	2	1	1	2	2

TABLE 11. Abelian quotient invariants of second order, $\tau^{(2)}D$, for $D = P_7 - \#4;\ell$

Identifier	Cat.	$\tau^{(2)}D = [1^2;(32;2^31,T_1),(32;2^31,T_2),(1^3;2^31,T_3),(1^3;2^31,T_4)]$			
$\ell =$		T_1	T_2	T_3	T_4
$11,14,19,21$	1	$(31^3)^3$	$(31^3)^3$	$(\mathbf{21^3})^\mathbf{3},(\mathbf{1^4})^\mathbf{9}$	$(\mathbf{21^3})^\mathbf{3},(\mathbf{1^4})^\mathbf{9}$
$33,36,39,43,48,49,59,62,$ $65,70,74,76,98,99,104,105$	2	$(31^3)^3$	$(31^3)^3$	$(2^21)^3,(21^2)^9$	$(\mathbf{21^3})^\mathbf{3},(\mathbf{1^4})^\mathbf{9}$
$113,116,118,125,126,130,143,146,$ $\mathbf{157},\mathbf{160},170,175,187,190,194,195$	3	$(31^3)^3$	$(31^3)^3$	$(2^21)^3,(21^2)^9$	$(2^21)^3,(21^2)^9$

Among the 16 members D of category three, 14 give rise to batches of 27 Schur σ-groups H each. Their identifiers in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4;\ell - \#2;k - \#4;j - \#1;i - \#2;1, \tag{4.7}$$

where ℓ is one of the counters different from **157** and **160** in category three of Table 11, $1 \le k \le 41$ has a unique value in dependence on ℓ (the unique σ-group among the immediate descendants of step size 2), j completely runs through the range $1 \le j \le 27$, and $1 \le i \le 5$ is a unique value in dependence on j.

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 20$, class $\mathrm{cl}(H) = 9$, coclass $\mathrm{cc}(H) = 11$, and derived length $\mathrm{dl}(H) = 3$. Their automorphism group is of uniform order $\#\mathrm{Aut}(H) = 2\cdot 3^{25}$. However, Table 13 shows that the centre, $\zeta_1 H$, the second derived subgroup, $\gamma_2^2 H$, and a component n of the IPAD of second order,

$$\tau^{(2)}H = [1^2;(32;2^31,(n1^3)^3),(32;2^31,(31^3)^3),(1^3;2^31,(2^21)^3,(21^2)^9)^2], \tag{4.8}$$

occur in the five variants of Table 12.

Table 13 also gives the number m of the metabelianization $H/H'' \simeq G = P_7 - \#2;m$ from Theorem 3.2, in dependence on ℓ.

4.2.4. *Type* F.13. Table 14 classifies the $36 = 4 + 16 + 16$ immediate descendants D of step size 4 of $P_7 := \langle 2187, 64\rangle$ with type F.13 into three categories, according to the IPAD of second order.

TABLE 12. Variants of $\zeta_1 H$, $\gamma_2^2 H$, and $\tau^{(2)} H$

Variant	I	II	III	IV	V
$\zeta_1 H$	2^2	21^2	21^2	2^2	21^2
$\gamma_2^2 H$	$32^3 1^2$	$32^3 1^2$	$3^2 21^3$	$3^2 21^3$	$32^3 1^2$
n	4	4	3	3	3

TABLE 13. Association of the values m, k, and the variant to each value of ℓ

ℓ	113	116	118	125	126	130	143	146	170	175	187	190	194	195
m	51	53	53	43	51	46	43	46	43	46	43	46	51	53
k	32	31	41	11	41	1	14	13	40	32	10	11	28	29
var.	III	III	IV	V	IV	V	IV	IV	I	I	II	II	I	I

TABLE 14. Abelian quotient invariants of second order, $\tau^{(2)} D$, for $D = P_7 - \#4; \ell$

Identifier	Cat.	$\tau^{(2)} D = [1^2; (32; 2^3 1, T_1), (32; 2^3 1, T_2), (1^3; 2^3 1, T_3), (1^3; 2^3 1, T_4)]$			
$\ell =$		T_1	T_2	T_3	T_4
$9, 15, 18, 20$	1	$(31^3)^3$	$(31^3)^3$	$\mathbf{(21^3)^3}, \mathbf{(1^4)^9}$	$\mathbf{(21^3)^3}, \mathbf{(1^4)^9}$
$32, 35, 38, 40, 47, 52, 60, 63, 66, 67, 75, 79, 95, 96, 107, 108$	2	$(31^3)^3$	$(31^3)^3$	$(2^2 1)^3, (21^2)^9$	$\mathbf{(21^3)^3}, \mathbf{(1^4)^9}$
$\mathbf{112}, \mathbf{115}, 122, 132, 135, 137, 141, 147, 158, 161, 163, 167, 171, 177, 185, 192$	3	$(31^3)^3$	$(31^3)^3$	$(2^2 1)^3, (21^2)^9$	$(2^2 1)^3, (21^2)^9$

Among the 16 members D of category three, 14 give rise to batches of 27 Schur σ-groups H each. Their identifiers in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4; \ell - \#2; k - \#4; j - \#1; i - \#2; 1, \tag{4.9}$$

where ℓ is one of the counters different from **112** and **115** in category three of Table 14, $1 \le k \le 41$ has a unique value in dependence on ℓ (the unique σ-group among the immediate descendants of step size 2), j completely runs through the range $1 \le j \le 27$, and $1 \le i \le 5$ is a unique value in dependence on j.

