SUCCESSIVE APPROXIMATION OF p-CLASS TOWERS

DANIEL C. MAYER

Abstract. Let F be a number field and p be a prime. In the successive approximation theorem, we prove that, for each integer $n \geq 1$, finitely many candidates for the Galois group $\mathrm{G}_p^n F$ of the nth stage $F_p^{(n)}$ of the p-class tower $F_p^{(\infty)}$ over F are determined by abelian type invariants of p-class groups $\mathrm{Cl}_p E$ of unramified extensions E/F with degree $[E : F] = p^{n-1}$. Illustrated by the most extensive numerical results available currently, the transfer kernels $\ker(T_{F,E})$ of the p-class extensions $T_{F,E} : \mathrm{Cl}_p F \to \mathrm{Cl}_p E$ from F to unramified cyclic degree-p extensions E/F are shown to be capable of narrowing down the number of contestants significantly. By determining the isomorphism type of the maximal subgroups $S < G$ of all 3-groups G with coclass $\mathrm{cc}(G) = 1$, and establishing a general theorem on the connection between the p-class towers of a number field F and of an unramified abelian p-extension E/F, we are able to provide a theoretical proof of the realization of certain 3-groups S with maximal class by 3-tower groups $\mathrm{G}_3^\infty E$ of dihedral fields E with degree 6, which could not be realized up to now.

1. Introduction

For a prime number p and an algebraic number field F, let $F_p^{(\infty)}$ be the *p-class tower*, more precisely the unramified Hilbert p-class field tower, that is the maximal unramified pro-p extension, of F. The individual stages $F_p^{(n)}$ and the Galois groups $\mathrm{Gal}(F_p^{(n)}/F)$ of the tower

$$F = F_p^{(0)} \leq F_p^{(1)} \leq F_p^{(2)} \leq \ldots \leq F_p^{(n)} \leq \ldots \leq F_p^{(\infty)}$$

are described by the derived quotients $\mathfrak{G}/\mathfrak{G}^{(n)} \simeq \mathrm{G}_p^n F := \mathrm{Gal}(F_p^{(n)}/F)$ with $n \geq 1$, of the *p-class tower group* $\mathfrak{G} := \mathrm{G}_p^\infty F := \mathrm{Gal}(F_p^{(\infty)}/F)$. The purpose of this paper is to report on the most up-to-date theoretical view of p-class towers and the state of the art of actual numerical investigations. After a summary of algebraic and arithmetic foundations in § 2, four crucial concepts will illuminate recent innovation and progress in a very ostensive way:

- the *Artin limit pattern* $(\tau^{(\infty)}F, \varkappa^{(\infty)}F)$ of the p-class tower $F_p^{(\infty)}$ in § 3,
- *successive approximation* and the current status of computational perspectives in § 4,
- *maximal subgroups* of 3-class tower groups with coclass one in § 5, and
- the realization of new 3-class tower groups over *dihedral fields* in § 6.

2. Algebraic and arithmetic foundations

2.1. **Abelian type invariants.** First, we recall the concepts of abelian type invariants and abelian quotient invariants in the context of finite p-groups and infinite pro-p groups, and we specify our conventions in their notation.

Let $p \geq 2$ be a prime number. It is well known that a finite abelian group A with order $|A|$ a power of p possesses a *unique* representation

$$A \simeq \oplus_{i=1}^{s} (\mathbb{Z}/p^{e_i}\mathbb{Z})^{r_i} \tag{2.1}$$

Date: October 11, 2017.

2010 *Mathematics Subject Classification.* Primary 11R37, 11R32, 11R29, 11R11, 11R20; secondary 20D15, 20F05, 20F12, 20F14, 20-04.

Key words and phrases. p-Class towers, Galois groups, second p-class groups, abelian type invariants of p-class groups, p-transfer kernel types, Artin limit pattern, quadratic fields, unramified cyclic extensions of degree p, dihedral fields of degree $2p$; finite p-groups, maximal nilpotency class, maximal subgroups, polycyclic pc-presentations, commutator calculus, central series.

Research supported by the Austrian Science Fund (FWF): P 26008-N25.

as a direct sum with integers $s \ge 0$, $r_i \ge 1$ for $1 \le i \le s$, and *strictly decreasing* $e_1 > \ldots > e_s \ge 1$.

Definition 2.1. The *abelian type invariants* of A are given either in *power form*,

$$\mathrm{ATI}(A) := [\overbrace{p^{e_1},\ldots,p^{e_1}}^{r_1 \text{ times}},\ldots,\overbrace{p^{e_s},\ldots,p^{e_s}}^{r_s \text{ times}}], \tag{2.2}$$

or in *logarithmic form* with formal exponents indicating iteration,

$$\mathrm{ATI}(A) := [e_1^{r_1},\ldots,e_s^{r_s}]. \tag{2.3}$$

Let G be a pro-p group with commutator subgroup G' and *finite* abelianization $G^{ab} := G/G'$.

Definition 2.2. The *abelian quotient invariants* of G are the abelian type invariants of the biggest abelian quotient of G

$$\mathrm{AQI}(G) := \mathrm{ATI}(G^{ab}). \tag{2.4}$$

2.1.1. *Higher abelian quotient invariants of a pro-p group.* Within the frame of group theory, abelian quotient invariants of *higher order* are defined recursively in the following manner.

Definition 2.3. The set of all maximal subgroups of G which contain the commutator subgroup,

$$\mathrm{Lyr}_1 G := \{S \lhd G \mid G' \le S,\ (G:S) = p\}, \tag{2.5}$$

is called the *first layer* of subgroups of G. For any positive integer $n \ge 1$, *abelian quotient invariants of nth order* of G are defined recursively by

$$\tau^{(1)}G := \mathrm{AQI}(G), \text{ and } \tau^{(n)}G := (\tau^{(1)}G; (\tau^{(n-1)}S)_{S\in\mathrm{Lyr}_1 G}) \text{ for } n \ge 2. \tag{2.6}$$

2.1.2. *Higher abelian type invariants of a number field.* Within the frame of algebraic number theory, abelian type invariants of *higher order* are defined recursively in the following way.

Let F be an algebraic number field, denote by $\mathrm{Cl}_p F$ the p-class group of F, and by $F_p^{(1)}$ the first Hilbert p-class field of F, that is, the maximal abelian unramified p-extension of F.

Definition 2.4. The set of all unramified cyclic extensions E/F of degree p which are contained in the p-class field,

$$\mathrm{Lyr}_1 F := \{E > F \mid E \le F_p^{(1)},\ [E:F] = p\} \tag{2.7}$$

is called the *first layer* of extension fields of F. For any positive integer $n \ge 1$, *abelian type invariants of nth order* of F are defined recursively by

$$\tau^{(1)}F := \mathrm{ATI}(\mathrm{Cl}_p F), \text{ and } \tau^{(n)}F := (\tau^{(1)}F; (\tau^{(n-1)}E)_{E\in\mathrm{Lyr}_1 F}) \text{ for } n \ge 2. \tag{2.8}$$

2.2. **Transfer kernel type.** Next, we explain the concept of transfer kernel type of finite p-groups and infinite pro-p groups.

2.2.1. *Transfer kernel type of a pro-p group.* Denote by $p \ge 2$ a prime number. Let G be a pro-p group with commutator subgroup G' and *finite* abelianization $G^{ab} = G/G'$.

Definition 2.5. By the *transfer kernel type* of G, we understand the finite family of kernels,

$$\varkappa(G) := (\ker(T_{G,S}))_{S\in\mathrm{Lyr}_1 G}, \tag{2.9}$$

where $T_{G,S} : G/G' \to S/S'$ denotes the transfer homomorphism from G to the normal subgroup S of finite index $(G:S) = p$, as given in Formula (3.1).

More specifically, suppose that $G^{ab} \simeq C_p \times C_p$ is elementary abelian of rank two. Then $\mathrm{Lyr}_1 G$ has $p+1$ elements $S_1,\ldots,S_{p+1}$, the transfer kernel type of G is described briefly by a family of non-negative integers $\varkappa(G) = (\varkappa_i)_{1\le i\le p+1} \in [0,p+1]^{p+1}$ such that

$$\varkappa_i := \begin{cases} 0 & \text{if } \ker(T_{G,S_i}) = G/G', \\ j & \text{if } \ker(T_{G,S_i}) = S_j/G' \text{ for some } 1 \le j \le p+1, \end{cases} \tag{2.10}$$

and the symmetric group S_{p+1} of degree $p+1$ acts on $[0,p+1]^{p+1}$ via $\varkappa \mapsto \varkappa^\pi := \pi_0^{-1} \circ \varkappa \circ \pi$, for each $\pi \in \mathrm{S}_{p+1}$, where the extension π_0 of π to $[0,p+1]$ fixes the zero.

Definition 2.6. The orbit $\varkappa(G)^{\mathrm{S}_{p+1}}$ is called the *invariant type* of G, but it is actually given by one of the orbit representatives $(\varkappa_i)_{1\le i\le p+1}$. Any two distinct orbit representatives $\lambda_1,\lambda_2\in\varkappa(G)^{\mathrm{S}_{p+1}}$ are called *equivalent*, denoted by the symbol $\lambda_1\sim\lambda_2$.

2.2.2. *Transfer kernel type of a number field.* Let F be an algebraic number field, and denote by Cl_pF the p-class group of F.

Definition 2.7. By the *transfer kernel type* of F, we understand the finite family of kernels,

$$\varkappa(F) := (\ker(T_{F,E}))_{E\in\mathrm{Lyr}_1F}, \tag{2.11}$$

where $T_{F,E}:\mathrm{Cl}_pF\to\mathrm{Cl}_pE$ denotes the transfer of p-classes from F to the unramified cyclic extension E of degree $[E:F]=p$, which is also known as the p-class extension homomorphism.

More specifically, suppose that $\mathrm{Cl}_pF\simeq C_p\times C_p$ is elementary abelian of rank two. Then Lyr_1F has $p+1$ elements $E_1,\dots,E_{p+1}$, the transfer kernel type of F is described briefly by a family of non-negative integers $\varkappa(F)=(\varkappa_i)_{1\le i\le p+1}\in[0,p+1]^{p+1}$ such that

$$\varkappa_i := \begin{cases} 0 & \text{if } \ker(T_{F,E_i})=\mathrm{Cl}_pF,\\ j & \text{if } \ker(T_{F,E_i})=\mathrm{Norm}_{E_j/F}(\mathrm{Cl}_pE_j) \text{ for some } 1\le j\le p+1,\end{cases} \tag{2.12}$$

and the symmetric group S_{p+1} of degree $p+1$ acts on $[0,p+1]^{p+1}$ via $\varkappa\mapsto\varkappa^\pi:=\pi_0^{-1}\circ\varkappa\circ\pi$, for each $\pi\in\mathrm{S}_{p+1}$, where the extension π_0 of π to $[0,p+1]$ fixes the zero.

Definition 2.8. The orbit $\varkappa(F)^{\mathrm{S}_{p+1}}$ is called the *invariant type* of F, but it is actually given by one of the orbit representatives $(\varkappa_i)_{1\le i\le p+1}$. Any two distinct orbit representatives $\lambda_1,\lambda_2\in\varkappa(F)^{\mathrm{S}_{p+1}}$ are called *equivalent*, denoted by the symbol $\lambda_1\sim\lambda_2$.

3. The Artin limit pattern

Let p be a prime number. For the recursive construction of the Artin limit pattern of a pro-p group G with commutator subgroup G' and *finite* abelianization $G^{ab}=G/G'$, we need the following considerations.

