3-CLASS FIELD TOWERS OF EXACT LENGTH 3

AUTHOR: NOMEN NOMINANDUM

The graphs on the following two pages show the *complete normal lattice* of two families of six isoclinic *non-metabelian* 3-groups which occur as Galois groups of 3-class field towers of exact length 3 over complex quadratic base fields with 3-class group of type $(3,3)$.

- The *upper* and *lower central series* of these groups form subgraphs whose relative position justifies the names of these series, as visualized impressively by the figures.
- In the second figure there even occur cases $6 \le j \le 7$ where the index $(\zeta_{m-j}(G) : \gamma_j(G))$ takes the value 3^2. ($m = 8$ denotes the index of nilpotency.)
- *Generators* $x, y \in G \setminus G'$, $s_2, s_3, t_3, \ldots \in G' \setminus G''$, and $u_5, u_7 \in G''$, are carefully selected independently from individual isomorphism types and placed in locations which illustrate the structure of the groups.
- Furthermore, the *normal lattice of the metabelianization* G/G'' is also included as a subgraph simply by putting $u_5 = 1$. The indices $(\zeta_{m-j}(G/G'') : \gamma_j(G/G'')) \le 3$ are bounded.
- Now it is easy to imagine how the figures for the *second, third, ... excited state* will look like. In particular, the indices $(\zeta_{m-j}(G) : \gamma_j(G))$ will be unbounded.

Power-commutator presentations for the six *ground states* of non-metabelian 3-groups G of derived length 3 and order 3^8 having second derived quotient $G/G'' = \langle 2187, i\rangle$,
where $i \in \{288, 290, 289, 304, 306, 302\}$.

$$
\begin{gathered}
G = \langle x, y\rangle = \langle x, y, s_2, s_3, t_3, s_4, s_5, u_5, \sigma_3, \tau_3, \sigma_4, \sigma_5 \mid \\
[y,x] = s_2, \quad [s_2,x] = s_3, \quad [s_2,y] = t_3, \quad [s_3,x] = s_4, \quad [s_4,x] = s_5, \\
[s_4,y] = u_5, \quad [s_3,y] = u_5, \quad [s_3,s_2] = u_5, \\
y^3 = \sigma_3, \quad x^3 = \tau_3, \quad [\sigma_3,x] = \sigma_4, \quad [\sigma_4,x] = \sigma_5, \\
s_2^3 = \sigma_4, \quad \sigma_3^3 = \sigma_5^{-1}, \quad s_3\sigma_3\sigma_4 \equiv \sigma_5^{\gamma}\tau_3^{\delta} \mod G'', \quad t_3^{-1}\tau_3 \equiv \sigma_5^{\alpha}\tau_3^{\beta} \mod G'', \\
y^3 = s_3 s_4^2 s_5^2, \quad x^3 = t_3 s_5, \quad s_2^3 = s_4^2 s_5 u_5, \quad t_3^3 = u_5^2, \quad s_3^3 = s_5^2\rangle
\end{gathered}
$$

Power-commutator presentations for the six *excited states* of non-metabelian 3-groups G of derived length 3 and order 3^{11} having second derived quotient $G/G'' = \langle 19683, \#\rangle$.

$$
\begin{gathered}
G = \langle x, y\rangle = \langle x, y, s_2, s_3, t_3, s_4, s_5, s_6, s_7, u_5, u_7, \sigma_3, \tau_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7 \mid \\
[y,x] = s_2, \quad [s_2,x] = s_3, \quad [s_2,y] = t_3, \quad [s_3,x] = s_4, \quad [s_4,x] = s_5, \quad [s_5,x] = s_6, \quad [s_6,x] = s_7, \\
[s_4,y] = u_5, \quad [s_6,y] = u_7, \quad [s_3,y] = u_5u_7^2, \quad [s_3,s_2] = u_5^2u_7^2, \quad [s_4,s_2] = u_7^2, \quad [s_4,s_3] = u_7, \\
[s_5,y] = u_7^2, \quad [s_5,s_2] = u_7^2, \\
y^3 = \sigma_3, \quad x^3 = \tau_3, \quad [\sigma_3,x] = \sigma_4, \quad [\sigma_4,x] = \sigma_5, \quad [\sigma_5,x] = \sigma_6, \quad [\sigma_6,x] = \sigma_7, \\
s_2^3 = \sigma_4, \quad \sigma_3^3 = \sigma_4^{-3}\sigma_5^{-1}, \quad \sigma_4^3 = \sigma_5^{-3}\sigma_6^{-1}, \quad \sigma_5^3 = \sigma_7^{-1}, \quad s_3\sigma_3\sigma_4 = \sigma_7^{\gamma}\tau_3^{\delta} \mod G'', \quad t_3^{-1}\tau_3 = \sigma_7^{\alpha}\tau_3^{\beta} \mod G'', \\
y^3 = s_3^2 t_3^2 s_4, \quad x^3 = t_3^2 s_7^2 u_7, \quad s_2^3 = s_4^2 s_5 u_5, \quad t_3^3 = u_5^2 u_7^2, \quad s_3^3 = s_5^2 s_6 u_7^2, \quad s_4^3 = s_6^2 s_7, \quad s_5^3 = s_7^2, \quad u_5^3 = u_7^2\rangle
\end{gathered}
$$

Exponents $-1 \le \alpha, \beta, \gamma, \delta \le 1$ depend on the isomorphism type.