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 20$, class $\mathrm{cl}(H) = 9$, coclass $\mathrm{cc}(H) = 11$, and derived length $\mathrm{dl}(H) = 3$. Their automorphism group is of uniform order $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$. However, Table 16 shows that the centre, $\zeta_1 H$, the second derived subgroup, $\gamma_2^2 H$, and a component n of the IPAD of second order,

$$\tau^{(2)} H = [1^2; (32; 2^3 1, (n1^3)^3), (32; 2^3 1, (31^3)^3), (1^3; 2^3 1, (2^2 1)^3, (21^2)^9)^2], \tag{4.10}$$

occur in the five variants of Table 15.

Table 16 also gives the number m of the metabelianization $H/H'' \simeq G = P_7 - \#2; m$ from Theorem 3.2, in dependence on ℓ.

4.2.5. *Schur σ-groups H with* $\mathrm{lo}(H) = 20$. We summarize the results about the *Shafarevich cover* $\mathrm{cov}(G, K)$ of metabelian σ-groups G of type F, logarithmic order $\mathrm{lo}(G) = 9$ and coclass $\mathrm{cc}(G) = 4$, with respect to *complex* quadratic fields K, from §§ 4.2.1 – 4.2.4 in the following theorem,

TABLE 15. Variants of $\zeta_1 H$, $\gamma_2^2 H$, and $\tau^{(2)} H$

Variant	I	II	III	IV	V
$\zeta_1 H$	2^2	21^2	21^2	2^2	21^2
$\gamma_2^2 H$	323^31^2	323^31^2	3^221^3	3^221^3	323^31^2
n	4	4	3	3	3

TABLE 16. Association of the values m, k, and the variant to each value of ℓ

ℓ	122	132	135	137	141	147	158	161	163	167	171	177	185	192
m	47	41	50	52	47	41	50	52	52	47	50	41	41	47
k	40	32	29	28	40	41	31	32	41	11	41	1	41	40
var.	I	I	I	I	II	II	III	III	IV	V	IV	V	IV	IV

disregarding several variants of the centre $\zeta_1 H$ and the second derived subgroup $\gamma_2^2 H$ of the non-metabelian contestants H.

Theorem 4.2. *Let $G := P_7 - \#2; m$ be a sporadic metabelian 3-group G of type* F *with coclass* $\mathrm{cc}(G) = 4$*. The following counters concern* 1359 *pairwise non-isomorphic Schur σ-groups H of logarithmic order* $\mathrm{lo}(H) = 20$ *and nilpotency class* $\mathrm{cl}(H) = 9$ *such that $H/H'' \simeq G$.*

(1) *For type* F.7*, there exist* 171*, in more detail,*
   81*,* 45*,* 45 *Schur σ-groups H satisfying Formula* (4.4) *in* $\mathrm{cov}(G, K)$*, for $m = 55, 56, 58$.*
   They all have $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$.
(2) *For type* F.11*, there exist* $108 + 324$*, in more detail,*
   (a) 54*,* 54 *Schur σ-groups H satisfying Formula* (4.6) *and having* $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$ *in* $\mathrm{cov}(G, K)$*, for $m = 36, 38$;*
   (b) 162*,* 162 *Schur σ-groups H satisfying Formula* (4.6) *and having* $\#\mathrm{Aut}(H) = 2 \cdot 3^{26}$ *in* $\mathrm{cov}(G, K)$*, for $m = 36, 38$.*
(3) *For type* F.12*, there exist* $216 + 162$*, in more detail,*
   (a) 54*,* 54*,* 54*,* 54 *Schur σ-groups H satisfying Formula* (4.8) *with $n = 3$ in* $\mathrm{cov}(G, K)$*, for $m = 43, 46, 51, 53$;*
   (b) 54*,* 54*,* 27*,* 27 *Schur σ-groups H satisfying Formula* (4.8) *with $n = 4$ in* $\mathrm{cov}(G, K)$*, for $m = 43, 46, 51, 53$.*
   They all have $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$.
(4) *For type* F.13*, there exist* $216 + 162$*, in more detail,*
   (a) 54*,* 54*,* 54*,* 54 *Schur σ-groups H satisfying Formula* (4.10) *with $n = 3$ in* $\mathrm{cov}(G, K)$*, for $m = 41, 47, 50, 52$;*
   (b) 54*,* 54*,* 27*,* 27 *Schur σ-groups H satisfying Formula* (4.10) *with $n = 4$ in* $\mathrm{cov}(G, K)$*, for $m = 41, 47, 50, 52$.*
   They all have $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$.

Remark 4.4. In Theorem 4.1 and Table 4, we have proved that the smallest non-trivial members of the Shafarevich cover $\mathrm{cov}(G, K)$ of the metabelian σ-groups $G = P_7 - \#2; m$ of type F, $\mathrm{lo}(G) = 9$, $\mathrm{cc}(G) = 4$, with respect to *real* quadratic fields K, are non-metabelian σ-groups $H = P_7 - \#3; \ell$ of $\mathrm{lo}(H) = 10$, $\mathrm{dl}(H) = 3$, with $d_2 H = 3$, a single such group for $m \in \{56, 58\}$, two groups otherwise. (In the Shafarevich cover of *complex* quadratic fields, these groups are forbidden.)

