PATTERN RECOGNITION VIA ARTIN TRANSFERS

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Abstract. The positions of higher p-class groups $\mathrm{G}_p^n(K)$, $2 \le n \le \infty$, on coclass graphs $\mathcal{G}(p,r)$ of finite p-groups of coclass $r \ge 1$, or at projective limits of their mainlines, are determined for various kinds of number fields K with p-class groups $\mathrm{Cl}_p(K)$ of types (p,p), (p^2,p) and (p,p,p), in the case of $p \in \{2,3,5,7\}$, by means of kernels and targets of Artin transfers.

1. Introduction

Guided by the principle *Per Aspera Ad Astra*, we emphasize the arduous long and winding road to a systematic strategy of pattern recognition via Artin transfers for identifying higher class groups of number fields.

1.1. Per Aspera: increasing experience in the last decade.

Let p be a prime number. For an algebraic number field K, we denote by $\mathrm{F}_p^2(K)$ the second Hilbert p-class field of K, that is the maximal metabelian unramified extension of K with a power of p as its relative degree $[\mathrm{F}_p^2(K) : K]$. The Galois group $\mathrm{G}_p^2(K) = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ is called the *second p-class group of K*.

In a series of four articles [43, 44, 45, 46], we have developed general strategies for identifying the metabelian p-group $\mathrm{G}_p^2(K)$ of a given number field K with the aid of information about abelian unramified extensions $L|K$ of p-power degree, i. e., intermediate fields $K \le L \le \mathrm{F}_p^1(K)$ between K and its first Hilbert p-class field.

In [43] it was proved for number fields K with p-class group $\mathrm{Cl}_p(K) = \mathrm{Syl}_p\mathrm{Cl}(K)$ of type (p,p) that the p-class numbers $\mathrm{h}_p(L)$ of at most three abelian unramified p-extensions $L|K$ determine the order $|G|$, class $\mathrm{cl}(G)$ and coclass $\mathrm{cc}(G)$ of $G = \mathrm{G}_p^2(K)$, generally in the cases $p \in \{2,3\}$, and for $p \ge 5$ when G is of coclass $\mathrm{cc}(G) = 1$.

In [44] we extended the information about unramified cyclic extensions $L|K$ of degree p by the kernels $\ker(j_{L|K})$ of the class extension homomorphisms $j_{L|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L)$ induced by the natural embedding of ideals of K into L. These kernels describe the *p-principalization* or *p-capitulation* of $L|K$. A complete description of the corresponding *transfer kernel types* (TKTs) $\varkappa(K) = \varkappa(G) = (\ker(T_{G,M_i}))_{1 \le i \le p+1}$ of the Artin transfers $T_{G,M_i} : G/G' \to M_i/M_i'$ [2] from the second p-class group $G = \mathrm{G}_p^2(K)$ into its maximal subgroups $M_i \lhd G$, $1 \le i \le p+1$, was presented in [44] for $\mathrm{Cl}_p(K)$ of type (p,p) and $p \in \{2,3\}$.

In [45] we completely settled the problem of determining the structure of the p-class groups $\mathrm{Cl}_p(L_i)$ of all unramified cyclic extensions $L_i|K$, $1 \le i \le p+1$, of degree p of a number field K with $\mathrm{Cl}_p(K)$ of type (p,p) if $p = 3$. It turned out that the corresponding *transfer target type* (TTT) $\tau(K) = \tau(G) = (M_i/M_i')_{1 \le i \le p+1}$ of the second p-class group $G = \mathrm{G}_p^2(K)$, consisting of the abelian quotient invariants of its maximal subgroups $M_i \lhd G$, uniquely characterizes the TKT $\varkappa(G)$ for those finite metabelian p-groups G with G/G' of type (p,p) which are populated most densely by second p-class group $\mathrm{G}_p^2(K)$ of number fields. This fact enabled the design of a new *principalization algorithm via class group structure.*

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1.2. **Ad Astra: the crystallized principle of pattern recognition.** Finally, we combined and generalized all techniques in [43, 44, 45] and arrived at the following universal strategy for identifying the second p-class group $\mathrm{G}_p^2(K)$ of a given number field K in [46, 47].

(1) We compute all abelian unramified p-extensions $K \le L \le \mathrm{F}_p^1(K)$ of K with the aid of Fieker's technique [30], which is implemented in the MAGMA computer algebra system [16, 17, 42], and we determine the structure of their p-class groups, $(\mathrm{Cl}_p(L))_{K\le L\le \mathrm{F}_p^1(K)} = \tau(K)$, and their p-capitulation kernels $\left(\ker(j_{L|K})\right)_{K\le L\le \mathrm{F}_p^1(K)} = \varkappa(K)$.