Date: March 04, 2013.

FIGURE 1. Full normal lattice, containing the upper and lower central series, of a **three-stage** non-metabelian 3-group G with TKT in section E, ground state.

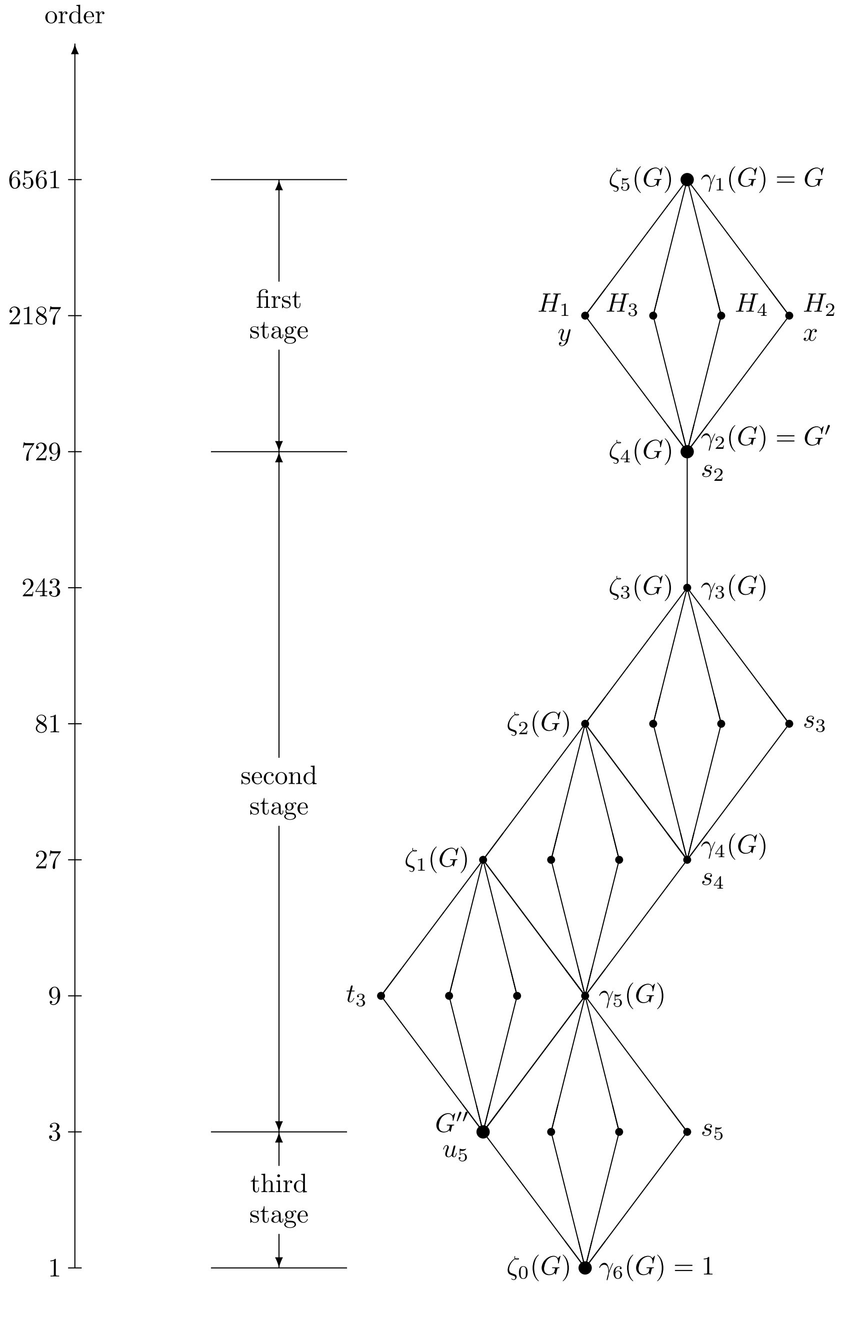
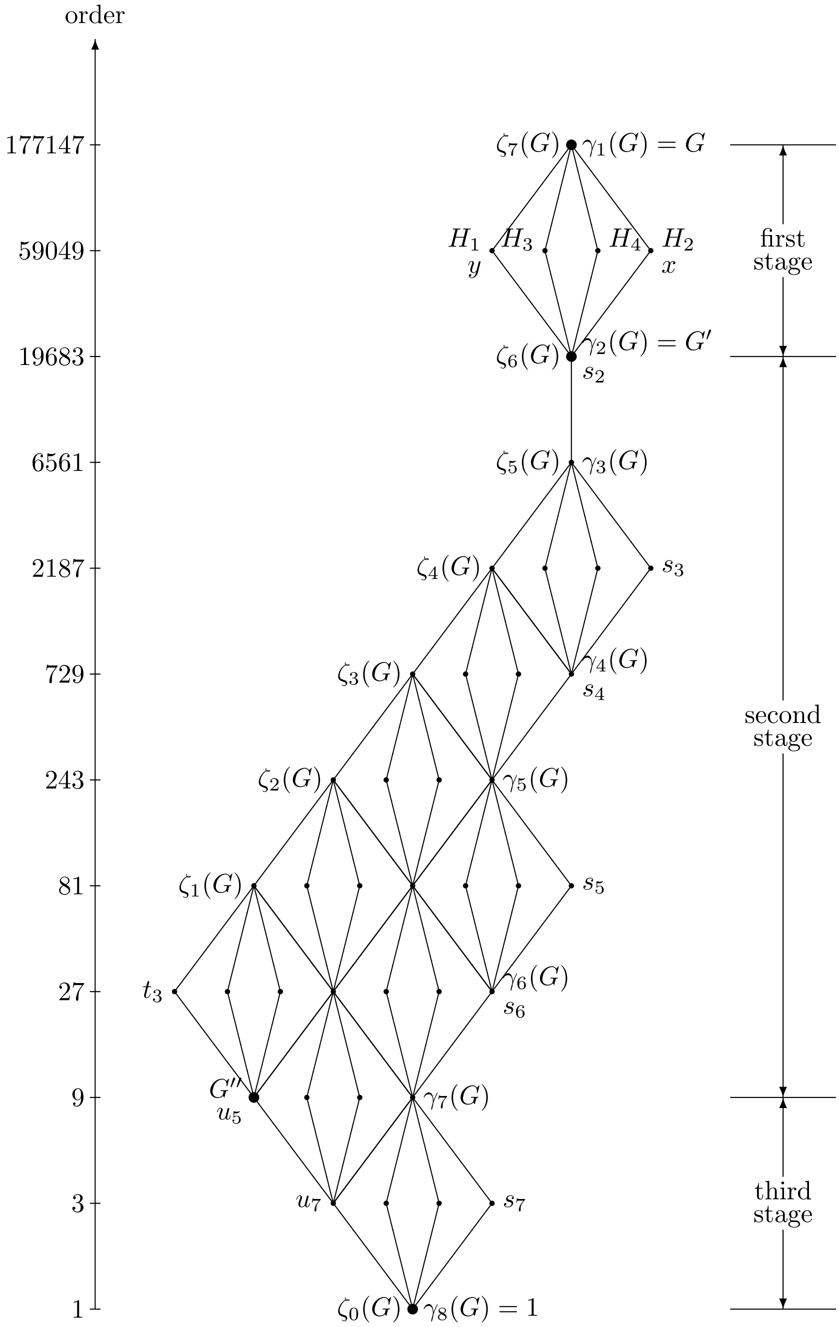