Of course, the Shafarevich cover $\mathrm{cov}(G, K)$ for *real* K also contains the suitable corresponding Schur σ-groups H of Theorem 4.2, which have $\mathrm{lo}(H) = 20$, $\mathrm{dl}(H) = 3$, and $d_2 H = 2$.

However, in Table 1, there do not occur any iterated IPADs of the Formulas (4.4), (4.6), (4.8), and (4.10). This means that *real* quadratic fields are happy with 3-class tower groups H having the minimal $\mathrm{lo}(H) = 10$ but only $d_2 H = 3$. They do not insist on Schur σ-groups.

This tendency can be made more precise with the aid of recent asymptotic densities, forming a non-abelian analogue of the heuristic by Cohen, Lenstra, and Martinet.

According to not yet published investigations by Boston, Bush, and Hajir, the probability $\mathrm{Prob}_K H$ that an assigned σ-group H of order a power of 3 occurs as the 3-class tower group $H \simeq \mathrm{G}_3^\infty K$ of a *real* quadratic field K is proportional to the reciprocal product $\#H \cdot \#\mathrm{Aut}(H)$:

$$\mathrm{Prob}_K H \sim \frac{1}{\#H \cdot \#\mathrm{Aut}(H)} \tag{4.11}$$

The groups H of Theorem 4.1 have $\#H = 3^{10}$ and $\#\mathrm{Aut}(H) = 2 \cdot 3^{14}$. The Schur σ-groups of Theorem 4.2 have $\#H = 3^{20}$ and usually $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$. Consequently, the probability for the former is

$$(3^{10} \cdot 2 \cdot 3^{14})^{-1} : (3^{20} \cdot 2 \cdot 3^{25})^{-1} = 3^{10} \cdot 3^{11} = 3^{21} = 10\,460\,353\,203$$

times bigger than the probability for the latter.

Conjecture 4.3. (Tower ground state) The complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d \in \{-225\,299, -343\,380, -423\,476, -486\,264\}$ of type F.7, resp. $d \in \{-27\,156, -241\,160, -477\,192, -484\,804\}$ of type F.11, resp. $d = -291\,220$ of type F.12, resp. $d \in \{-167\,064, -296\,407, -317\,747, -401\,603\}$ of type F.13, have 3-class field towers of exact length $\ell_3 K = 3$ with a suitable Schur σ-group in Theorem 4.2.

For all types, F.7, F.11, F.12, F.13, the tower group $H = \mathrm{G}_3^\infty K$ has $\mathrm{lo}(H) = 20$, $\mathrm{cl}(H) = 9$, $\mathrm{cc}(H) = 11$, $\zeta_1 H = (9,9)$ or $(9,3,3)$, $\gamma_2^2 H = (27,27,9,3,3,3)$ or $(27,9,9,9,3,3)$, and usually $\#\mathrm{Aut}(H) = 2 \cdot 3^{25}$, rarely $2 \cdot 3^{26}$.

Remark 4.5. Figure 5 shows one of the possible tree topologies, the one with the highest probability, for the complex quadratic field $K = \mathbb{Q}(\sqrt{-225\,299})$ of type F.7, expressing the mutual location of $G = \mathrm{G}_3^2 K$ and $H = \mathrm{G}_3^3 K = \mathrm{G}_3^\infty K$, connected by the fork $\pi G = \pi^5 H = P_7$ of type b.10.

4.2.6. *Schur σ-groups H with* $\mathrm{lo}(H) = 26$. Among the 10 members D of *category three* in Table 5, three reveal an *exceptional behaviour*. They give rise to a total of 29, 30, 72, respectively, Schur σ-groups H of smallest order $\#H = 3^{26}$. The identifiers of these non-metabelian groups H in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4; \ell - \#2; k - \#4; j - \#2; i - \#4; h - \#1; 1 - \#2; 1, \tag{4.12}$$