(2) Due to the Artin reciprocity law [1] and further investigations by Artin and Miyake [2, 50], we can identify $\tau(K)$ with $\tau(G) = (U/U')_{G'\lhd U\lhd G}$ and $\varkappa(K)$ with $\varkappa(G) = (\ker(T_{G,U}))_{G'\lhd U\lhd G}$ for the second p-class group $G = \mathrm{G}_p^2(K)$ of K and its normal subgroups $G' \lhd U \lhd G$ containing the commutator subgroup G'.

(3) We use the p-group generation algorithm by O'Brien [54] to construct all descendants H of the abelian p-group $\mathrm{Cl}_p(K)$ having abelianization $H/H' \simeq \mathrm{Cl}_p(K)$ and we prune the descendant tree with respect to $\tau(H)$ and $\varkappa(H)$ compatible with $\tau(K)$ and $\varkappa(K)$, in the sense that $\tau(H) \le \tau(K)$ and $\varkappa(H) \ge \varkappa(K)$ component-wise. Finally, we can stop this *process of pattern recognition* whenever $\tau(G) = \tau(K)$ and $\varkappa(G) = \varkappa(K)$ for some vertex G of the descendant tree but $\tau(H) > \tau(K)$ or $\varkappa(H) < \varkappa(K)$ for all immediate descendants H of G.

It is the purpose of the present article and of several forthcoming papers to apply the described strategy of pattern recognition among TTTs and TKTs to extensive series of number fields K, each series having a fixed type of p-class group $\mathrm{Cl}_p(K)$ in $\{(3,3),(9,3),(3,3,3),(4,2),(2,2,2)\}$.

Quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with $\mathrm{Cl}_p(K)$ of type (p,p) have been investigated in detail for $p \in \{2,3\}$ in [43, 45] and for $p \in \{5,7\}$ in [46].

Complex bicyclic biquadratic fields $K = \mathbb{Q}(\sqrt{-3},\sqrt{D})$ with $\mathrm{Cl}_3(K)$ of type $(3,3)$ will be treated in [3] and quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with $\mathrm{Cl}_3(K)$ of type $(9,3)$ will be studied in [48].

Results for cyclic cubic fields K with $\mathrm{Cl}_3(K)$ of type $(3,3)$, $(3,3,3)$ or $(9,3)$ are in preparation.

By investigating complex and totally real S_3-fields K with $\mathrm{Cl}_3(K)$ of type $(3,3,3)$ or $(9,3)$, we plan to shed light on the exact length of the 3-class tower of a special series of quadratic fields for which this problem was raised by Bartholdi and Bush [12] and is still completely open up to now.

Here we are going to focus on quadratic fields $K = \mathbb{Q}(\sqrt{D})$ and complex bicyclic biquadratic fields $K = \mathbb{Q}(\sqrt{-1},\sqrt{D})$ having $\mathrm{Cl}_2(K)$ of type $(2,2,2)$ or $(4,2)$.

We briefly describe the layout of this article.

In § 2, we begin with the most general notion of Artin transfers from a group to a (not necessarily normal) subgroup of finite index, which will enable us to consider TTTs and TKTs *of second order* as a sophisticated tool for coping with particularly hard boiled cases, where invariants of first order are unable to make a final decision.

The central ideas which enable a pattern recognition via Artin transfers are presented in § 3, where we show that every descendant tree of finite p-groups admits a partial order of TTTs and TKTs, provided that the concept of a parent is defined by means of the last non-trivial term of the usual lower central series.

In § 4, the inheritance of TTTs and TKTs from parents is used to prove remarkable stabilization criteria for p-groups located on important coclass graphs. This new approach sheds light on properties of well-known p-groups from another perspective.

Finally, we devote § 5 to an extensive application of the preceding theory to the problem of finding 3-class field towers of exact length three.

2. Artin transfers

For the further continuation of realizing our intention to identify second p-class groups $G = \mathrm{G}_p^2(K)$ of number fields K with p-class groups $\mathrm{Cl}_p(K)$ of type (p,p,p) or (p^2,p) in the present article, our definition of the transfer given in [44, eqn. (4), p. 470] is not sufficient any longer, since it is restricted to transfers from a p-group G with abelianization G/G' of type (p,p) to its maximal normal subgroups M of index $(G:M)=p$. Therefore we begin with a more sophisticated theory of Artin transfers from groups to arbitrary (not necessarily normal) subgroups of any finite index.