3.1. **Mappings of the Artin limit pattern.** Due to our assumptions, the first layer Lyr_1G of subgroups of G is a finite set consisting of maximal normal subgroups S of G with abelian quotients G/S. Consequently, the *Artin transfer homomorphism* from G to $S\in\mathrm{Lyr}_1G$ is distinguished by a very simple mapping law:

$$T_{G,S}:G/G'\to S/S',\ g\cdot G'\mapsto\begin{cases} g^p\cdot S' & \text{if } g\in(G/G')\setminus(S/G'),\\ g^{1+h+h^2+\ldots+h^{p-1}}\cdot S' & \text{if } g\in S/G',\end{cases} \tag{3.1}$$

where h denotes an arbitrary element in $G\setminus S$ [24, § 4.1, p. 76].

The Artin limit pattern encapsulates particular group theoretic information (connected with Artin transfers) about the lattice of subgroups of G, where each element U has at least one predecessor, except the root G itself. We select a unique predecessor in the following way: for $U\in\mathrm{Lyr}_1S$ we put $\pi(U):=S$, and we add the formal definition $\pi(G):=G$. This enables a recursive construction, as follows:

Definition 3.1. The *collection of Artin transfers up to order* n of G is defined recursively by

$$\alpha^{(1)}G:=T_{\pi(G),G},\ \text{and}\ \alpha^{(n)}G:=[\alpha^{(1)}G;(\alpha^{(n-1)}S)_{S\in\mathrm{Lyr}_1G}]\ \text{for}\ n\ge 2. \tag{3.2}$$

The limit of this infinite recursive nesting process is denoted by

$$\alpha^{(\infty)}G:=\lim_{n\to\infty}\alpha^{(n)}G \tag{3.3}$$

and is called the *Artin transfer collection* of G.

Remark 3.1. By means of the collection of Artin transfers up to order three,

$$\alpha^{(3)}G = [T_{G,G}; (\alpha^{(2)}S)_{S\in\mathrm{Lyr}_1 G}] = [T_{G,G}; ([T_{G,S}; (T_{S,U})_{U\in\mathrm{Lyr}_1 S}])_{S\in\mathrm{Lyr}_1 G}],$$

it should be emphasized that our definition of stepwise relative mappings $T_{G,S}$ and $T_{S,U}$ admits finer information than the corresponding absolute mappings $T_{G,U} = T_{S,U} \circ T_{G,S}$ [24, Thm. 3.3, p. 72], since in general the kernel of $T_{S,U}$ cannot be reconstructed from $T_{G,U}$ and $T_{G,S}$.

3.2. **Objects of the Artin limit pattern.** The infinite collection of mappings $\alpha^{(\infty)}G$ is only the foundation for the objects $\tau^{(\infty)}G$ and $\varkappa^{(\infty)}G$ we are really interested in.

Definition 3.2. The *iterated abelian quotient invariants up to order* n of G are defined recursively by

$$\tau^{(1)}G := \mathrm{AQI}(G), \text{ and } \tau^{(n)}G := [\tau^{(1)}G; (\tau^{(n-1)}S)_{S\in\mathrm{Lyr}_1 G}] \text{ for } n \ge 2. \tag{3.4}$$

Similarly, the *iterated transfer kernels up to order* n of G are defined recursively by

$$\varkappa^{(1)}G := \ker(T_{\pi(G),G}), \text{ and } \varkappa^{(n)}G := [\varkappa^{(1)}G; (\varkappa^{(n-1)}S)_{S\in\mathrm{Lyr}_1 G}] \text{ for } n \ge 2. \tag{3.5}$$

Both are collected in the nth *order Artin pattern* $\mathrm{AP}^{(n)}G := (\tau^{(n)}G, \varkappa^{(n)}G)$ of G. The limits of these infinite recursive nesting processes are called the *abelian invariant collection* of G,

$$\tau^{(\infty)}G := \lim_{n\to\infty} \tau^{(n)}G, \tag{3.6}$$

and the *transfer kernel collection* of G,

$$\varkappa^{(\infty)}G := \lim_{n\to\infty} \varkappa^{(n)}G. \tag{3.7}$$

Finally, the pair $\mathrm{ALP}(G) := (\tau^{(\infty)}G, \varkappa^{(\infty)}G)$ is called the *Artin limit pattern* of G.

Remark 3.2. For a finite p-group G, the recursive nesting processes in the definition of the Artin limit pattern are actually finite.

The abelian quotient invariants are a *unary* concept, since $\tau^{(1)}G = \mathrm{AQI}(G) = \mathrm{ATI}(G/G')$ depends on G only. The first order abelian quotient invariants $\tau^{(1)}G$ already contain non-trivial information on the abelianization of G.

The transfer kernels are a *binary* concept for $S < G$, since $\varkappa^{(1)}S = \ker(T_{\pi(S),S})$ depends on $\pi(S)$ and S. The first order transfer kernel of G is trivial: $\varkappa^{(1)}G = \ker(T_{\pi(G),G}) = \ker(T_{G,G}) = \ker(\mathrm{id}_{G/G'}) = 1$, and non-trivial information starts with the transfer kernels of second order $\varkappa^{(1)}S = \ker(T_{\pi(S),S}) = \ker(T_{G,S})$ for $S \in \mathrm{Lyr}_1 G$ which are members of $\varkappa^{(2)}G$.

The analogous constructions for a *number field* F instead of a pro-p group G, along the lines of §§ 2.1.2 and 2.2.2, lead to the *Artin limit pattern* $\mathrm{ALP}(F) := (\tau^{(\infty)}F, \varkappa^{(\infty)}F)$ of F.

3.3. **Connection between pro-p groups and number fields.** Let $F_p^{(\infty)}$ be the Hilbert p-class tower of the number field F, that is, the maximal unramified pro-p extension of F, and denote by $\mathrm{G}_p^\infty F = \mathrm{Gal}(F_p^{(\infty)}/F)$ its Galois group, which is briefly called the *p-tower group* of F. Now we are going to employ the abelian type invariant collection $\tau^{(\infty)}F$ of F, and the abelian quotient invariant collection $\tau^{(\infty)}(\mathrm{G}_p^\infty F)$ of $\mathrm{G}_p^\infty F$, i.e., the first component of the respective Artin limit pattern. The transfer kernel collections $\varkappa^{(\infty)}$ will be considered further in § 5.

Theorem 3.1. *For each integer $n \ge 1$, the abelian quotient invariants of nth order of the p-tower group $\mathrm{G}_p^\infty F$ of F are equal to the abelian type invariants of nth order of the number field F*

$$(\forall n \ge 1) \quad \tau^{(n)}(\mathrm{G}_p^\infty F) = \tau^{(n)}F, \text{ and thus } \tau^{(\infty)}(\mathrm{G}_p^\infty F) = \tau^{(\infty)}F. \tag{3.8}$$

The invariant type of the p-tower group $\mathrm{G}_p^\infty F$ of F coincides with the invariant type of the number field F

$$\varkappa(\mathrm{G}_p^\infty F)^{S_{p+1}} = \varkappa(F)^{S_{p+1}}. \tag{3.9}$$

Even the orbit representatives of the transfer kernel types of $\mathrm{G}_p^\infty F$ and F coincide,

$$\varkappa(\mathrm{G}_p^\infty F) = (\ker(T_{\mathrm{G}_p^\infty F,U_i}))_{1\le i\le p+1} = (\ker(T_{F,E_i}))_{1\le i\le p+1} = \varkappa(F), \tag{3.10}$$

provided that the $U_i \in \mathrm{Lyr}_1(\mathrm{G}_p^\infty F)$ and the $E_i \in \mathrm{Lyr}_1 F$ are connected by $U_i = \mathrm{Gal}(F_p^{(\infty)}/E_i)$, for each $1 \le i \le p+1$. Otherwise, we only have equivalence $\varkappa(\mathrm{G}_p^\infty F) \sim \varkappa(F)$.

Proof. The claims are well-known consequences of the Artin reciprocity law of class field theory [1, 2]. □

In contrast to the full p-tower group $\mathfrak{G} = \mathrm{G}_p^\infty F$, the Galois groups $\mathrm{G}_p^m F := \mathrm{Gal}(F_p^{(m)}/F) \simeq \mathfrak{G}/\mathfrak{G}^{(m)}$ of the finite stages $F_p^{(m)}$ of the p-class tower of F, that is, of the higher Hilbert p-class fields of the number field F, in general fail to reveal the abelian type invariants of nth order of the number field F. More precisely, there is a strict upper bound on the order n of the ATI of F which coincide with the AQI of order n of the mth p-class group $\mathrm{G}_p^m F$ of F with a fixed integer $m \ge 0$, namely the bound $n \le m$.

Theorem 3.2. ***(Successive Approximation Theorem.)***
Let F be a number field, p a prime, and m, n integers. The abelian invariant collection $\tau^{(\infty)}F$ of F is approximated successively by the iterated AQI of sufficiently high p-class groups of F:

$$(3.11) \qquad (\forall n \ge 1) \quad (\forall m \ge n) \quad \tau^{(n)}(\mathrm{G}_p^m F) = \tau^{(n)} F.$$

However, the transfer kernel type is a phenomenon of second order:

$$(3.12) \qquad (\forall m \ge 2) \quad \varkappa(\mathrm{G}_p^m F) \sim \varkappa(F),$$

in particular, the metabelian second p-class group $\mathfrak{M} := \mathrm{G}_p^2 F \simeq \mathfrak{G}/\mathfrak{G}''$ of F is sufficient for determining the transfer kernel type of F.

Proof. This is one of the main results of [27, Thm. 1.19, p. 78] and [28, p. 13]. □

In general, the upper bound on the order n of the ATI of F in Theorem 3.2 seems to be sharp, in the following sense, where $m = n - 1$.

Conjecture 3.1. ***(Stage Separation Criterion.)***
Denote by $\ell_p F$ the length of the p-class tower of F, that is the derived length $\mathrm{dl}(\mathrm{G}_p^\infty F)$ of the p-tower group of F. It is determined in terms of iterated AQI of higher p-class groups of F by the following condition:

$$(3.13) \qquad (\forall n \ge 1) \quad [\ \ell_p F \ge n \iff \tau^{(n)}(\mathrm{G}_p^{n-1} F) < \tau^{(n)} F\ ].$$

The sufficiency of the condition in Conjecture 3.1 is a proven theorem [28, p. 13].

4. Successive approximation of the p-class tower

4.1. **Computational perspectives.** Our first attempt to find sound asymptotic tendencies in the distribution of higher non-abelian p-class groups $\mathrm{G}_p^n F = \mathrm{Gal}(F_p^{(n)}/F)$, with $n \ge 2$, among the finite p-groups was planned in 1991 already [16, § 3, Remark, p. 77]. However, the insurmountable obstacles in the required computations limited the progress for twenty years. In 2012, we finally succeeded in the significant break-through of computing the second 3-class groups $\mathfrak{M} = \mathrm{G}_3^2 F$, that is, the metabelianizations $\mathfrak{G}/\mathfrak{G}^{(2)}$ of the 3-class tower groups $\mathfrak{G} = \mathrm{Gal}(F_3^{(\infty)}/F)$ of all 4596 quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants in the remarkable range $-10^6 < d < 10^7$ and elementary bicyclic 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$ of rank two [18, § 6, pp. 495–499]. The underlying computational techniques were based on the *principalization algorithm via class group structure* which we had invented in 2009 and implemented by means of the number theoretic computer algebra system PARI/GP [35] in 2010, as described in [19, §§ 5–6, pp. 446–455].