FIGURE 2. Full normal lattice, including the upper and lower central series, of a **three-stage** non-metabelian 3-group G with TKT in section E, first excited state.



Finally, it should be recalled that the figures on the previous two pages suggest the following conjecture which provides a *complete solution for section E.*

Conjecture 1. Let $n \geq 2$ be a positive integer. There exist exactly six Schur σ-groups G of order 3^{3n+2}, class $2n+1$, coclass $n+1$, having fixed derived length 3, such that

(1) the factors of their lower central series are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3,3) & \text{for odd } 1 \leq j \leq 2n+1, \\ (3) & \text{for even } 2 \leq j \leq 2n, \end{cases}$$

(2) the factors of their upper central series are given by

$$\zeta_{j+1}(G)/\zeta_j(G) \simeq \begin{cases} (3,3) & \text{for } j = 2n, \\ (3) & \text{for } 1 \leq j \leq 2n-1, \\ (3,3^n) & \text{for } j = 0, \end{cases}$$

(3) their metabelianization G/G'' is of order 3^{2n+3} and of fixed coclass 2, where G'' is cyclic of order 3^{n-1},

(4) their generalized parent $G/\gamma_{2n+1}(G)$ is of order 3^{3n}, class $2n$, coclass n, and still of derived length 3, except for $n = 2$,

(5) their biggest metabelian generalized predecessor, that is the $(2n-3)$rd generalized parent, is given by either $\langle 729, 49\rangle$ or $\langle 729, 54\rangle$.