Suppose G is a group and $H \le G$ is a subgroup of finite index $n = (G:H)$. Let $\{g_1,\ldots,g_n\}$ be a *left transversal*, that is a system of representatives for the left cosets, of H in G, such that $G = \dot\cup_{i=1}^n g_iH$ is a disjoint union.

Then we can construct a mapping $\pi:\ G \to S_n,\ x \mapsto \pi_x$ in the following way. Given a fixed element $x \in G$ and an integer $1 \le j \le n$, there exists a unique integer $1 \le \ell = \pi_x(j) \le n$ such that $xg_j \in g_\ell H$, resp. $g_\ell^{-1}xg_j \in H$. The mapping π_x is a permutation of the symmetric group S_n acting on the set $\{1,\ldots,n\}$, since the relation $xg_k \in g_\ell H$ for another integer $1 \le k \le n$, that is $\pi_x(k) = \ell = \pi_x(j)$, implies $xg_j = g_\ell u$ and $xg_k = g_\ell v$ with $u, v \in H$ and thus

$$g_j^{-1}g_k = g_j^{-1}x^{-1}xg_k = (xg_j)^{-1}xg_k = (g_\ell u)^{-1}g_\ell v = u^{-1}g_\ell^{-1}g_\ell v = u^{-1}v \in H.$$

Consequently, we obtain $g_kH = g_jH$ and therefore $k = j$, that is, π is bijective.

Definition 2.1. The *Artin transfer* $T_{G,H}$ from G to H is defined as the mapping

$$T_{G,H}:\ G \to H/H',\ x \mapsto \prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_iH'.$$

Remark 2.1. In a slightly more general setting, we can replace the commutator subgroup H' by any normal subgroup $H' \lhd N \lhd H$, that is, by a normal subgroup N for which the quotient H/N is abelian.

Huppert [37, 1.4. Hauptsatz, pp. 413–414] proves the following theorem by means of considerations involving the *wreath product*. However, we prefer to give an immediate proof.

Theorem 2.1. *Let G be a group and $H \le G$ be a subgroup of finite index.*

1. *The mapping $\pi:\ G \to S_n$ is a homomorphism, called permutation representation of G.*
2. *The Artin transfer $T_{G,H}:\ G \to H/H'$ is a homomorphism.*

Proof. For $x, y \in G$ with $T_{G,H}(x) = \prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_iH'$ and $T_{G,H}(y) = \prod_{j=1}^n g_{\pi_y(j)}^{-1}yg_jH'$, we have

$$T_{G,H}(x)\cdot T_{G,H}(y) = \left(\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_iH'\right)\cdot\left(\prod_{j=1}^n g_{\pi_y(j)}^{-1}yg_jH'\right).$$

Since H/H' is abelian and π_y is a permutation of $\{1,\ldots,n\}$, we can change the order of the factors in the first product by putting $i = \pi_y(j)$:

$$T_{G,H}(x)\cdot T_{G,H}(y) = \left(\prod_{j=1}^n g_{\pi_x(\pi_y(j))}^{-1}xg_{\pi_y(j)}H'\right)\cdot\left(\prod_{j=1}^n g_{\pi_y(j)}^{-1}yg_jH'\right).$$

Using the commutativity of H/H' once again, we can merge the two products into one product:

$$T_{G,H}(x)\cdot T_{G,H}(y) = \prod_{j=1}^n g_{\pi_x(\pi_y(j))}^{-1}xg_{\pi_y(j)}g_{\pi_y(j)}^{-1}yg_jH' = \prod_{j=1}^n g_{(\pi_x\circ\pi_y)(j)}^{-1}xyg_jH'.$$

This relation simultaneously shows that $\pi_x \circ \pi_y = \pi_{xy}$ and $T_{G,H}(x)\cdot T_{G,H}(y) = T_{G,H}(xy)$. □

3. TTT and TKT of parents and descendants

For a fixed prime number p, the set of all isomorphism classes of finite p-groups can be endowed with various kinds of structure by defining a *parent* $\pi(G)$ of each group G. Then the set becomes a *directed graph* (digraph) with finite p-groups as vertices and with a directed edge $G \to \pi(G)$ from each group to its parent. Three kinds of parents have been suggested by M. F. Newman in [51, § 2, p. 52].

The first implicitly goes back to P. Hall's article [34], where the concept of isoclinism of groups was introduced, and defines $\pi(G) = G/\zeta_1(G)$ as the central quotient of G, where the upper (or ascending) central series

$$1 = \zeta_0(G) < \zeta_1(G) = \mathrm{Centre}(G) < \ldots < \zeta_c(G) = G$$

of G is given recursively by $\zeta_0(G) = 1$ and $\zeta_j(G)/\zeta_{j-1}(G) = \mathrm{Centre}(G/\zeta_{j-1}(G))$ for $j \geq 1$. Here and in the following, c denotes the nilpotency class $\mathrm{cl}(G)$ of G.