Throughout this paper, isomorphism classes of finite groups G are characterized uniquely by their identifier in the SmallGroups Database [3, 4], which is denoted by a pair $\langle o, i \rangle$ consisting of the order $o = \mathrm{ord}(G)$ and a positive integer i, delimited with angle brackets. The counter $1 \le i \le N(o)$ is unique for a fixed value of the order o. In the computational algebra system MAGMA [7, 8, 15], the upper bound $N(o)$ can be obtained as return value of the function `NumberOfSmallGroups`(o), provided that `IsInSmallGroupDatabase`(o) returns `true`. The identifier of a given finite group G can be retrieved as return value of the function `IdentifyGroup`(G), provided that `CanIdentifyGroup`(o) returns `true`.

4.2. **Trivial towers with $\ell_pF = 0$.** For the decision if the p-class tower of a number field F is trivial with length $\ell_pF = 0$ it suffices to compute the class number $h(F)$ of the field.

Theorem 4.1. ***(Trivial p-class tower.)***
The p-class tower of a number field F is trivial, $F_p^{(\infty)} = F$, with length $\ell_pF = 0$, if and only if the class number $h(F) = \#\mathrm{Cl}(F)$ is not divisible by p, i. e., the p-class number is $h_pF = 1$.

Proof. The proof consists of a sequence of equivalent statements: The class number satisfies $p \nmid h(F)$. $\iff$ The p-valuation of $h(F)$ is $v_p(h(F)) = 0$. $\iff$ The p-class number is $\#\mathrm{Cl}_pF = h_pF = p^{v_p(h(F))} = 1$. $\iff$ The p-class group $\mathrm{Cl}_pF = 1$ is trivial. $\iff$ The p-class rank $\rho_p = \dim_{\mathbb{F}_p}(\mathrm{Cl}(F)/\mathrm{Cl}(F)^p)$ is equal to zero. $\iff$ The number of unramified cyclic extensions E/F of degree p is $\frac{p^{\rho_p}-1}{p-1} = \frac{p^0-1}{p-1} = \frac{1-1}{p-1} = 0$. $\iff$ The maximal unramified p-extension $F_p^{(\infty)}$ of F coincides with F. $\iff$ The Galois group $\mathrm{G}_p^\infty F = \mathrm{Gal}(F_p^{(\infty)}/F) = \mathrm{Gal}(F/F) = 1$ is trivial. $\iff$ The length of the p-class tower is $\ell_pF = \mathrm{dl}(\mathrm{G}_p^\infty F) = \mathrm{dl}(1) = 0$. □

Already C. F. Gauss was able to compute class numbers $h(F)$ of quadratic fields $F = \mathbb{Q}(\sqrt{d})$, at a time when the concept of class field theory was not yet coined. Nowadays, there exist extensive tables of quadratic class numbers which even contain the structures of the associated class groups $\mathrm{Cl}(F)$. In 1998, Jacobson [13] covered all real quadratic fields with positive discriminants in the range $0 < d < 10^9$, and in 2016, Mosunov and Jacobson [33] investigated all imaginary quadratic fields with negative discriminants $-10^{12} < d < 0$. Now we apply these results to class field theory.

Corollary 4.1. ***(Statistics for $p = 3$.)*** *The asymptotic proportion of imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$, with negative discriminants $d < 0$, whose class number $h(F)$ is, respectively is not, divisible by $p = 3$ is given as 43.99%, respectively 56.01%, by the heuristics of Cohen, Lenstra and Martinet. In Table 1, the approximations of these theoretical limits by relative frequencies in various ranges $L < d < 0$ are shown.*

TABLE 1. Imaginary quadratic fields F with non-trivial, resp. trivial, 3-class tower

L	$\#(3 \mid h(F))$	rel. fr.	$\#(3 \nmid h(F))$	rel. fr.	w. r. t. #total
-10^6	121 645	40.02%	182 323	59.98%	303 968
-10^{11}	13 206 088 529	43.45%	17 190 266 523	56.55%	30 396 355 052
-10^{12}	132 584 350 621	43.62%	171 379 200 091	56.38%	303 963 550 712

Proof. The heuristic asymptotic limits are given in [12, § 2, (1.1.c), p. 126]. Their approximation by discriminants $L < d < 0$ with $L = -10^6$ in [17, Example, p. 843] and [16, § 2, Remark, and § 3, Remark, p. 77], where $118\,455 + 3\,190 = 121\,645$, is still rather far away from the limits. In contrast, the approximations associated with the bounds $L = -10^{11}$ and $L = -10^{12}$ in [33, p. 2001] are very close already. □

4.3. **Abelian single-stage towers with $\ell_pF = 1$.** The first stage of the p-class tower of a number field F is determined by the structure of the p-class group Cl_pF of F as a finite abelian p-group. This is exactly the first order Artin pattern

$$\mathrm{AP}^{(1)}F = (\tau^{(1)}F, \varkappa^{(1)}F) = (\mathrm{ATI}(\mathrm{Cl}_pF), \ker(T_{F,F})), \tag{4.1}$$

since the trivial $\ker(T_{F,F}) = 1$ does not contain information. However, only in the case of p-class rank one, $\rho_p = \dim_{\mathbb{F}_p}(\mathrm{Cl}(F)/\mathrm{Cl}(F)^p) = 1$, it is warranted that the exact length of the tower is $\ell_pF = 1$. A statistical example [16, § 2, Remark, p. 77] is shown in Table 2.

Theorem 4.2. *A number field F with non-trivial cyclic p-class group Cl_pF has an abelian p-class tower of exact length $\ell_pF = 1$, in fact, the Galois group $\mathrm{G}_p^\infty F \simeq \mathrm{G}_p^1 F \simeq \mathrm{Cl}_pF$ is cyclic.*

Proof. Suppose that $\mathrm{Cl}_pF > 1$ is non-trivial and cyclic. If the p-class tower had a length $\ell_pF \geq 2$, the second p-class group $\mathfrak{M} = \mathrm{G}_p^2F$ would be a non-abelian finite p-group with cyclic abelianization $\mathfrak{M}/\mathfrak{M}' \simeq \mathrm{Cl}_pF$. However, it is well known that a nilpotent group with cyclic abelianization is abelian, which contradicts the assumption of a length $\ell_pF \geq 2$. □

TABLE 2. Imaginary quadratic fields F with cyclic 3-class tower for $-10^6 < d < 0$

$\mathrm{Cl}_3F \simeq$	abs. fr.	rel. fr.	w. r. t. $\#(\rho_3 = 1)$
C_3	80 115	67.63%	118 455
C_9	26 458	22.34%	118 455
C_{27}	8 974	7.58%	118 455
C_{81}	2 472	2.09%	118 455
C_{243}	393	0.33%	118 455
C_{729}	43	0.04%	118 455

Remark 4.1. We interpret the computation of abelian type invariants $\tau^{(1)}F$ of the Sylow 3-subgroup Cl_3F of the ideal class group $\mathrm{Cl}(F)$ of a quadratic field $F = \mathbb{Q}(\sqrt{d})$ as the determination of the single-stage approximation $\mathfrak{G}/\mathfrak{G}' \simeq \mathrm{G}_3^1F \simeq \mathrm{Cl}_3F$ of the 3-class tower group $\mathfrak{G} = \mathrm{G}_3^\infty F$ of F. This step yields complete information about the lattice of all unramified abelian 3-extensions E/F within the Hilbert 3-class field F_3^1F of F.

4.4. **Metabelian two-stage towers with $\ell_pF = 2$.** According to the Successive Approximation Theorem 3.2, the second stage $F_p^{(2)}$ of the p-class tower of a number field F is determined by the second order Artin pattern

(4.2)

$$\mathrm{AP}^{(2)}F = (\tau^{(2)}F, \varkappa^{(2)}F) = ([\mathrm{ATI}(\mathrm{Cl}_pF); (\mathrm{ATI}(\mathrm{Cl}_pE))_{E\in\mathrm{Lyr}_1F}], [\ker(T_{F,F}); (\ker(T_{F,E}))_{E\in\mathrm{Lyr}_1F}]).$$

The determination of $\mathrm{AP}^{(2)}F$ for a quadratic field F with 3-class rank $\rho_3 = 2$ requires the computation of four 3-class groups Cl_3E_i of unramified cyclic cubic extensions $E_1, \ldots, E_4$ and of four transfer kernels $\ker(T_{F,E_i})$.

Whereas Mosunov and Jacobson [33] were able to determine the class groups $\mathrm{Cl}(F)$ of more than 300 billion, precisely 303 963 550 712, imaginary quadratic fields F with discriminants $-10^{12} < d < 0$ by parallel processes on multiple cores of a supercomputer in several years of total CPU time, it is currently definitely out of scope to compute the class groups $\mathrm{Cl}(E_i)$, $1 \leq i \leq 4$, for the 22 757 307 168 unramified cyclic cubic extensions E_i/F, of absolute degree six, of the 5 689 326 792 imaginary quadratic fields F with discriminants $-10^{12} < d < 0$ and 3-class rank $\rho_3 = 2$.

Therefore, it must not be underestimated that Boston, Bush and Hajir [9] succeeded in completing this task for the smaller range $-10^8 < d < 0$ with 461 925 imaginary quadratic fields F having 3-class rank $\rho_3 = 2$, and 1 847 700 associated *totally complex dihedral fields* E_i of degree six [18, Prp. 4.1, p. 482]. For this purpose the authors used the computational algebra system MAGMA [7, 8, 15] in a distributed process involving several processors with multiple cores. 276 375 of these quadratic fields F have a 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$.

Imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with negative discriminants $d < 0$ are the simplest number fields with respect to their unit group U_F, which is a finite torsion group of Dirichlet unit rank zero. This fact has considerable consequences for their p-class tower groups, according to the Shafarevich theorem [36], corrected in [23, Thm. 5.1, p. 28], [22].

Theorem 4.3. *Among the finite 3-groups G with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$ of rank two, there exist only two metabelian groups with GI-action and relation rank $d_2G = 2$ (so-called Schur σ-groups [14, 9]), namely $\langle 243, 5\rangle$ and $\langle 243, 7\rangle$.*

(1) *These are the groups of smallest order which are admissible as 3-class tower groups $G \simeq \mathrm{G}_3^\infty F$ of imaginary quadratic fields F with 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$.*

(2) *Generally, for any number field F, these groups are determined uniquely by the second order Artin pattern.*
  (a) *If* $\mathrm{AP}^{(2)}F = ([1^2;(21,21,1^3,21)],[1;(2241)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 243,5\rangle$.
  (b) *If* $\mathrm{AP}^{(2)}F = ([1^2;(1^3,21,1^3,21)],[1;(4224)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 243,7\rangle$.
(3) *The actual distribution of these 3-class tower groups G among the* 276 375 *imaginary quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with 3-class group* $\mathrm{Cl}_3F \simeq C_3\times C_3$ *and discriminants* $-10^8 < d < 0$ *is presented in Table 3.*

TABLE 3. Frequencies of metabelian 3-class tower groups G for $-10^8 < d < 0$

$G\simeq$	abs. fr.	rel. fr.	w. r. t.	rel. fr.	w. r. t.	measure [9]	$\lvert d\rvert_{\min}$
$\langle 243,5\rangle$	83 353	30.16%	276 375	18.04%	461 925	$128/729 \approx 17.56\%$	4 027
$\langle 243,7\rangle$	41 398	14.98%	276 375	8.96%	461 925	$64/729 \approx 8.78\%$	12 131

Proof. All finite 3-groups G with abelianization $G/G' \simeq C_3\times C_3$ are vertices of the descendant tree $\mathcal{T}(R)$ with abelian root $R=\langle 9,2\rangle \simeq C_3\times C_3$. A search for metabelian vertices with relation rank $d_2G=2$ in this tree yields three hits $\langle 27,4\rangle$, $\langle 243,5\rangle$, and $\langle 243,7\rangle$, but only the latter two of them possess a GI-action.