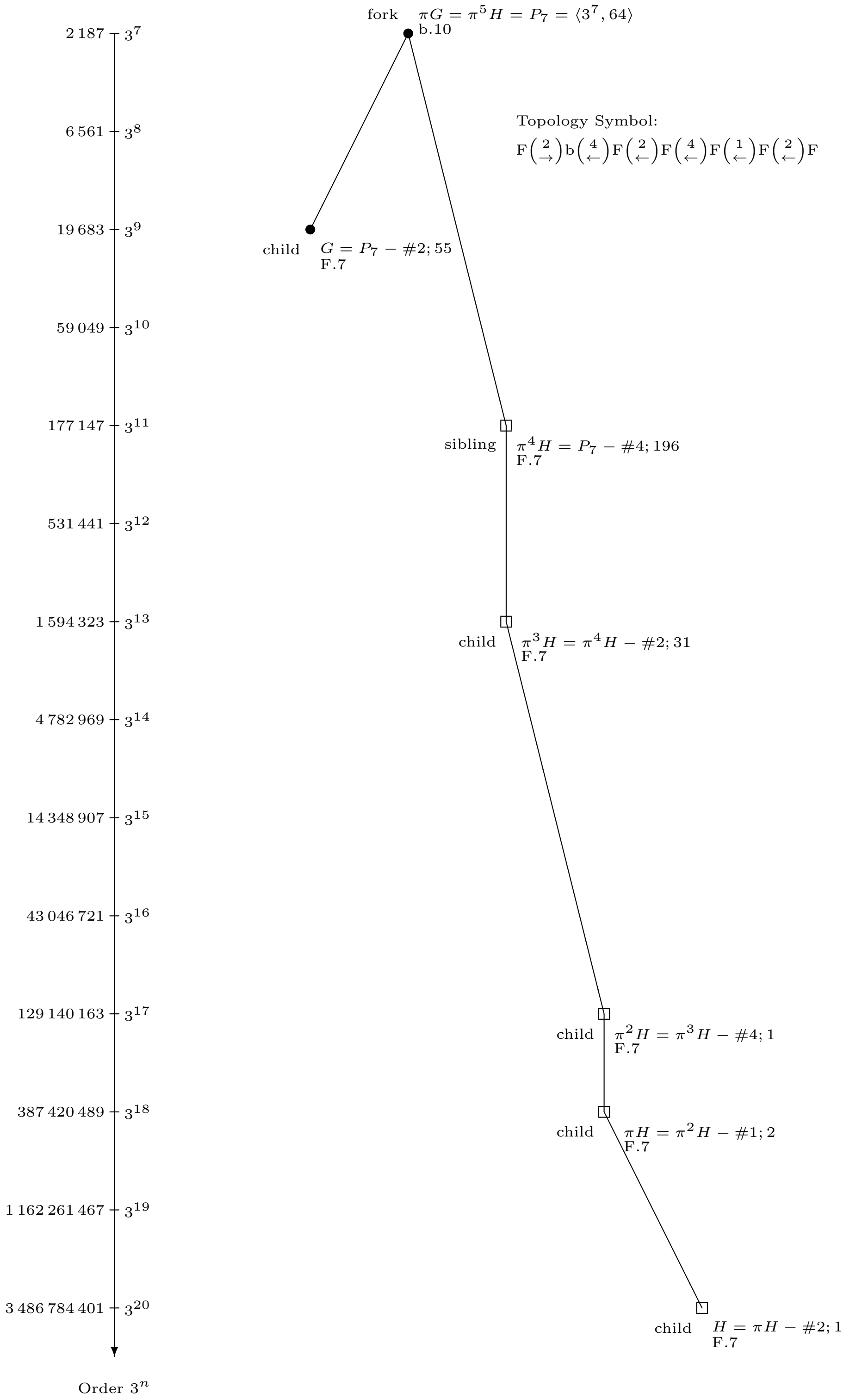
where ℓ is one of the counters 123, 128, 165 in category three of Table 5, k has the unique value 12, 12, 29, in dependence on ℓ (the unique σ-group among the immediate descendants of step size 2), j takes selected values in the range $1 \le j \le 18$ for the counters 123, 128, resp. $1 \le j \le 27$ for the counter 165, $1 \le i \le 41$ is a unique value in dependence on j, and h takes selected values in dependence on i. The number m of the metabelianization $H/H'' \simeq G = P_7 - \#2; m$ from Theorem 3.2 is given by 56, 58, 55, in dependence on ℓ.

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 26$, class $\mathrm{cl}(H) = 11$, coclass $\mathrm{cc}(H) = 15$, derived length $\mathrm{dl}(H) = 4$, and IPAD of second order,

$$\tau^{(2)} H = [1^2; (32; 2^3 1, (31^3)^3)^2, (1^3; 2^3 1, (2^2 1)^3, (21^2)^9)^2]. \tag{4.13}$$

However, Tables 18, 19, and 20, show that the centre, $\zeta_1 H$, the abelian quotient invariants of the second derived subgroup, $\gamma_2^2 H$, the third derived subgroup, $\gamma_2^3 H$, and the order of the automorphism group $\#\mathrm{Aut}(H)$ occur in three variants of Table 17. Since the third derived subgroup $\gamma_2^3 H$ coincides with the last, resp. last but one, non-trivial lower central of H, the third derived quotient $H/\gamma_2^3 H$ is isomorphic to the parent πH, resp. the grandparent $\pi^2 H$, of H.

Figure 5. Possible fork topology of $K = \mathbb{Q}(\sqrt{-225\,299})$



Remark 4.6. Figure 6 shows another of the possible tree topologies, one with significantly lower probability, for the complex quadratic field $K = \mathbb{Q}(\sqrt{-225\,299})$ of type F.7, expressing the mutual

TABLE 17. Variants of $\zeta_1 H$, $\gamma_2^2 H/\gamma_2^3 H$, $\gamma_2^3 H$, and $\#\mathrm{Aut}(H)$

Variant	I	II	III
$\zeta_1 H$	2^2	2^2	21^2
$\gamma_2^2 H/\gamma_2^3 H$	$3^3 2^3$	$3^3 2^3$	$3^2 2^4$
$\gamma_2^3 H$	1^2	1^2	1^3
$\#\mathrm{Aut}(H)$	$2^2 \cdot 3^{30}$	$2 \cdot 3^{30}$	$2 \cdot 3^{30}$

TABLE 18. Associations for $P_7 - \#4; 123$

j	i	h	var.
3	21	1,14,17	I
5	20	2,14	II
8	37	2,4,9	III
9	38	19,24,26	III
11	20	7,12,16	I
13	11	2,4,9	III
14	40	1,5,9,12,13,17,20,24,25	III
15	21	1	II
17	21	9	II
18	19	9	II

TABLE 19. Associations for $P_7 - \#4; 128$

j	i	h	var.
1	21	7,16	II,I
2	40	8,15,19	III
4	41	6,10,26	III
5	20	3,14	II,I
7	38	4,10,25	III
8	10	7,13,19	III
10	19	6,18	II,I
12	29	3,18,24	III
14	21	6,18	II,I
15	31	1,17,24	III
17	21	9,12	II,I
18	20	1,17	II,I

location of $G = \mathrm{G}_3^2 K$, $\pi^2 H = \mathrm{G}_3^3 K$, and $H = \mathrm{G}_3^4 K = \mathrm{G}_3^\infty K$, connected by the fork $\pi G = \pi^7 H = P_7$ of type b.10. We point out that this would be the first example of a *four-stage tower*.