The second kind is due to a conference presentation by C. R. Leedham-Green in 1974 and puts $\pi(G) = G/\gamma_c(G)$ the last non-trivial lower central quotient of G, where the lower (or decending) central series

$$G = \gamma_1(G) < \gamma_2(G) = G' = [G, G] < \ldots < \gamma_c(G) < \gamma_{c+1}(G) = 1$$

of G is defined inductively by $\gamma_1(G) = G$ and $\gamma_j(G) = [G, \gamma_{j-1}(G)]$ for $j \geq 2$.

For computational and theoretical purposes, a third kind of parent definition turned out to be adequate [51, p. 53]. It puts $\pi(G) = G/P_{\ell-1}(G)$ the last non-trivial lower exponent-p central quotient of G, where the lower exponent-p central series

$$G = P_0(G) < P_1(G) = \Phi(G) = G' \cdot G^p < \ldots < P_{\ell-1}(G) < P_\ell(G) = 1$$

of G is given inductively by $P_0(G) = G$ and $P_j = [G, P_{j-1}(G)] \cdot P_{j-1}(G)^p$ for $j \geq 1$, and ℓ denotes the p-class $\mathrm{cl}_p(G)$ of G.

The common feature shared by all these kinds of parent definitions is that the parent $\pi(G)$, and more generally any ancestor (iterated parent), of a finite p-group G is a quotient G/N of G by some normal subgroup $N \lhd G$. Thus, an equivalent definition can be given by selecting an epimorphism φ from G onto a p-group $\tilde{G}$ whose kernel $\ker(\varphi)$ plays the role of the normal subgroup N. We are going to use this point of view in the following general investigation of connections between the TKTs and TTTs of ancestors and descendants.

Proposition 3.1. *If $\varphi : G \to A$ is a homomorphism from a group G to an abelian group A, then there exists a unique homomorphism $\tilde{\varphi} : G/G' \to A$ such that $\varphi = \tilde{\varphi} \circ \omega$, where $\omega : G \to G/G'$ denotes the canonical projection.*

Proof. Firstly, the uniqueness of $\tilde{\varphi}$ is a consequence of the condition $\varphi = \tilde{\varphi} \circ \omega$, which implies that $\tilde{\varphi}$ must be defined by $\tilde{\varphi}(xG') = \tilde{\varphi}(\omega(x)) = (\tilde{\varphi} \circ \omega)(x) = \varphi(x)$, for any $x \in G$. Secondly, the relation $\tilde{\varphi}(xG'yG') = \tilde{\varphi}(xyG') = \varphi(xy) = \varphi(x)\varphi(y) = \tilde{\varphi}(xG')\tilde{\varphi}(yG')$, for any $x, y \in G$, shows that $\tilde{\varphi}$ is a homomorphism, provided it is a well-defined map. Finally, it remains to show that the definition of $\tilde{\varphi}$ is independent of the coset representatives modulo G'. For the commutators of elements $x, y \in G$, we have $\varphi([x, y]) = [\varphi(x), \varphi(y)] = 1$, since A is abelian. Thus, the commutator subgroup G' is contained in the kernel $\ker(\varphi)$, and for any $x, y \in G$, we obtain $xG' = yG' \implies x^{-1}y \in G' \leq \ker(\varphi) \implies \tilde{\varphi}(xG')^{-1}\tilde{\varphi}(yG') = \tilde{\varphi}(x^{-1}yG') = \varphi(x^{-1}y) = 1 \implies \tilde{\varphi}(xG') = \tilde{\varphi}(yG')$, that is, $\tilde{\varphi}$ is indeed well-defined. □

Our aim is now to investigate relations between the targets and kernels of Artin transfers of groups and their epimorphic images. We begin with transfer targets with respect to a single subgroup, that is with singulets of TTTs.

Theorem 3.1. *(TTT singulets)*
Suppose $\tilde{G}$ and G are groups, $\tilde{G} = \varphi(G)$ is the image of G under an epimorphism $\varphi : G \to \tilde{G}$, and $\tilde{H} = \varphi(H)$ is the image of a subgroup $H \leq G$.