The abelianization G/G' of a finite 3-group G which is realized as the 3-class tower group $\mathrm{G}_p^\infty F$ of an algebraic number field F is isomorphic to the 3-class group Cl_3F of F. When F is imaginary quadratic, it possesses signature $(r_1,r_2)=(0,1)$ and torsionfree Dirichlet unit rank $r=r_1+r_2-1=0$. If $G/G' \simeq \mathrm{Cl}_3F \simeq C_3\times C_3$, then the generator rank of G is $d_1G=2$ and the Shafarevich theorem implies bounds for the relation rank $2=d_1G\le d_2G\le d_1G+r=2$.

The entries of Table 3 have been taken from [9]. □

More recently, Boston, Bush and Hajir [10] used MAGMA [15] for computing the class groups of the 481 756 real quadratic fields F having 3-class rank $\varrho_3=2$ and discriminants in the range $0<d<10^9$, and the class groups of the 1 927 024 associated *totally real dihedral fields* E_i of degree six, arising from unramified cyclic cubic extensions E_i/F [18, Prp. 4.1, p. 482]. 415 698 of these quadratic fields F have a 3-class group $\mathrm{Cl}_3F \simeq C_3\times C_3$ (415 699 according to [13, Tbl. 7]).

Real quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with positive discriminants $d>0$ are the second simplest number fields with respect to their unit group U_F, which is an infinite group of torsionfree Dirichlet unit rank one. Again, there are remarkable consequences for their p-tower groups, by the Shafarevich theorem [23, Thm. 5.1, p. 28].

Theorem 4.4. *Among the finite* 3*-groups G with elementary bicyclic abelianization* $G/G' \simeq C_3\times C_3$ *of rank two, there exist infinitely many metabelian groups with GI-action and relation rank $d_2G=3$ (so-called Schur+1 σ-groups [10]), but only three of minimal order 3^4, namely* $\langle 81,7\rangle$, $\langle 81,8\rangle$ *and* $\langle 81,10\rangle$.

(1) *These are the groups of smallest order which are admissible as* 3*-class tower groups* $G \simeq \mathrm{G}_3^\infty F$ *of real quadratic fields F with 3-class group* $\mathrm{Cl}_3F\simeq C_3\times C_3$.
(2) *Generally, for any number field F, these groups are determined uniquely by the second order Artin pattern.*
  (a) *If* $\mathrm{AP}^{(2)}F = ([1^2;(1^3,1^2,1^2,1^2)],[1;(2000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 81,7\rangle$.
  (b) *If* $\mathrm{AP}^{(2)}F = ([1^2;(21,1^2,1^2,1^2)],[1;(2000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 81,8\rangle$.
  (c) *If* $\mathrm{AP}^{(2)}F = ([1^2;(21,1^2,1^2,1^2)],[1;(1000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 81,10\rangle$.
(3) *The actual distribution of these* 3*-class tower groups G among the* 415 698 *real quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with 3-class group* $\mathrm{Cl}_3F\simeq C_3\times C_3$ *and discriminants* $0<d<10^9$ *is presented in Table 4. Additionally, the frequencies of the groups* $\langle 243,5\rangle$ *and* $\langle 243,7\rangle$ *in Theorem 4.3 are given.*

TABLE 4. Frequencies of metabelian 3-class tower groups G for $0 < d < 10^9$

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	rel. fr.	w. r. t.	measure [10]	$d_{\min}$
$\langle 81, 7\rangle$	122 955	29.58%	415 698	25.52%	481 756	$1664/6561 \approx 25.36\%$	142 097
$\langle 81, 8\rangle$ or $\langle 81, 10\rangle$	208 236	50.09%	415 698	43.22%	481 756	$8320/19683 \approx 42.27\%$	32 009
$\langle 243, 5\rangle$	13 712	3.30%	415 698	2.85%	481 756	$1664/59049 \approx 2.82\%$	422 573
$\langle 243, 7\rangle$	6 691	1.61%	415 698	1.39%	481 756	$832/59049 \approx 1.41\%$	631 769

Proof. A search for metabelian vertices G of minimal order with relation rank $d_2G = 3$ in the descendant tree $\mathcal{T}(R)$ with abelian root $R = \langle 9, 2\rangle \simeq C_3 \times C_3$ yields three hits $\langle 27, 7\rangle$, $\langle 27, 8\rangle$, and $\langle 27, 10\rangle$. All of them possess a GI-action.

The abelianization G/G' of a finite 3-group G which is realized as the 3-class tower group $\mathrm{G}_p^\infty F$ of an algebraic number field F is isomorphic to the 3-class group Cl_3F of F. When F is real quadratic, it possesses signature $(r_1, r_2) = (2, 0)$ and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 1$. If $G/G' \simeq \mathrm{Cl}_3F \simeq C_3 \times C_3$, then the generator rank of G is $d_1G = 2$ and the Shafarevich theorem implies bounds for the relation rank $2 = d_1G \le d_2G \le d_1G + r = 3$.

The entries of Table 4 have been taken from [10]. □

In [10], Boston, Bush and Hajir only computed the first component of the second order Artin pattern $\mathrm{AP}^{(2)}F = (\tau^{(2)}F, \varkappa^{(2)}F)$ in Formula (4.2), that is, the abelian type invariants $\tau^{(2)}F$ of second order of real quadratic fields F with discriminants $0 < d < 10^9$. Determining the second component $\varkappa^{(2)}F$, the transfer kernel type of F, is considerably harder with respect to the computational expense. Consequently, the most extensive numerical results on transfer kernels available currently, have been computed by ourselves for the smaller ranges $0 < d < 10^8$ in [25, 26], and, even computing third order Artin patterns, for $0 < d < 10^7$ in [30, 31]. With the aid of these results, we now illustrate that the transfer kernels $\ker(T_{F,E})$ of 3-class extensions $T_{F,E} : \mathrm{Cl}_3F \to \mathrm{Cl}_3E$ from real quadratic fields F to unramified cyclic cubic extensions E/F are capable of narrowing down the number of contestants for the 3-tower group $\mathrm{G}_3^\infty F$ significantly, and thus of refining the statistics in [10].

TABLE 5. Frequencies of metabelian 3-class tower groups G for $0 < d < 10^8$ resp. 10^7

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	$d_{\min}$
$\langle 81, 7\rangle$	10 244	29.58%	34 631	142 097
$\langle 81, 8\rangle$	10 514	30.36%	34 631	32 009
$\langle 81, 10\rangle$	7 104	20.51%	34 631	72 329
$\langle 729, 96\rangle$	242	0.70%	34 631	790 085
$\langle 729, 97\rangle$ or $\langle 729, 98\rangle$	713	2.06%	34 631	494 236
$\langle 729, 99\rangle$	66	2.56%	2 576	62 501
$\langle 729, 100\rangle$	42	1.63%	2 576	152 949
$\langle 729, 101\rangle$	42	1.63%	2 576	252 977

Corollary 4.2. (1) *If* $\mathrm{AP}^{(2)}F = ([1^2; (32, 1^2, 1^2, 1^2)], [1; (1000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 729, 96\rangle$.
(2) *If* $\mathrm{AP}^{(2)}F = ([1^2; (32, 1^2, 1^2, 1^2)], [1; (2000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 729, i\rangle$ *with* $i \in \{97, 98\}$.
(3) *If* $\mathrm{AP}^{(2)}F = ([1^2; (2^2, 1^2, 1^2, 1^2)], [1; (0000)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 729, i\rangle$ *with* $i \in \{99, 100, 101\}$.
The actual distribution of these 3*-class tower groups* G *among the* 34 631*, respectively* 2 576*, real quadratic fields* $F = \mathbb{Q}(\sqrt{d})$ *with* 3*-class group* $\mathrm{Cl}_3F \simeq C_3 \times C_3$ *and discriminants* $0 < d < 10^8$*, respectively* $0 < d < 10^7$*, is presented in Table 5.*

4.5. **Non-metabelian three-stage towers with $\ell_p F = 3$.** According to the Successive Approximation Theorem 3.2, the third stage $F_p^{(3)}$ of the p-class tower of a number field F is usually determined by the third order Artin pattern

$$\mathrm{AP}^{(3)}F = (\tau^{(3)}F, \varkappa^{(3)}F) = ([\tau^{(1)}F; (\tau^{(2)}E)_{E\in\mathrm{Lyr}_1 F}], [\varkappa^{(1)}F; (\varkappa^{(2)}E)_{E\in\mathrm{Lyr}_1 F}]). \tag{4.3}$$

It is interesting, however, that there are extensive collections of quadratic fields F with 3-class towers of exact length $\ell_3 F = 3$, which can be characterized by the second order Artin pattern already. We begin with imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with discriminants $d < 0$.

Theorem 4.5. *Among the finite 3-groups G with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$ of rank two, there exist infinitely many non-metabelian groups with GI-action and relation rank $d_2 G = 2$ (so-called Schur σ-groups [14, 9]), but only seven of minimal order 3^8, namely $\langle 6561, i\rangle$ with $i \in \{606, 616, 617, 618, 620, 622, 624\}$.*

1. *These are the groups of smallest order which are admissible as non-metabelian 3-class tower groups $G \simeq \mathrm{G}_3^\infty F$ of imaginary quadratic fields F with 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$.*
2. *Exceptionally, for an imaginary quadratic field F, the trailing six of these groups are determined by the second order Artin pattern already.*
   (a) *If $\mathrm{AP}^{(2)}F = ([1^2; (32, 21, 1^3, 21)], [1; (1313)])$ then $\mathrm{G}_3^\infty F \simeq \langle 6561, 616\rangle$.*
   (b) *If $\mathrm{AP}^{(2)}F = ([1^2; (32, 21, 1^3, 21)], [1; (2313)])$ then $\mathrm{G}_3^\infty F \simeq \langle 6561, i\rangle$ with $i \in \{617, 618\}$.*
   (c) *If $\mathrm{AP}^{(2)}F = ([1^2; (32, 21, 21, 21)], [1; (1231)])$ then $\mathrm{G}_3^\infty F \simeq \langle 6561, 622\rangle$.*
   (d) *If $\mathrm{AP}^{(2)}F = ([1^2; (32, 21, 21, 21)], [1; (2231)])$ then $\mathrm{G}_3^\infty F \simeq \langle 6561, i\rangle$ with $i \in \{620, 624\}$.*
3. *The actual distribution of these 3-class tower groups G among the 24 476 imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$ and discriminants $-10^7 < d < 0$ is presented in Table 6.*

Table 6. Frequencies of non-metabelian 3-class tower groups G for $-10^7 < d < 0$

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	type	$\varkappa$	$\lvert d\rvert_{\min}$
$\langle 6561, 616\rangle$	760	3.11%	24 476	E.6	(1313)	15 544
$\langle 6561, 617\rangle$ or $\langle 6561, 618\rangle$	1572	6.42%	24 476	E.14	(2313)	16 627
$\langle 6561, 622\rangle$	798	3.26%	24 476	E.8	(1231)	34 867
$\langle 6561, 620\rangle$ or $\langle 6561, 624\rangle$	1583	6.47%	24 476	E.9	(2231)	9 748

Proof. By a similar but more extensive search than in the proof of Theorem 4.3. Data for Table 6 has been computed by ourselves in June 2016 using MAGMA [15]. □

Remark 4.2. It should be pointed out that items (1) and (2) of Theorem 4.5 are *not valid for real quadratic fields*, as documented in [29, Thm. 7.8, p. 162, and Thm. 7.12, p. 165].