However, according to the main result of [8] (the complex analogue of Formula (4.11)), the probability for the situation in Figure 5 is

$$(2 \cdot 3^{25})^{-1} : (2 \cdot 3^{30})^{-1} = 3^5 = 243$$

times bigger than the probability for the situation in Figure 6.

Table 20. Associations for $P_7 - \#4; 165$

j	i	h	var.
4	11	2,6,7,12,13,17,19,23,27	III
5	41	1,18,23	II
6	41	12,14,16	III
8	31	11,13,18	III
9	31	6,11,25	II
10	37	12,14,16	III
11	40	1,18,23	II
16	29	8,13,21	II
17	32	3,5,7	III
18	5	2,6,7,12,13,17,19,23,27	III
19	38	1,6,8	III
20	14	3,4,8,10,14,18,20,24,25	III
21	41	1,18,23	II
22	37	2,16,24	II
23	13	3,4,8,10,14,18,20,24,25	III
24	40	3,5,7	III

4.2.7. *Schur σ-groups H with* $\mathrm{lo}(H) = 23$. There do not arise any exceptions among the 8 members D of type F.11 and category three in Table 8.

However, among the 16 members D of *category three* in both Tables 11 (type F.12) and 14 (type F.13), two reveal an *exceptional behaviour.*

Remark 4.7. (**Open problems No. 2**)
We were unable to find Schur σ-groups among the descendants of $P_7 - \#4; \ell$ with $\ell \in \{157, 160\}$, type F.12, and with $\ell \in \{112, 115\}$, type F.13. They all belong to category three.

Even more annoying, we were unable to find any Schur σ-groups among the descendants of the numerous roots in *category one and two* of the Tables 5, 8, 11, and 14.

4.2.8. *The cover of sporadic groups of coclass* 6. Finally we celebrate our priority in discovering the smallest *non-metabelian* σ-groups H with relation rank $d_2 H = 2$ (*Schur* σ-groups) [37], which are contestants for 3-class tower groups $\mathrm{G}_3^\infty K$ of *complex* quadratic fields K with $G = \mathrm{G}_3^2 K \simeq H/H''$ of *elevated coclass* $\mathrm{cc}(G) = 6$.

Exemplarily, we restrict our investigations to the transfer kernel type F.11. The crucial idea how to start the path from the mandatory fork P_7 to the Schur σ-group H was inspired by the *symmetry of the topology symbol around the fork* (independently of step sizes):

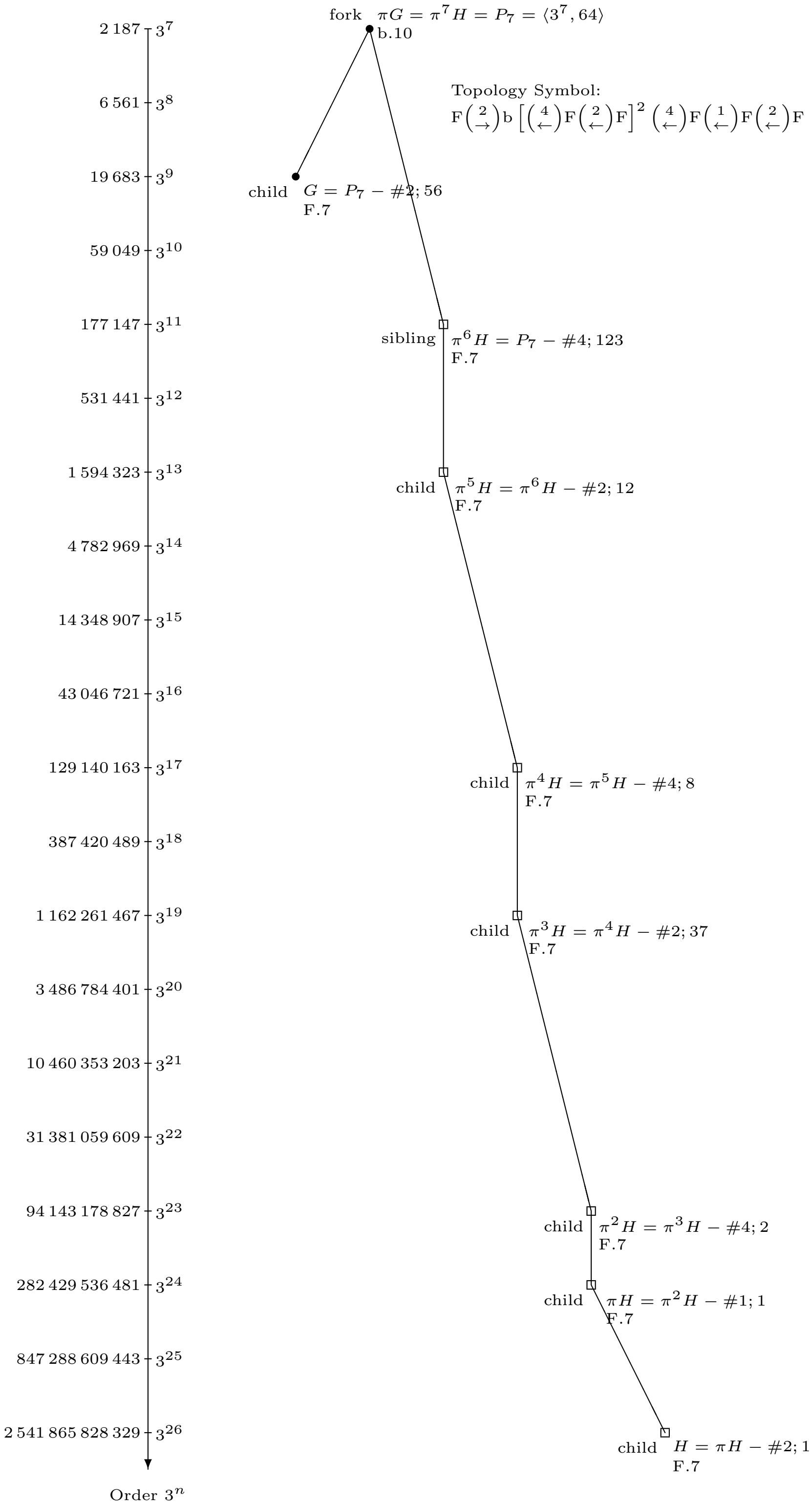
$$\mathrm{F}\binom{2}{\rightarrow}\mathrm{b}\binom{2}{\rightarrow}\mathrm{b}\binom{2}{\rightarrow}\underline{\mathbf{b}}\binom{4}{\leftarrow}\mathrm{b}\binom{2}{\leftarrow}\mathrm{b}\binom{4}{\leftarrow}\mathrm{F}\ldots,$$