1. *If H is abelian, then $\tilde{H}$ is also abelian.*
2. *The commutator subgroup of $\tilde{H}$, $\tilde{H}' = \varphi(H')$, is image of the commutator subgroup of H.*

(3) *If* $\ker(\varphi) \le H$ *then* $\tilde{H} \simeq H/\ker(\varphi)$, φ *induces a unique epimorphism* $\tilde{\varphi}: H/H' \to \tilde{H}/\tilde{H}'$, *and* $\tilde{H}/\tilde{H}'$ *is epimorphic image of* H/H'*, that is, a quotient of* H/H'.

(4) *If* $\ker(\varphi) \le H'$ *then* $\tilde{H}' \simeq H'/\ker(\varphi)$, $\tilde{\varphi}$ *is an isomorphism, and* $\tilde{H}/\tilde{H}' \simeq H/H'$.

The various mappings which appear in Theorem 3.1 and its proof are illustrated by the diagram in Table 1. By ω we denote the canonical projection from a group to its abelianization.

TABLE 1. Epimorphisms and commutator quotients

$$
\begin{array}{ccccc}
 & & \omega_{\tilde{H}} & & \\
 & \tilde{H} & \longrightarrow & \tilde{H}/\tilde{H}' & \\
\varphi|_H & \uparrow & \backslash\backslash\backslash & \uparrow & \tilde{\varphi} \\
 & H & \longrightarrow & H/H' & \\
 & & \omega_H & &
\end{array}
$$

Proof. (1) If $xy = yx$ for all $x, y \in H$, then we also generally have $\varphi(x)\varphi(y) = \varphi(xy) = \varphi(yx) = \varphi(y)\varphi(x)$ in $\varphi(H)$.

(2) $\varphi(H') = \varphi([H,H]) = \varphi(\langle [h_1,h_2] \mid h_1,h_2 \in H\rangle)$ $= \langle [\varphi(h_1),\varphi(h_2)] \mid h_1,h_2 \in H\rangle = [\varphi(H),\varphi(H)] = \varphi(H)' = \tilde{H}'$.

(3) If $\ker(\varphi) \le H$ then φ can be restricted to an epimorphism $\varphi|_H: H \to \tilde{H}$ such that $\tilde{H} = \varphi(H) \simeq H/\ker(\varphi)$. According to Proposition 3.1, the composite epimorphism $\omega_{\tilde{H}} \circ \varphi|_H: H \to \tilde{H}/\tilde{H}'$ from H onto the abelian group $\tilde{H}/\tilde{H}'$ factors through H/H' by means of a uniquely determined epimorphism $\tilde{\varphi}: H/H' \to \tilde{H}/\tilde{H}'$ such that $\omega_{\tilde{H}} \circ \varphi|_H = \tilde{\varphi} \circ \omega_H$.

(4) According to item (2), the commutator subgroup of $\tilde{H} = \varphi(H)$ is given by $\tilde{H}' = \varphi(H')$. If $\ker(\varphi) \le H'$ then $\tilde{H}' = \varphi(H') \simeq H'/\ker(\varphi)$ and $\tilde{\varphi}$ is an isomorphism, since $\tilde{\varphi}(xH') = (\tilde{\varphi} \circ \omega_H)(x) = (\omega_{\tilde{H}} \circ \varphi|_H)(x) = \varphi(x)\tilde{H}' = \tilde{H}'$ implies $\varphi(x) \hat{=} x\ker(\varphi) \in \tilde{H}' \simeq H'/\ker(\varphi)$ and thus $x \in H'$.

□

Definition 3.1. We define a partial order on the set of abelian quotient invariants by putting $\tilde{H}/\tilde{H}' \le H/H'$ when $\tilde{H}/\tilde{H}'$ is epimorphic image of H/H', and by putting $\tilde{H}/\tilde{H}' = H/H'$ when $\tilde{H}/\tilde{H}' \simeq H/H'$.

Next, we study transfer kernels with respect to a single subgroup, that is, singulets of TKTs.

Theorem 3.2. *(TKT singulets)*
Suppose $\tilde{G}$ *and* G *are groups,* $\tilde{G} = \varphi(G)$ *is the image of* G *under an epimorphism* $\varphi: G \to \tilde{G}$, *and* $\tilde{H} = \varphi(H)$ *is the image of a subgroup* $H \le G$ *of finite index* $n = (G:H)$.

(1) *If* $H \lhd G$ *is a normal subgroup of* G *then* $\tilde{H} \lhd \tilde{G}$ *is also a normal subgroup of* $\tilde{G}$.