The group $\langle 6561, 606\rangle$ belongs to the infinite Shafarevich cover of the metabelian group $\langle 729, 45\rangle$ with respect to imaginary quadratic fields [20, Cor. 6.2, p. 301], [21]. It shares a common second order Artin pattern with all other elements of the Shafarevich cover. Third order Artin patterns must be used for its identification, as shown in [29, Thm. 7.14, p. 168].

Now we turn to real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with discriminants $d > 0$.

Theorem 4.6. *Among the finite 3-groups G with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$ of rank two, there exist infinitely many non-metabelian groups with GI-action and relation rank $d_2 G = 3$ (so-called Schur+1 σ-groups [10]), but only nine of minimal order 3^7, namely $\langle 2187, i\rangle$ with $i \in \{270, 271, 272, 273, 284, 291, 307, 308, 311\}$.*

(1) *These are the groups of smallest order which are admissible as non-metabelian* 3*-class tower groups* $G \simeq \mathrm{G}_3^\infty F$ *of real quadratic fields* F *with* 3*-class group* $\mathrm{Cl}_3 F \simeq C_3 \times C_3$*.*
(2) *Exceptionally, for a real quadratic field* F*, four of these groups are determined by the second order Artin pattern already.*
   (a) *If* $\mathrm{AP}^{(2)}F = ([1^2;(2^2,21,1^3,21)],[1;(0313)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 2187, i\rangle$ *with* $i \in \{284, 291\}$*.*
   (b) *If* $\mathrm{AP}^{(2)}F = ([1^2;(2^2,21,21,21)],[1;(0231)])$ *then* $\mathrm{G}_3^\infty F \simeq \langle 2187, i\rangle$ *with* $i \in \{307, 308\}$*.*
(3) *The actual distribution of these* 3*-class tower groups* G *among the* 415 698 *real quadratic fields* $F = \mathbb{Q}(\sqrt{d})$ *with* 3*-class group* $\mathrm{Cl}_3 F \simeq C_3 \times C_3$ *and discriminants* $1 < d < 10^9$ *is presented in Table 7.*

TABLE 7. Frequencies of non-metabelian 3-class tower groups G for $0 < d < 10^9$

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	type	$\varkappa$	$d_{\min}$
$\langle 2187, 284\rangle$ or $\langle 2187, 291\rangle$	4318	1.04%	415 698	c.18	(0313)	534 824
$\langle 2187, 307\rangle$ or $\langle 2187, 308\rangle$	4377	1.05%	415 698	c.21	(0231)	540 365

Proof. The claims for transfer kernel type c.18, $\varkappa(F) \sim (0313)$, are a consequence of [23, Prp. 7.1, p. 32, Thm. 7.1, p. 33, and Rmk. 7.1, p. 35], those for type c.21, $\varkappa(F) \sim (0231)$, have been proved in [23, Prp. 8.1, p. 42, Thm. 8.1, p. 44, and Rmk. 8.2, p. 45]. A slightly stronger result is the Main Theorem [23, Thm. 2.1, p. 22]. □

Remark 4.3. The groups $\langle 2187, i\rangle$ with $i \in \{270, 271, 272, 273\}$ are elements of the infinite Shafarevich cover of the metabelian group $\langle 729, 45\rangle$ with respect to real quadratic fields.

The group $\langle 2187, 311\rangle$ belongs to the infinite Shafarevich cover of the metabelian group $\langle 729, 57\rangle$ with respect to real quadratic fields.

These five groups share a common second order Artin pattern with all other elements of the relevant Shafarevich cover. Third order Artin patterns must be employed for their identification, as shown in [29, Thm. 7.13, p. 167, and Thm. 7.15, p. 169].

5. MAXIMAL SUBGROUPS OF 3-GROUPS OF COCLASS ONE

Let $(\gamma_i(G))_{i\geq 1}$ be the descending lower central series of the group G, defined recursively by $\gamma_1(G) := G$ and $\gamma_i(G) := [\gamma_{i-1}(G), G]$ for $i \geq 2$, in particular, $\gamma_2(G) = G'$ is the commutator subgroup of G. A finite p-group G is nilpotent with $\gamma_1(G) > \gamma_2(G) > \ldots > \gamma_c(G) > \gamma_{c+1}(G) = 1$ for some integer $c \geq 1$, which is called the *nilpotency class* $\mathrm{cl}(G) = c$ of G. When G is of order p^n, for some integer $n \geq 1$, the *coclass* of G is defined by $\mathrm{cc}(G) := n - c$ and $\mathrm{lo}(G) := n$ is called the *logarithmic order* of G.

Finite 3-groups G with coclass $\mathrm{cc}(G) = 1$ were investigated by N. Blackburn [6] in 1958. All of these CF-groups, which exclusively have *cyclic factors* $\gamma_i(G)/\gamma_{i+1}(G)$ of their descending central series for $i \geq 2$, are necessarily metabelian with second derived subgroup $G'' = 1$ and abelian commutator subgroup G' and possess abelianization $G/G' \simeq C_3 \times C_3$, according to Blackburn [5].

For the statement of Theorem 5.1, we need a precise ordering of the four maximal subgroups $H_1, \ldots, H_4$ of the group $G = \langle x, y\rangle$, which can be generated by two elements x, y, according to the Burnside basis theorem. For this purpose, we select the generators x, y such that

$$H_1 = \langle y, G'\rangle, \quad H_2 = \langle x, G'\rangle, \quad H_3 = \langle xy, G'\rangle, \quad H_4 = \langle xy^2, G'\rangle, \tag{5.1}$$

and $H_1 = \chi_2(G)$, provided that G is of nilpotency class $\mathrm{cl}(G) \geq 3$. Here we denote by

$$\chi_2(G) := \{g \in G \mid (\forall\ h \in \gamma_2(G))\ [g, h] \in \gamma_4(G)\} \tag{5.2}$$

the *two-step centralizer* of G' in G.

5.1. **Parametrized presentations of metabelian 3-groups.** The identification of the groups will be achieved with the aid of parametrized polycyclic power-commutator presentations, as given by Blackburn [6], Miech [32], and Nebelung [34]:

$$\begin{aligned}(5.3)\quad & G^n_a(z,w) := \langle x,y,s_2,\ldots,s_{n-1} \mid s_2=[y,x],\ (\forall_{i=3}^{n})\ s_i=[s_{i-1},x],\ s_n=1,\ [y,s_2]=s_{n-1}^a,\\ & (\forall_{i=3}^{n-1})\ [y,s_i]=1,\ x^3=s_{n-1}^w,\ y^3s_2^3s_3=s_{n-1}^z,\ (\forall_{i=2}^{n-3})\ s_i^3s_{i+1}^3s_{i+2}=1,\ s_{n-2}^3=s_{n-1}^3=1\ \rangle,\end{aligned}$$

where $a\in\{0,1\}$ and $w,z\in\{-1,0,1\}$ are bounded parameters, and the *index of nilpotency* $n=\mathrm{cl}(G)+1=\mathrm{cl}(G)+\mathrm{cc}(G)=\log_3(\mathrm{ord}(G))=:\mathrm{lo}(G)$ is an unbounded parameter.

The following lemma generalizes relations for second and third powers of generators in [30, Lem. 3.1], [31].

Lemma 5.1. *Let $G=\langle x,y\rangle$ be a finite 3-group with two generators $x,y\in G$. Denote by $s_2:=[y,x]$ the main commutator, and by $s_3:=[s_2,x]$ and $t_3:=[s_2,y]$ the two iterated commutators. Then the second and third power of the element xy, respectively xy^2, are given by*

$$\begin{aligned}(5.4)\quad & (xy)^2=x^2y^2s_2t_3 \quad \text{and } (xy)^3=x^3y^3s_2^3s_3t_3^2, \text{ respectively}\\ & (xy^2)^2=x^2y^4s_2^2t_3^2 \quad \text{and } (xy^2)^3=x^3y^6s_2^6s_3^2t_3^2,\end{aligned}$$

provided that $t_3\in\zeta(G)$ is central, $t_3^3=1$, and $[s_3,y]=[s_3,s_2]=1$.

Proof. We begin by preparing three commutator relations:

$$(5.5)\qquad yx=xy[y,x]=xys_2,\quad s_2x=xs_2[s_2,x]=xs_2s_3,\quad \text{and}\quad s_2y=ys_2[s_2,y]=ys_2t_3.$$

Now we prove the power relations by expanding the power expressions by iterated substitution of the commutator relations in Formula (5.5), always observing that t_3 belongs to the centre, $t_3^3=1$, and $s_3y=ys_3$ commute:

$$(xy)^2=xyxy=xxys_2y=x^2yys_2t_3=x^2y^2s_2t_3, \text{ and thus}$$

$$\begin{aligned}(xy)^3 &= (xy)^2xy=x^2y^2s_2t_3xy=x^2y^2s_2xyt_3=x^2yyxs_2s_3yt_3=x^2yxys_2s_2ys_3t_3=\\ &= x^2xys_2ys_2ys_2t_3s_3t_3=x^3yys_2t_3ys_2t_3s_2s_3t_3^2=x^3y^2s_2ys_2s_2s_3t_3^4=\\ &= x^3y^2ys_2t_3s_2^2s_3t_3=x^3y^3s_2^3s_3t_3^2, \text{ respectively}\end{aligned}$$

$$\begin{aligned}(xy^2)^2 &= xyyxyy=xyxys_2yy=xxys_2yys_2t_3y=x^2yys_2t_3ys_2yt_3=x^2y^2s_2yys_2t_3t_3^2=\\ &= x^2y^2ys_2t_3ys_2t_3^3=x^2y^3s_2ys_2t_3=x^2y^3ys_2t_3s_2t_3=x^2y^4s_2^2t_3^2, \text{ and thus}\end{aligned}$$

$$\begin{aligned}(xy^2)^3 &= (xy^2)^2xy^2=x^2y^4s_2^2t_3^2xy^2=x^2y^4s_2s_2xyyt_3^2=x^2y^4s_2xs_2s_3yyt_3^2=\\ &= x^2yyyyxs_2s_3s_2yys_3t_3^2=x^2yyyxys_2s_2ys_2t_3ys_3^2t_3^2=x^2yyxys_2ys_2ys_2t_3s_2ys_3^2t_3^3=\\ &= x^2yxys_2yys_2t_3ys_2t_3s_2ys_2t_3s_3^2t_3^4=x^2xys_2yys_2t_3ys_2ys_2t_3^2ys_2t_3s_2s_3^2t_3^2=\\ &= x^3yys_2t_3ys_2yys_2t_3s_2t_3^3ys_2^2s_3^2t_3^3=x^3y^2s_2ys_2yys_2t_3^2s_2ys_2^2s_3^2=\\ &= x^3y^2ys_2t_3ys_2t_3ys_2t_3^2ys_2t_3s_2^2s_3^2=x^3y^3s_2ys_2t_3^2ys_2yt_3^3s_2^3s_3^2=x^3y^3ys_2t_3s_2t_3^2yys_2t_3s_2^3s_3^2=\\ &= x^3y^4s_2s_2yys_2t_3^4s_2^3s_3^2=x^3y^4s_2s_2yys_2^4s_3^2t_3=x^3y^4s_2ys_2t_3ys_2^4s_3^2t_3=x^3y^4s_2ys_2ys_2^4s_3^2t_3^2=\\ &= x^3y^4ys_2t_3ys_2t_3s_2^4s_3^2t_3^2=x^3y^5s_2ys_2t_3^2s_2^4s_3^2t_3^2=x^3y^5ys_2t_3s_2^5s_3^2t_3^4=x^3y^6s_2^6s_3^2t_3^2.\end{aligned}$$

□

Theorem 5.1. *Let $G=\langle x,y\rangle\simeq G^n_a(z,w)$ be a finite 3-group of coclass $\mathrm{cc}(G)=1$ and order $|G|=3^n$ with generators x,y such that $y\in\chi_2(G)$ is contained in the two-step centralizer of G, whereas $x\in G\setminus\chi_2(G)$, given by a polycyclic power commutator presentation with parameters $a\in\{0,1\}$, $w,z\in\{-1,0,1\}$, and index of nilpotency $n\geq 4$.*

Then three of the four maximal subgroups, $H_i=\langle xy^{i-2},G'\rangle<G$, $2\leq i\leq 4$, are non-abelian 3-groups of coclass $\mathrm{cc}(H_i)=1$, as listed in Table 8 in dependence on the parameters n,a,z,w.