which suggests that two vertices with type b.10 must be found on the path before a chain of vertices with type F.11 leads to the leaf H.

Indeed, we found two descendants of step size $s = 4$ of the fork P_7 where paths of the desired shape can be constructed as described in the Tables 21 and 22. They give rise to a total of $4 \cdot 18 \cdot 3 = 216$ Schur σ-groups H of smallest order $\#H = 3^{29}$, each. The identifiers of these non-metabelian groups H in the sense of the ANUPQ package [14], which is implemented in GAP [15] and MAGMA [18], are given in the following shape:

$$H = P_7 - \#4; \ell - \#2; k - \#4; j - \#2; i - \#4; h - \#1; g - \#2; f - \#1; e - \#2; 1, \tag{4.14}$$

FIGURE 6. Another possible fork topology of $K = \mathbb{Q}(\sqrt{-225\,299})$



where (ℓ, k) is one of the pairs $(148, 11)$, $(179, 11)$, j and i take the unique values in the tables, h is restricted to 18 values in dependence on (ℓ, j), g takes a unique value in dependence on (ℓ, j, h),

f uniformly runs through $1 \le f \le 3$, and e takes a unique value in dependence on (ℓ, j, h, f). The number m of the metabelianization $H/H'' \simeq G = P_{11} - \#2; m$ from Theorem 3.3 is also given by the tables, in dependence on j.

TABLE 21. Associations for $P_7 - \#4; 148 - \#2; 11$

j	i	h	m
1	40		40
18	40		42
33	32		40
35	29	2,3,4,5,7,9,10,12,14,15,16,17,19,20,22,24,26,27	42

TABLE 22. Associations for $P_7 - \#4; 179 - \#2; 11$

j	i	h	m
1	40		40
18	40		42
33	32		40
35	29	2,3,4,6,7,8,10,11,14,15,16,18,19,21,22,23,26,27	42

All the Schur σ-groups H share a common logarithmic order $\mathrm{lo}(H) = 29$, class $\mathrm{cl}(H) = 13$, coclass $\mathrm{cc}(H) = 16$, derived length $\mathrm{dl}(H) = 3$, and IPAD of second order,

$$\tau^{(2)}H = [1^2; (43; 3^32, (521^2)^3), (43; 3^32, (421^2)^3), (1^3; 3^32, (2^21)^3, (21^2)^9)^2]. \tag{4.15}$$

Remark 4.8. Figure 7 shows one of the possible tree topologies for the complex quadratic field $K = \mathbb{Q}(\sqrt{-4\,838\,891})$ of type F.11, expressing the mutual location of $G = \mathrm{G}_3^2K$ and $H = \mathrm{G}_3^3K = \mathrm{G}_3^\infty K$, connected by the fork $\pi^3G = \pi^9H = P_7$ of type b.10.

Remark 4.9. (**Open problems No. 3**)
We intend another section § 3.3 after the section § 3.2 on sporadic vertices outside of coclass trees. In section § 3.3 we shall explore periodic infinite sequences of vertices on coclass trees, in particular of coclass 4, where numerous arithmetical realizations are known. Proceeding in this manner, we shall encounter three new types d.19, d.23, d.25, which act like a scaffold or struts for type F. However, the central fork will remain at $P_7 = \langle 3^7, 64\rangle$ of type b.10.