(2) *If* $\ker(\varphi) \le H$ *then*
   (a) *the image* $(\varphi(g_1),\ldots,\varphi(g_n))$ *of a transversal* $(g_1,\ldots,g_n)$ *for* H *in* G *is a transversal for* $\tilde{H}$ *in* $\tilde{G}$*, and thus* $\tilde{H}$ *is of finite index* $n = (\tilde{G}:\tilde{H})$ *in* $\tilde{G}$,
   (b) $\tilde{\varphi}(T_{G,H}(x)) = T_{\tilde{G},\tilde{H}}(\varphi(x))$ *for all* $x \in G$,
   (c) *in particular,* $\varphi(\ker(T_{G,H})) \le \ker(T_{\tilde{G},\tilde{H}})$.

(3) *If* $\ker(\varphi) \le H'$ *then* $\varphi(\ker(T_{G,H})) = \ker(T_{\tilde{G},\tilde{H}})$.

The various mappings which appear in Theorem 3.2 and its proof are illustrated by the diagram in Table 2.

Proof. (1) For any $g \in \tilde{G}$, we have $g = \varphi(x)$ for some $x \in G$, and thus $g^{-1}\varphi(H)g = \varphi(x)^{-1}\varphi(H)\varphi(x) = \varphi(x^{-1}Hx) = \varphi(H)$, whence $\varphi(H) \lhd \varphi(G)$.

Table 2. Epimorphisms and Artin transfers

$$\begin{array}{ccccc} & & T_{\tilde{G},\tilde{H}} & & \\ & \tilde{G} & \longrightarrow & \tilde{H}/\tilde{H}' & \\ \varphi & \uparrow & \backslash\backslash\backslash & \uparrow & \tilde{\varphi} \\ & G & \longrightarrow & H/H' & \\ & & T_{G,H} & & \end{array}$$

(2) (a) Let $\{g_1,\ldots,g_n\}$ be a left transversal of H in G. Then $G=\dot\cup_{i=1}^n g_iH$ is a disjoint union, but $\varphi(G)=\cup_{i=1}^n\varphi(g_i)\varphi(H)$ is not necessarily disjoint. For $1\le j,k\le n$, we have $\varphi(g_j)\varphi(H)=\varphi(g_k)\varphi(H)\iff\varphi(H)=\varphi(g_j)^{-1}\varphi(g_k)\varphi(H)=\varphi(g_j^{-1}g_k)\varphi(H)$ $\iff\varphi(g_j^{-1}g_k)=\varphi(h)$ for some $h\in H\iff h^{-1}g_j^{-1}g_k=x_0\in\ker(\varphi)$. Here, we need the condition $\ker(\varphi)\lhd H$ to be able to conclude that $g_j^{-1}g_k=hx_0\in H$ and thus $j=k$, so the union is also disjoint. Consequently, $\ker(\varphi)\lhd H$ implies that $\{\varphi(g_1),\ldots,\varphi(g_n)\}$ is a left transversal of $\varphi(H)$ in $\varphi(G)$.

(b) We have $\varphi(T_{G,H}(x))=\varphi\left(\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_iH'\right)=\prod_{i=1}^n\varphi(g_{\pi_x(i)}^{-1})\varphi(x)\varphi(g_i)\varphi(H')=\prod_{i=1}^n\varphi(g_{\pi_x(i)})^{-1}\varphi(x)\varphi(g_i)\varphi(H)'$, where the right hand side equals $T_{\varphi(G),\varphi(H)}(\varphi(x))$, for $x\in G$, provided that $\{\varphi(g_1),\ldots,\varphi(g_n)\}$ is a left transversal of $\varphi(H)$ in $\varphi(G)$, which is the case, when $\ker(\varphi)\lhd H$.

(c) For $x\in G$, we have $x\in\ker(T_{G,H})\iff T_{G,H}(x)=\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_iH'=H'\iff\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_i\in H'$, which implies that
$T_{\varphi(G),\varphi(H)}(\varphi(x))=\prod_{i=1}^n\varphi(g_{\pi_x(i)})^{-1}\varphi(x)\varphi(g_i)\varphi(H)'=\varphi(H)'$,
that is $\varphi(x)\in\ker(T_{\varphi(G),\varphi(H)})$. Therefore, $\varphi(\ker(T_{G,H}))\le\ker(T_{\tilde{G},\tilde{H}})$.

(3) The opposite inclusion $\ker(T_{\tilde{G},\tilde{H}})\le\varphi(\ker(T_{G,H}))$,
that is, $\prod_{i=1}^n\varphi(g_{\pi_x(i)})^{-1}\varphi(x)\varphi(g_i)\in\varphi(H')$ implies $x\in\ker(T_{\tilde{G},\tilde{U}})$, for $x\in G$,
can be proved under the sufficient condition that $\ker(\varphi)\le H'$:
then, $\varphi\left(\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_i\right)=\varphi(h')$ for some $h'\in H'$ is equivalent with
$\varphi\left((h')^{-1}\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_i\right)=1\iff(h')^{-1}\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_i=x_0\in\ker(\varphi)$,
whence $\prod_{i=1}^n g_{\pi_x(i)}^{-1}xg_i=h'x_0\in H'$, that is, $T_{G,H}(x)=H'$.