The supplementary Table 9 shows the abelian maximal subgroups of the remaining two extra special 3-group of coclass $\mathrm{cc}(G)=1$ and order $|G|=3^3$.

TABLE 8. Non-abelian maximal subgroups $H_i < G$ of 3-groups G of coclass 1

$G \simeq$	n	a	z	w	$H_2 = \langle x, G' \rangle$	$H_3 = \langle xy, G' \rangle$	$H_4 = \langle xy^2, G' \rangle$
$G_0^n(0,0)$	≥ 4	0	0	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$
$G_0^n(0,1)$	≥ 4	0	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_0^n(1,0)$	≥ 4	0	1	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_0^n(-1,0)$	≥ 4	0	-1	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_1^n(0,-1)$	≥ 5	1	0	-1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$
$G_1^n(0,0)$	≥ 5	1	0	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_1^n(0,1)$	≥ 5	1	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$

TABLE 9. Abelian maximal subgroups $H_i < G$ of extra special 3-groups G

$G \simeq$	n	a	z	w	$H_1 = \langle y, G' \rangle$	$H_2 = \langle x, G' \rangle$	$H_3 = \langle xy, G' \rangle$	$H_4 = \langle xy^2, G' \rangle$
$G_0^3(0,0)$	3	0	0	0	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$
$G_0^3(0,1)$	3	0	0	1	$\simeq C_3 \times C_3$	$\simeq C_9$	$\simeq C_9$	$\simeq C_9$

Proof. For an index of nilpotency $n \geq 4$, the first maximal subgroup $H_1 = \langle y, G' \rangle$ of G coincides with the two-step centralizer $\chi_2(G)$ of G, which is a *nearly homocyclic abelian* 3-group $A(3, n-1)$ of order 3^{n-1}, when $a = 0$. For $a = 1$, we have $H_1/H_1' \simeq A(3, n-1)$.

We transform all relations of the group $G \simeq G_a^n(z, w)$ into relations of the remaining three maximal subgroups $H \simeq G_\alpha^{n-1}(\zeta, \omega)$ of G.

The *polycyclic commutator relations* $s_2 = [y, x]$, $s_i = [s_{i-1}, x]$ for $3 \leq i \leq n$, and the *nilpotency relation* $s_n = 1$ for the group $G = \langle x, y \rangle$, with lower central series $\gamma_i G = \langle s_i, \gamma_{i+1} G \rangle$ for $i \geq 2$, can be used immediately for the subgroup $H_2 = \langle x, G' \rangle = \langle x, s_2 \rangle$ with lower central series $\gamma_i H_2 = \langle t_i, \gamma_{i+1} H_2 \rangle$, where $t_i := s_{i+1}$ for $i \geq 2$, and $t_{n-1} = 1$.

For the lower central series of $H_3 = \langle xy, G' \rangle$ and $H_4 = \langle xy^2, G' \rangle$, we must employ the *main commutator relation* $[y, s_2] = s_{n-1}^a$, and $[y, s_i] = 1$ for $i \geq 3$. According to the *right product rule* for commutators, we have $[s_{i-1}, xy] = [s_{i-1}, y] \cdot [s_{i-1}, x]^y = 1 \cdot s_i^y = s_i [s_i, y] = s_i \cdot 1 = s_i$, for $i \geq 4$, but $[s_2, xy] = [s_2, y] \cdot [s_2, x]^y = s_{n-1}^{-a} s_3^y = s_{n-1}^{-a} s_3 [s_3, y] = s_{n-1}^{-a} s_3$, and in a similar fashion $[s_{i-1}, xy^2] = [s_{i-1}, y] \cdot [s_{i-1}, xy]^y = 1 \cdot s_i^y = s_i [s_i, y] = s_i \cdot 1 = s_i$, for $i \geq 4$, but again exceptionally $[s_2, xy^2] = [s_2, y] \cdot [s_2, xy]^y = s_{n-1}^{-a} y^{-1} s_{n-1}^{-a} s_3 y = s_{n-1}^{-2a} s_3 = s_{n-1}^a s_3$. For $a = 1$, the *left product rule* for commutators shows $[s_{n-1}^{\mp 1} s_3, xy^{\pm 1}] = [s_{n-1}^{\mp 1}, xy^{\pm 1}]^{s_3} \cdot [s_3, xy^{\pm 1}] = s_4$, that is, the slight anomaly for the main commutator disappears in the next step. Thus, the lower central series is $\gamma_i H_j = \langle t_i, \gamma_{i+1} H_j \rangle$ for $i \geq 2$, $3 \leq j \leq 4$, where generally $t_i := s_{i+1}$ for $i \geq 3$, and $t_2 := s_3$ for $a = 0$, $t_2 := s_{n-1}^{2-j} s_3$ for $a = 1$. In particular, $H_3 = \langle xy, s_2 \rangle$ and $H_4 = \langle xy^2, s_2 \rangle$.

The main commutator relation for all three subgroups H_2, H_3, H_4 of any group $G \simeq G_a^n(z, w)$ with $n \geq 4$ is $[s_2, t_2] = 1 = t_{n-2}^\alpha$, that is $\alpha = 0$, generally, and it remains to determine ζ, ω.

For this purpose, we come to the *power relations* of G, $x^3 = s_{n-1}^w$, $y^3 s_2^3 s_3 = s_{n-1}^z$, and $s_i^3 s_{i+1}^3 s_{i+2} = 1$ for $i \geq 2$, supplemented by (5.4): $(xy)^3 = x^3 y^3 s_2^3 s_3 s_{n-1}^{-2a} = s_{n-1}^w s_{n-1}^z s_{n-1}^{-2a}$ and $(xy^2)^3 = x^3 (y^3 s_2^3 s_3)^2 s_{n-1}^{-2a} = s_{n-1}^w s_{n-1}^{2z} s_{n-1}^{-2a}$, and we use these relations to determine ζ, ω in dependence on w, z, a. Generally, we have $s_2^3 t_2^3 t_3 = s_2^3 s_3^3 s_4 = 1$ for $a = 0$, $s_2^3 t_2^3 t_3 = s_2^3 s_{n-1}^{3(2-j)} s_3^3 s_4 = s_2^3 s_3^3 s_4 = 1$ for $a = 1$, and thus uniformly $\zeta = 0$.

For $G_0^n(0,0)$, we uniformly have $x^3 = (xy)^3 = (xy^2)^3 = 1$, and thus $\omega = 0$ for all three subgroups. For $G_0^n(0,1)$, we uniformly have $x^3 = (xy)^3 = (xy^2)^3 = s_{n-1}$, and thus $\omega = 1$ for all three subgroups. For $G_0^n(\pm 1, 0)$, we have $x^3 = 1$, but $(xy)^3 = s_{n-1}^{\pm 1}$, $(xy^2)^3 = s_{n-1}^{\pm 2} = s_{n-1}^{\mp 1}$, and thus $\omega = 0$ for H_2 but $\omega = 1$ for H_3, H_4, since $G_0^n(0,-1) \simeq G_0^n(0,1)$.

For $G_1^n(0,-1)$, we have $x^3 = s_{n-1}^{-1}$, but $(xy)^3 = (xy^2)^3 = s_{n-1}^{-3} = 1$, and thus $\omega = 1$ for H_2 but $\omega = 0$ for H_3, H_4. For $G_1^n(0,0)$, we have $x^3 = 1$, but $(xy)^3 = (xy^2)^3 = s_{n-1}^{-2} = s_{n-1}$, and thus

$\omega = 0$ for H_2 but $\omega = 1$ for H_3, H_4. For $G_1^n(0,1)$, we have $x^3 = s_{n-1}$, $(xy)^3 = (xy^2)^3 = s_{n-1}^{-1}$, and thus $\omega = 1$ for all three subgroups, again observing that $G_0^n(0,-1) \simeq G_0^n(0,1)$.

The only 3-groups G of coclass $\mathrm{cc}(G) = 1$ and order $|G| = 3^3$ are the two extra special groups $G_0^3(0,0)$ and $G_0^3(0,1)$. Since $t_2 = s_3 = 1$, all their four maximal subgroups, $H_1 = \langle y, s_2\rangle$, $H_2 = \langle x, s_2\rangle$, $H_3 = \langle xy, s_2\rangle$, $H_4 = \langle xy^2, s_2\rangle$, are abelian. For $w = z = 0$, s_2 is independent of the other generator, and $H_i \simeq C_3 \times C_3$ for $1 \le i \le 4$. However, for $w = 1$, $z = 0$, we have $x^3 = (xy)^3 = (xy^2)^3 = s_2$, $s_2^3 = 1$, and thus $H_2 \simeq H_3 \simeq H_4 \simeq C_9$, whereas $H_1 \simeq C_3 \times C_3$. □

6. A general theorem for arbitrary base fields

Suppose that p is a prime, F is an algebraic number field with non-trivial p-class group $\mathrm{Cl}_pF > 1$, and E is one of the unramified abelian p-extensions of F. We show that, even in this general situation, a finite p-class tower of F exerts a very severe restriction on the p-class tower of E.

Theorem 6.1. *Assume that F possesses a p-class tower $F_p^{(\infty)} = F_p^{(n)}$ of exact length $\ell_pF = n$ for some integer $n \ge 1$. Then the Galois group $\mathrm{Gal}(E_p^{(\infty)}/E)$ of the p-class tower of E is a subgroup of index $[E : F]$ of the p-class tower group $\mathrm{Gal}(F_p^{(\infty)}/F)$ of F and the length of the p-class tower of E is bounded by $\ell_pE \le n$.*

Proof. According to the assumptions, there exists a tower of field extensions,

$$F < E \le F_p^{(1)} \le E_p^{(1)} \le F_p^{(2)} \le E_p^{(2)} \le \ldots \le F_p^{(n)} \le E_p^{(n)} \le F_p^{(n+1)},$$

where $\ell_pF = n$ enforces the coincidence $F_p^{(n)} = E_p^{(n)} = F_p^{(n+1)}$ of the trailing three fields. Since $\mathrm{Gal}(F_p^{(n)}/F)/\mathrm{Gal}(F_p^{(n)}/E) \simeq \mathrm{Gal}(E/F)$, the group index of $\mathrm{Gal}(E_p^{(n)}/E) = \mathrm{Gal}(F_p^{(n)}/E)$ in $\mathrm{Gal}(F_p^{(n)}/F)$ is equal to the field degree $[E : F]$ and $\mathrm{Gal}(E_p^{(\infty)}/E) = \mathrm{Gal}(E_p^{(n)}/E)$ is a subgroup of index $[E : F]$ of $\mathrm{Gal}(F_p^{(n)}/F) = \mathrm{Gal}(F_p^{(\infty)}/F)$. The equality $E_p^{(n)} = E_p^{(n+1)}$ implies the bound $\ell_pE \le n$. □

We shall apply Theorem 6.1 to the situation where $p = 3$, $n = 2$, and E is an unramified cyclic cubic extension of F, whence $\mathrm{Gal}(E_3^{(\infty)}/E)$ is a maximal subgroup of $\mathrm{Gal}(F_3^{(\infty)}/F)$.