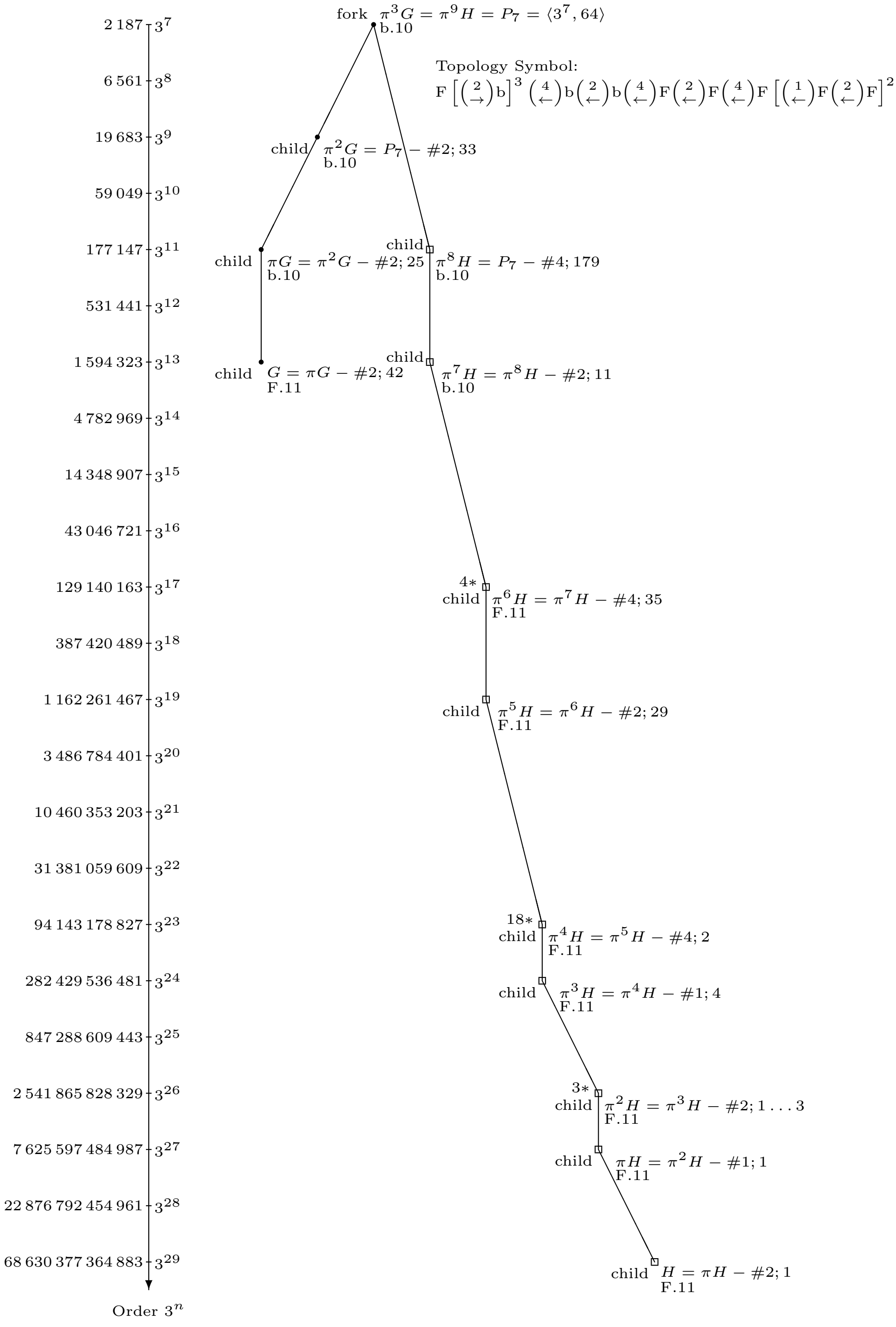
In § 4.2.6 on exceptional cases of type F.7, it turned out that the iterated IPAD of second order is not able to distinguish between Schur σ-groups H of $\mathrm{lo}(H) = 20$, $\mathrm{dl}(H) = 3$, and $\mathrm{lo}(H) = 26$, $\mathrm{dl}(H) = 4$, since the Formulas (4.4) and (4.13) are identical. So the length $\ell_3K \ge 3$ of the 3-class tower of complex quadratic fields in the *tower ground state* remains unknown.

Therefore, we hope that it will be possible to show that the numerous *excited tower states* which occur in the Tables 2 and 3 are realized exclusively by Schur σ-groups H with derived length $\mathrm{dl}(H) \ge 4$. This would prove the long desired $\ell_3K \ge 4$ for $K = \mathbb{Q}(\sqrt{d})$ with $d \in \{-124\,363, -260\,515, -160\,403, -224\,580\}$, and $d \in \{-2\,383\,059, -5\,765\,812\}$.

5. ACKNOWLEDGEMENTS

The first author gratefully acknowledges that his research is supported financially by the Austrian Science Fund (FWF): P 26008-N25.

Figure 7. Possible fork topology of $K = \mathbb{Q}(\sqrt{-4\,838\,891})$



References

[1] J. A. Ascione, *On 3-groups of second maximal class*, Ph. D. Thesis, Australian National University, Canberra, 1979.

[2] J. A. Ascione, G. Havas and C. R. Leedham-Green, *A computer aided classification of certain groups of prime power order*, Bull. Austral. Math. Soc. **17** (1977), 257–274, Corrigendum 317–319, Microfiche Supplement p. 320, DOI 10.1017/s0004972700010467.