□

Definition 3.2. We define a partial order of transfer kernels by putting
$\ker(T_{G,H})\le\ker(T_{\tilde{G},\tilde{H}})$ when $\varphi(\ker(T_{G,H}))\le\ker(T_{\tilde{G},\tilde{H}})$, and by putting
$\ker(T_{G,H})=\ker(T_{\tilde{G},\tilde{H}})$ when $\varphi(\ker(T_{G,H}))=\ker(T_{\tilde{G},\tilde{H}})$.

After this general investigation of TTT and TKT singulets, we apply the results for showing that TTT and TKT multiplets of descendants are inherited from ancestors, in the sense of the partial orders in Definitions 3.1 and 3.2.

Definition 3.3. Suppose that G is a group with finite abelianization G/G' and let $(U)_{G'\lhd U\lhd G}$ be the family of its (necessarily normal) subgroups which contain the commutator subgroup G'. For each U, let $T_{G,U}:G\to U/U'$ be the Artin transfer from G to U.

(1) The family $\varkappa(G)=(\ker(T_{G,U}))_{G'\lhd U\lhd G}$ is called the *transfer kernel type* (TKT) of G.
(2) The family $\tau(G)=(U/U')_{G'\lhd U\lhd G}$ is called the *transfer target type* (TTT) of G.

Corollary 3.1. *(TTT and TKT multiplets)*
Suppose $\tilde{G}$ and G are groups with isomorphic finite abelianizations and $\tilde{G}=\varphi(G)$ is the image of G under an epimorphism $\varphi:G\to\tilde{G}$. Let $s\ge1$ be an integer and let $(U_i)_{1\le i\le s}$ be a family of (necessarily normal) subgroups $G'\lhd U_i\lhd G$ which contain the commutator subgroup G'. For each

U_i, let $T_{G,U_i} : G \to U_i/U_i'$ be the Artin transfer from G to U_i, $\tilde{U}_i = \varphi(U_i)$ be the image of U_i, and $T_{\tilde{G},\tilde{U}_i} : \tilde{G} \to \tilde{U}_i/\tilde{U}_i'$ be the Artin transfer from $\tilde{G}$ to $\tilde{U}_i$.

(1) *If $\ker(\varphi) \le \cap_{i=1}^{s} U_i$ then*
$\tau(\tilde{G})_i \le \tau(G)_i$, *i. e., $\tilde{U}_i/\tilde{U}_i'$ is epimorphic image of U_i/U_i', and*
$\varkappa(G)_i \le \varkappa(\tilde{G})_i$, *i. e., $\varphi(\ker(T_{G,U_i})) \le \ker(T_{\tilde{G},\tilde{U}_i})$ for $1 \le i \le s$.*

(2) *If $\ker(\varphi) \le \cap_{i=1}^{s} U_i'$ then*
$\tau(\tilde{G})_i = \tau(G)_i$, *i. e., $\tilde{U}_i/\tilde{U}_i' \simeq U_i/U_i'$, and*
$\varkappa(G)_i = \varkappa(\tilde{G})_i$, *i. e., $\varphi(\ker(T_{G,U_i})) = \ker(T_{\tilde{G},\tilde{U}_i})$ for $1 \le i \le s$.*

Proof. These statements are immediate consequences of Theorems 3.1 and 3.2 and Definitions 3.1 and 3.2. □

4. Stabilization criteria for TTT and TKT

First we would like to point out the reason why we are going to use the parent definition by means of the last non-trivial term of the lower central series.

Theorem 4.1. *Let p be a prime number. Suppose that G is a non-abelian finite p-group of nilpotency class $c = \mathrm{cl}(G) \ge 2$ and its parent is defined as the image $\pi(G)$ of G under the natural epimorphism $\pi : G \to G/\gamma_c(G)$. Then the TTT and TKT of G and $\pi(G)$ are comparable in the sense that $\tilde{\pi}(\tau(G)) \ge \tau(\pi(G))$ and $\pi(\varkappa(G)) \le \varkappa(\pi(G))$.*

Proof. For any subgroup $G' \lhd U \lhd G$, we have $\ker(\pi) = \gamma_c(G) \le \gamma_2(G) = G' \le U$, since we assume $c \ge 2$. Therefore, our Theorems 3.1 and 3.2 and Corollary 3.1 prove the claims. □