6.1. **Application to quadratic base fields.**

Proposition 6.1. *Let G be a finite 3-group with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1. *The transfer kernel type of G is* D.10, $\varkappa(G) \sim (2241)$.
2. *The abelian quotient invariants of the four maximal subgroups $H_1, \ldots, H_4$ of G are $\tau(G) \sim (21, 21, 1^3, 21)$.*
3. *The isomorphism types of the four maximal subgroups of G are $H_1 \simeq H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$ and $H_3 \simeq \langle 3^4, 13\rangle$.*
4. *The group G is isomorphic to the Schur σ-group $\langle 3^5, 5\rangle$ with relation rank $d_2 = 2$.*

Proof. We put $G := \langle 243, 5\rangle$ and use the presentation [15]

$$G = \langle x, y, s_2, s_3, t_3 \mid s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y], x^3 = s_3, y^3 = s_3\rangle.$$

Then we obtain the maximal subgroups
$H_1 = \langle y, G'\rangle = \langle y, s_2, s_3\rangle$, since $t_3 = [s_2, y]$,
$H_2 = \langle x, G'\rangle = \langle x, s_2, t_3\rangle$, since $s_3 = [s_2, x]$,
$H_3 = \langle xy, G'\rangle = \langle xy, s_2, s_3\rangle$, since $[s_2, xy] = s_3t_3$,
$H_4 = \langle xy^2, G'\rangle = \langle xy^2, s_2, s_3\rangle$, since $[s_2, xy^2] = s_3t_3^2$.
Using Lemma 5.1, and comparing to the abstract presentations [15]
$\langle 81, 3\rangle = \langle \xi, \upsilon, \sigma_2, \tau \mid \sigma_2 = [\upsilon, \xi], \tau = \xi^3\rangle$ and
$\langle 81, 13\rangle = \langle \xi, \upsilon, \zeta, \sigma_2 \mid \sigma_2 = [\upsilon, \xi], \xi^3 = \sigma_2, \upsilon^3 = \zeta^3 = 1\rangle$,
we conclude
$H_1 = \langle y, s_2, s_3\rangle = \langle y, s_2\rangle \simeq \langle 81, 3\rangle$, since $y^3 = s_3 \ne [s_2, y] = t_3$,
$H_2 = \langle x, s_2, t_3\rangle \simeq \langle 81, 13\rangle$, since $x^3 = s_3 = [s_2, x]$,

$H_3 = \langle xy, s_2, s_3\rangle = \langle xy, s_2\rangle \simeq \langle 81, 3\rangle$, since $(xy)^3 = t_3^2 \neq [s_2, xy] = s_3 t_3$,
$H_4 = \langle xy^2, s_2, s_3\rangle = \langle xy^2, s_2\rangle \simeq \langle 81, 3\rangle$, since $(xy^2)^3 = s_3^2 t_3^2 \neq [s_2, xy^2] = s_3 t_3^2$. □

Theorem 6.2. *Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field with elementary bicyclic 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1. *The transfer kernel type of F is* D.10, $\varkappa(F) \sim (2241)$.
2. *The abelian type invariants of the 3-class groups $\mathrm{Cl}_3 E_i$ of the four unramified cyclic cubic extensions E_i/F are $\tau(F) \sim (21, 21, 1^3, 21)$.*
3. *The second 3-class group $\mathrm{G}_3^2 F$ of F has the maximal subgroups $H_1 \simeq H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$ and $H_3 \simeq \langle 3^4, 13\rangle$.*
4. *The 3-class tower group $\mathrm{G}_3^\infty F$ of F is the Schur σ-group $\langle 3^5, 5\rangle$ with relation rank $d_2 = 2$.*

Proof. The claims follow from Proposition 6.1 by applying the Successive Approximation Theorem 3.2 of first order. □

Corollary 6.1. *Let F be a quadratic field which satisfies one of the equivalent conditions in Theorem 6.2. Then the length of the 3-class tower of F is $\ell_3 F = 2$. The four unramified cyclic cubic extensions E_i/F are absolutely dihedral of degree 6, with torsionfree Dirichlet unit rank $r \geq 2$, and possess 3-class towers of length $\ell_3 E_i = 2$. More precisely, $\mathrm{Cl}_3 E_3 \simeq C_3 \times C_3 \times C_3$ and $\mathrm{G}_3^\infty E_3 \simeq \langle 3^4, 13\rangle$ with relation rank $d_2 = 5$, but $\mathrm{Cl}_3 E_i \simeq C_9 \times C_3$ and $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 3\rangle$ with relation rank $d_2 = 4$ for $i \in \{1, 2, 4\}$.*

Proof. This is a consequence of Theorems 6.1 and 6.2, satisfying the Shafarevich theorem. □

Proposition 6.2. *Let G be a finite 3-group with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1. *The transfer kernel type of G is* D.5, $\varkappa(G) \sim (4224)$.
2. *The abelian quotient invariants of the four maximal subgroups $H_1, \ldots, H_4$ of G are $\tau(G) \sim (1^3, 21, 1^3, 21)$.*
3. *The isomorphism types of the four maximal subgroups of G are $H_1 \simeq H_3 \simeq \langle 3^4, 13\rangle$ and $H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$.*
4. *The group G is isomorphic to the Schur σ-group $\langle 3^5, 7\rangle$ with relation rank $d_2 = 2$.*

Proof. We put $G := \langle 243, 7\rangle$ and use the presentation [15]

$$G = \langle x, y, s_2, s_3, t_3 \mid s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y], x^3 = s_3, y^3 = s_3^2\rangle.$$

Similarly as in Proposition 6.1, we obtain the maximal subgroups
$H_1 = \langle y, G'\rangle = \langle y, s_2, s_3\rangle$, $H_2 = \langle x, G'\rangle = \langle x, s_2, t_3\rangle$,
$H_3 = \langle xy, G'\rangle = \langle xy, s_2, s_3\rangle$, and $H_4 = \langle xy^2, G'\rangle = \langle xy^2, s_2, s_3\rangle$.
Using Lemma 5.1, and comparing to the abstract presentations
$\langle 81, 3\rangle = \langle \xi, \upsilon, \sigma_2, \tau \mid \sigma_2 = [\upsilon, \xi], \tau = \xi^3\rangle$ and
$\langle 81, 13\rangle = \langle \xi, \upsilon, \zeta, \sigma_2 \mid \sigma_2 = [\upsilon, \xi], \xi^3 = \sigma_2, \upsilon^3 = \zeta^3 = 1\rangle$,
we conclude
$H_1 = \langle y, s_2, s_3\rangle = \langle y, s_2\rangle \simeq \langle 81, 3\rangle$, since $y^3 = s_3^2 \neq [s_2, y] = t_3$,
$H_2 = \langle x, s_2, t_3\rangle \simeq \langle 81, 13\rangle$, since $x^3 = s_3 = [s_2, x]$,
$H_3 = \langle xy, s_2, s_3\rangle = \langle xy, s_2\rangle \simeq \langle 81, 3\rangle$, since $(xy)^3 = s_3 t_3^2 \neq [s_2, xy] = s_3 t_3$,
$H_4 = \langle xy^2, s_2, s_3\rangle \simeq \langle 81, 13\rangle$, since $(xy^2)^3 = s_3 t_3^2 = [s_2, xy^2]$. □

Theorem 6.3. *Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field with elementary bicyclic 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1. *The transfer kernel type of F is* D.5, $\varkappa(F) \sim (4224)$.
2. *The abelian type invariants of the 3-class groups $\mathrm{Cl}_3 E_i$ of the four unramified cyclic cubic extensions E_i/F are $\tau(F) \sim (1^3, 21, 1^3, 21)$.*
3. *The second 3-class group $\mathrm{G}_3^2 F$ of F has the maximal subgroups $H_1 \simeq H_3 \simeq \langle 3^4, 13\rangle$ and $H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$.*
4. *The 3-class tower group $\mathrm{G}_3^\infty F$ of F is the Schur σ-group $\langle 3^5, 7\rangle$ with relation rank $d_2 = 2$.*

Proof. The claims follow from Proposition 6.2 by applying the Successive Approximation Theorem 3.2 of first order. □

Corollary 6.2. *Let F be a quadratic field which satisfies one of the equivalent conditions in Theorem 6.3. Then the length of the 3-class tower of F is $\ell_3 F = 2$. The four unramified cyclic cubic extensions E_i/F are absolutely dihedral of degree 6, with torsionfree Dirichlet unit rank $r \ge 2$, and possess 3-class towers of length $\ell_3 E_i = 2$. More precisely, $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3 \times C_3$ and $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 13\rangle$ with relation rank $d_2 = 5$ for $i \in \{1,3\}$, but $\mathrm{Cl}_3 E_i \simeq C_9 \times C_3$ and $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 3\rangle$ with relation rank $d_2 = 4$ for $i \in \{2,4\}$.*

Proof. This is a consequence of Theorems 6.1 and 6.3, satisfying the Shafarevich theorem. □

6.2. **Application to dihedral fields.** We recall that a dihedral field E of degree 6 is an absolute Galois extension $E/\mathbb{Q}$ with group $\mathrm{Gal}(E/\mathbb{Q}) = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1}\rangle$. It is a cyclic cubic relative extension E/F of its unique quadratic subfield $F = E^\sigma$, and it contains three isomorphic, conjugate non-Galois cubic subfields $L = E^\tau$, L^σ, L^{σ^2}. The conductor c of E/F is a nearly squarefree positive integer with special prime factors, and the discriminants satisfy the relations $d_E = c^4 d_F^3$ and $d_L = c^2 d_F$. Here, we shall always be concerned with unramified extensions, characterized by the conductor $c = 1$, and thus $d_E = d_F^3$, a perfect cube, and equal $d_L = d_F$.

6.2.1. *Totally complex dihedral fields.* The computational information on 3-tower groups $G := \mathrm{G}_3^\infty F$ of imaginary quadratic fields F in Table 3 admits the purely theoretical deduction of impressive statistics for 3-tower groups $S := \mathrm{G}_3^\infty E$ of totally complex dihedral fields E in Table 10 by means of the Corollaries 6.1 and 6.2. We use the crucial new insight that the groups $S \lhd G$ are maximal subgroups of G, because the extensions E/F are unramified cyclic of degree 3.

TABLE 10. Frequencies of dihedral 3-class tower groups S for $-10^{24} < d_E < 0$

$G \simeq$	$\tau^{(1)}G$	abs. fr.	$S \simeq$	$\tau^{(1)}S$	abs. fr.	$\lvert d_E\rvert_{\min}$
$\langle 243, 5\rangle$	1^2	83 353	$\langle 81, 3\rangle$	21	250 059	$4\,027^3$
$\langle 243, 5\rangle$	1^2	83 353	$\langle 81, 13\rangle$	1^3	83 353	$4\,027^3$
$\langle 243, 7\rangle$	1^2	41 398	$\langle 81, 3\rangle$	21	82 796	$12\,131^3$
$\langle 243, 7\rangle$	1^2	41 398	$\langle 81, 13\rangle$	1^3	82 796	$12\,131^3$

6.2.2. *Totally real dihedral fields.* The computational information on 3-tower groups $G := \mathrm{G}_3^\infty F$ of real quadratic fields F in Table 4 admits the purely theoretical deduction of impressive statistics for 3-tower groups $S := \mathrm{G}_3^\infty E$ of totally real dihedral fields E in Table 11 by means of Theorem 5.1 and Theorem 6.1. Again, we use the innovative result that the groups $S \lhd G$ are maximal subgroups of G, since the extensions E/F are unramified cyclic cubic.