[3] H. U. Besche, B. Eick, and E. A. O'Brien, *A millennium project: constructing small groups*, Int. J. Algebra Comput. **12** (2002), 623-644.
[4] H. U. Besche, B. Eick, and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.
[5] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.
[6] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.
[7] W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), *Handbook of Magma functions* (Edition 2.22, Sydney, 2016).
[8] N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of imaginary quadratic fields*, to appear in Math. Annalen, 2016. (arXiv: 1111.4679v2 [math.NT] 10 Dec 2014.)
[9] J. R. Brink, *The class field tower for imaginary quadratic number fields of type* $(3,3)$, Dissertation, Ohio State University, 1984.
[10] J. R. Brink and R. Gold, *Class field towers of imaginary quadratic fields*, manuscripta math. **57** (1987), 425–450.
[11] M. R. Bush and D. C. Mayer, *3-class field towers of exact length* 3, J. Number Theory **147** (2015), 766–777, DOI 10.1016/j.jnt.2014.08.010.
[12] M. du Sautoy, *Counting p-groups and nilpotent groups*, Inst. Hautes Études Sci. Publ. Math. **92** (2000) 63–112.
[13] B. Eick and C. Leedham-Green, *On the classification of prime-power groups by coclass*, Bull. London Math. Soc. **40** (2) (2008), 274–288.
[14] G. Gamble, W. Nickel, and E. A. O'Brien, *ANU p-Quotient — p-Quotient and p-Group Generation Algorithms*, 2006, an accepted GAP package, available also in MAGMA.
[15] The GAP Group, *GAP – Groups, Algorithms, and Programming — a System for Computational Discrete Algebra*, Version 4.8.5, Aachen, Braunschweig, Fort Collins, St. Andrews, 2016, (http://www.gap-system.org).
[16] F.-P. Heider und B. Schmithals, *Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen*, J. Reine Angew. Math. **336** (1982), 1–25.
[17] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Discrete mathematics and its applications, Chapman and Hall/CRC Press, Boca Raton, 2005.
[18] The MAGMA Group, *MAGMA Computational Algebra System*, Version 2.22-5, Sydney, 2016, (http://magma.maths.usyd.edu.au).
[19] D. C. Mayer, *Principalization in complex S_3-fields*, Congressus Numerantium **80** (1991), 73–87. (Proceedings of the Twentieth Manitoba Conference on Numerical Mathematics and Computing, Univ. of Manitoba, Winnipeg, Canada, 1990.)
[20] D. C. Mayer, *The second p-class group of a number field*, Int. J. Number Theory **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.
[21] D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.
[22] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456, DOI 10.5802/jtnb.842. (27ièmes Journées Arithmétiques, Faculty of Mathematics and Informatics, University of Vilnius, Lithuania, July 2011.)
[23] D. C. Mayer, *Principalization algorithm via class group structure*, J. Théor. Nombres Bordeaux **26** (2014), no. 2, 415–464, DOI 10.5802/jtnb.874.
[24] D. C. Mayer and M. F. Newman, *Finite 3-groups as viewed from class field theory*, Groups St Andrews 2013, Univ. of St Andrews, Fife, Scotland.
[25] D. C. Mayer, M. R. Bush, and M. F. Newman, *3-class field towers of exact length* 3, 18th ÖMG Congress and 123rd Annual DMV Meeting 2013, Univ. of Innsbruck, Tyrol, Austria.
[26] D. C. Mayer, M. R. Bush, and M. F. Newman, *Class towers and capitulation over quadratic fields*, West Coast Number Theory 2013, Asilomar Conference Center, Pacific Grove, Monterey, California, USA.
[27] D. C. Mayer, *Periodic bifurcations in descendant trees of finite p-groups*, Adv. Pure Math. **5** (2015), no. 4, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015.
[28] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015), no. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015. (29ièmes Journées Arithmétiques, University of Debrecen, Hungary, July 2015.)
[29] D. C. Mayer, *Periodic sequences of p-class tower groups*, J. Appl. Math. Phys. **3** (2015), no. 7, 746–756, DOI 10.4236/jamp.2015.37090. (1st International Conference on Groups and Algebras, Shanghai, China, July 2015.)
[30] D. C. Mayer, *New number fields with known p-class tower*, Tatra Mountains Math. Publ. **64** (2015), 21 – 57, DOI 10.1515/tmmp-2015-0040, Special Issue on Number Theory and Cryptology '15. (22nd Czech and Slovak International Conference on Number Theory, Liptovský Ján, Slovakia, August 2015.)
[31] D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66 – 104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.

[32] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Math. Phys. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135. (2nd International Conference on Groups and Algebras, Suzhou, China, July 2016.)

[33] B. Nebelung, *Klassifikation metabelscher* 3*-Gruppen mit Faktorkommutatorgruppe vom Typ* $(3,3)$ *und Anwendung auf das Kapitulationsproblem*, Inauguraldissertation, Universität zu Köln, 1989.

[34] M. F. Newman, *Determination of groups of prime-power order*, pp. 73–84, in: Group Theory, Canberra, 1975, Lecture Notes in Math., vol. 573, Springer, Berlin, 1977.

[35] E. A. O'Brien, *The p-group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698.

[36] A. Scholz und O. Taussky, *Die Hauptideale der kubischen Klassenkörper imaginär quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluß auf den Klassenkörperturm*, J. Reine Angew. Math. **171** (1934), 19–41.

[37] I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

Naglergasse 53, 8010 Graz, Austria
E-mail address: algebraic.number.theory@algebra.at
URL: http://www.algebra.at

Australian National University, Canberra, Australian Capital Territory
E-mail address: newman2603@gmail.com