TABLE 11. Frequencies of dihedral 3-class tower groups S for $0 < d_E < 10^{27}$

$G \simeq$	$\tau^{(1)}G$	abs. fr.	$S \simeq$	$\tau^{(1)}S$	abs. fr.	$(d_E)_{\min}$
$\langle 81, 7\rangle$	1^2	122 955	$\langle 27, 3\rangle$	1^2	122 955	$142\,097^3$
$\langle 81, 7\rangle$	1^2	122 955	$\langle 27, 4\rangle$	1^2	245 910	$142\,097^3$
$\langle 81, 7\rangle$	1^2	122 955	$\langle 27, 5\rangle$	1^3	122 955	$142\,097^3$

The first row of Table 11 reveals extensive realizations of the extraspecial group $S = \langle 27, 3\rangle$ as 3-tower group of dihedral fields. This is the first time that $S = \langle 27, 3\rangle$ occurs as a 3-tower group. It is forbidden for quadratic fields, and it did not occur for cyclic cubic fields and bicyclic biquadratic fields, up to now.

Theorem 6.4. ***(A new realization as 3-tower group.)*** *The extraspecial* 3*-group* $S = \langle 27, 3\rangle$ *of coclass* 1 *and exponent* 3 *occurs as* 3*-class tower group* $\mathrm{G}_3^\infty E$ *of totally real dihedral fields* E *of degree* 6.

Proof. The group $S = \langle 27, 3\rangle$ possesses the relation rank $d_2 S = 4$. According to the Shafarevich Theorem, it is therefore excluded as 3-tower group $\mathrm{G}_3^\infty F$ of both, imaginary and real quadratic fields F. However, the combination of Theorem 5.1 and Theorem 6.1 proves its occurrence as 3-class tower group $\mathrm{G}_3^\infty E$ of totally real dihedral fields E of degree 6, as visualized in Table 11. □

Theorem 6.5. ***(3-class tower groups of totally real dihedral fields.)*** *Let* $F = \mathbb{Q}(\sqrt{d})$ *be a real quadratic field with* 3*-class group* $\mathrm{Cl}_3 F \simeq C_3 \times C_3$ *and fundamental discriminant* $d > 1$. *Suppose the second order Artin pattern* $\mathrm{AP}^{(2)} F = (\tau^{(2)}(F), \varkappa^{(2)}(F))$ *is given by the abelian type invariants* $\tau^{(2)}(F) = [1^2; (2^2, 1^2, 1^2, 1^2)]$ *and the transfer kernel type* $\varkappa^{(2)}(F) = [1; (0000)]$. *Let* E_2, E_3, E_4 *be the three unramified cyclic cubic relative extensions of* F *with* 3*-class group* $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3$.

Then $E_i/\mathbb{Q}$ *is a totally real dihedral extension of degree* 6*, for each* $2 \le i \le 4$*, and the connection between the component* $\#\varkappa^{(3)}(F)_i = \#\ker(T_{E_i, F_3^{(1)}})$ *of the third order transfer kernel type* $\varkappa^{(3)}(F)$ *and the* 3*-class tower group* $S_i = G_3^\infty E_i = \mathrm{Gal}((E_i)_3^{(\infty)}/E_i)$ *of* E_i *is given in the following way:*

$$\begin{aligned} \#\varkappa^{(3)}(F)_i = 3 \quad &\Longleftrightarrow \quad S_i \simeq \langle 243, 27\rangle \quad \text{with } \varkappa(S_i) = (1000), \\ \#\varkappa^{(3)}(F)_i = 9 \quad &\Longleftrightarrow \quad S_i \simeq \langle 243, 26\rangle \quad \text{with } \varkappa(S_i) = (0000). \end{aligned} \tag{6.1}$$

Proof. This theorem was expressed as a conjecture in [30, 31], and is now an immediate consequence of Theorems 5.1 and 6.1. □

Remark 6.1. Recall that each unramified cyclic cubic relative extension E_i/F, $1 \le i \le 4$, gives rise to a dihedral absolute extension $E_i/\mathbb{Q}$ of degree 6, that is an S_3-extension [18, Prp. 4.1, p. 482]. For the trailing three fields E_i, $2 \le i \le 4$, in the stable part of $\tau^{(2)}(F) = [1^2; (2^2, 1^2, 1^2, 1^2)]$, i.e. with $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3$, we have constructed the unramified cyclic cubic extensions $\tilde{E}_{i,j}/E_i$, $1 \le j \le 4$, and determined the Artin pattern $\mathrm{AP}^{(2)} E_i$ of E_i, in particular, the transfer kernel type of E_i in the fields $\tilde{E}_{i,j}$ of absolute degree 18. The dihedral fields E_i of degree 6 share a common polarization $\tilde{E}_{i,1} = F_3^{(1)}$, the Hilbert 3-class field of F, which is contained in the relative 3-genus field $(E_i/F)^*$, whereas the other extensions $\tilde{E}_{i,j}$ with $2 \le j \le 4$ are non-abelian over F, for each $2 \le i \le 4$. Our computational results underpin Theorem 6.5 concerning the infinite family of totally real dihedral fields E_i for varying real quadratic fields F.

7. Conclusion

Guided by the Successive Approximation Theorem 3.2 in terms of the Artin limit pattern, we have given a most up-to-date survey concerning the finite 3-groups which are populated most densely by 3-class tower groups $\mathrm{G}_3^\infty F$ of quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ in sections 4.2 – 4.5. In particular, the discovery of non-metabelian 3-class towers with exact length $\ell_3 F = 3$, which is currently the maximal proven finite length, in Theorems 4.5 and 4.6, is entirely due to our cooperation with M. R. Bush, initiated by our joint paper [11]. With Theorems 5.1 and 6.1, we have finally presented a new technique for deriving theoretical conclusions on 3-class towers of dihedral fields with degree six from corresponding results for quadratic fields.

8. Acknowledgements

The author gratefully acknowledges that his research was supported by the Austrian Science Fund (FWF): P 26008-N25. Indebtedness is expressed for valuable suggestions by the referees.

References

[1] E. Artin, *Beweis des allgemeinen Reziprozitätsgesetzes*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 353–363, DOI 10.1007/BF02952531.

[2] E. Artin, *Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 46–51, DOI 10.1007/BF02941159.

[3] H. U. Besche, B. Eick, and E. A. O'Brien, *A millennium project: constructing small groups*, Int. J. Algebra Comput. **12** (2002), 623-644.

[4] H. U. Besche, B. Eick, and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.

[5] N. Blackburn, *On prime-power groups in which the derived group has two generators*, Proc. Camb. Phil. Soc. **53** (1957), 19–27.

[6] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.

[7] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[8] W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), *Handbook of Magma functions* (Edition 2.23, Sydney, 2017).

[9] N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of imaginary quadratic fields*, Math. Ann. **368** (2017), no. 1, 633–669, DOI 10.1007/s00208-016-1449-3.

[10] N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of real quadratic fields*, preprint, 2017.

[11] M. R. Bush and D. C. Mayer, *3-class field towers of exact length* 3, J. Number Theory **147** (2015), 766–777, DOI 10.1016/j.jnt.2014.08.010.

[12] H. Cohen and J. Martinet, *Class groups of number fields: numerical heuristics*, Math. Comp. **48** 1987, no. 177, 123–137.

[13] M. J. Jacobson, Jr., *Experimental results on class groups of real quadratic fields*, In: J. P. Buhler (ed.), Algorithmic Number Theory, ANTS 1998, Lecture Notes in Computer Science, vol. **1423**, Springer, Berlin, Heidelberg.

[14] H. Koch und B. B. Venkov, *Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers*, Astérisque **24–25** (1975), 57–67.

[15] MAGMA Developer Group, *MAGMA Computational Algebra System*, Version 2.23-4, Sydney, 2017, (`http://magma.maths.usyd.edu.au`).

[16] D. C. Mayer, *Principalization in complex S_3-fields*, Congressus Numerantium **80** (1991), 73–87. (Proceedings of the Twentieth Manitoba Conference on Numerical Mathematics and Computing, Univ. of Manitoba, Winnipeg, Canada, 1990.)

[17] D. C. Mayer, *Multiplicities of dihedral discriminants*, Math. Comp. **58** (1992), no. 198, 831–847, supplements section S55–S58, DOI 10.2307/2153221.

[18] D. C. Mayer, *The second p-class group of a number field*, Int. J. Number Theory **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.

[19] D. C. Mayer, *Principalization algorithm via class group structure*, J. Théor. Nombres Bordeaux **26** (2014), no. 2, 415–464, DOI 10.5802/jtnb.874.

[20] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015) no. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015.

[21] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, 29ièmes Journées Arithmétiques, Univ. of Debrecen, Hungary, presentation delivered on July 09, 2015.

[22] D. C. Mayer, *New number fields with known p-class tower*, 22nd Czech and Slovak International Conference on Number Theory 2015, Liptovský Ján, Slovakia, presentation delivered on August 31, 2015, `http://www.algebra.at/22CSICNT2015.pdf`.

[23] D. C. Mayer, *New number fields with known p-class tower*, Tatra Mt. Math. Pub. **64** (2015), 21–57, DOI 10.1515/tmmp-2015-0040, Special Issue on Number Theory and Cryptology '15.

[24] D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66–104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.

[25] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Math. Phys. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135.

[26] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, 2nd International Conference on Groups and Algebras (ICGA) 2016, Suzhou, China, invited lecture delivered on July 26, 2016, `http://www.algebra.at/ICGA2016Suzhou.pdf`.

[27] D. C. Mayer, *Recent progress in determining p-class field towers*, Gulf J. Math. **4** (2016), no. 4, 74–102, ISSN 2309-4966.

[28] D. C. Mayer, *Recent progress in determining p-class field towers*, 1st International Colloquium of Algebra, Number Theory, Cryptography and Information Security (ANCI) 2016, Faculté Polydisciplinaire de Taza, Université Sidi Mohamed Ben Abdellah, Fès, Morocco, invited keynote delivered on November 12, 2016, `http://www.algebra.at/ANCI2016DCM.pdf`.

[29] D. C. Mayer, *Criteria for three-stage towers of p-class fields*, Adv. Pure Math. **7** (2017), no. 2, 135 – 179, DOI 10.4236/apm.2017.72008, Special Issue on Number Theory, February 2017.

[30] D. C. Mayer, *Deep transfers of p-class tower groups*, to appear in J. Appl. Math. Phys.

[31] D. C. Mayer, *Deep transfers of p-class tower groups*, 3nd International Conference on Groups and Algebras (ICGA) 2016, Sanya, China, invited lecture due on January 14, 2018.

[32] R. J. Miech, *Metabelian p-groups of maximal class*, Trans. Amer. Math. Soc. **152** (1970), 331–373.

[33] A. S. Mosunov and M. J. Jacobson, Jr., *Unconditional class group tabulation of imaginary quadratic fields to* $|\Delta| < 2^{40}$, Math. Comp. **85** (2016), no. 300, 1983–2009.

[34] B. Nebelung, *Klassifikation metabelscher* 3*-Gruppen mit Faktorkommutatorgruppe vom Typ* $(3, 3)$ *und Anwendung auf das Kapitulationsproblem*, Inauguraldissertation, Universität zu Köln, 1989.

[35] PARI Developer Group, PARI/GP *Computer Algebra System*, Version 2.9.2, Bordeaux, 2017, (http://pari.math.u-bordeaux.fr).

[36] I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

Naglergasse 53, 8010 Graz, Austria
E-mail address: algebraic.number.theory@algebra.at
URL: http://www.algebra.at