Let p be any prime number and assume that G is a metabelian p-group of coclass $\mathrm{cc}(G) = 1$ with order $|G| = p^n$, $n \ge 2$, and nilpotency class $\mathrm{cl}(G) = n - 1 \ge 1$. Then G has an abelianization $G/\gamma_2(G)$ of type (p, p), is generated by two elements x, y, and possesses $p + 1$ maximal subgroups $U_i = \langle g_i, \gamma_2(G)\rangle$ with generators $g_1 = y$ and $g_i = xy^{i-2}$ for $2 \le i \le p + 1$. According to [43, Cor. 3.1, p. 476], the commutator subgroups of the last p maximal subgroups are given uniformly by $\gamma_2(U_i) = \gamma_3(G)$ for $2 \le i \le p + 1$.

Theorem 4.2. *(Partial stabilization criterion for maximal class)*
A metabelian p-group G of coclass $\mathrm{cc}(G) = 1$ and of class $c = \mathrm{cl}(G) \ge 3$ shares the last p components of the TTT $\tau(G)$ and of the TKT $\varkappa(G)$ with its parent $\pi(G) = G/\gamma_c(G)$.

Proof. Let $\pi : G \to \pi(G) = G/\gamma_c(G)$ denote the canonical projection onto the parent. If $c = \mathrm{cl}(G) \ge 3$, then $\ker(\pi) = \gamma_c(G) \le \gamma_3(G) = \gamma_2(U_i)$ for $2 \le i \le p + 1$ [43, Cor. 3.1, p. 476]. Thus, our Theorems 3.1 and 3.2 state that $\tilde{\pi}(\tau_i(G)) = \tau_i(\pi(G))$ and $\pi(\varkappa_i(G)) = \varkappa_i(\pi(G))$ for $2 \le i \le p + 1$. □

Remark 4.1. For all odd primes $p \ge 3$, the last p components of the TTT $\tau(G)$ and of the TKT $\varkappa(G)$ of a metabelian p-group G of coclass $\mathrm{cc}(G) = 1$ and of class $\mathrm{cl}(G) \ge 3$ are uniformly given by $\tau(G)_i = U_i/\gamma_2(U_i) \simeq (p, p)$ [43, Thm. 3.2, p. 477], [15, Thm. 3.4, p. 68] and $\varkappa(G)_i = \ker(T_{G,U_i}) = G$, briefly denoted by $\varkappa(G)_i = 0$ [44, Thm. 2.5, pp. 478–479], for $2 \le i \le p + 1$.
However, the behaviour of the prime $p = 2$ is exceptional, since, for a metabelian 2-group G of coclass $\mathrm{cc}(G) = 1$ and of class $\mathrm{cl}(G) \ge 3$, the abelianizations are not of the nearly homocyclic form in [15, Thm. 3.4, p. 68], $\tau(G)_i = U_i/\gamma_2(U_i) \simeq (2, 2)$ for $2 \le i \le 3$ [43, Tbl. 6–9, pp. 502–503], and the transfer kernels are only partial, $\varkappa(G)_2 = \ker(T_{G,U_2}) = U_3$, $\varkappa(G)_3 = \ker(T_{G,U_3}) = U_2$, briefly denoted by the transposition $\varkappa(G)_2 = 3$, $\varkappa(G)_3 = 2$ [44, Thm. 2.6, p. 481].

Remark 4.2. The extra special p-group G of order p^3 and exponent p^2 shows that the condition $\mathrm{cl}(G) \ge 3$ in Theorem 4.2 is sharp. It is of class $\mathrm{cl}(G) = 2$, its parent is the abelian p-group $\pi(G) = C_p \times C_p$ and we have $\tau(G)_i = C_p \times C_p > C_p = \tau(\pi(G))_i$ $\pi(\varkappa(G)_i) = \pi(U_1) < \pi(G) = \varkappa(\pi(G))_i$ for $2 \le i \le p+1$ when $p \ge 3$, and $\tau(G)_i = C_4 > C_2 = \tau(\pi(G))_i$ $\pi(\varkappa(G)_i) = \pi(U_i) < \pi(G) = \varkappa(\pi(G))_i$ for $2 \le i \le 3$ when $p = 2$, i. e., G is the quaternion group.

Theorem 4.3. *(Total stabilization criterion for maximal class and positive defect)*
A metabelian p-group G of coclass $\mathrm{cc}(G) = 1$ *and of class* $m - 1 = \mathrm{cl}(G) \geq 4$ *with positive defect of commutativity* $k = k(G) \geq 1$ [46, 3.1.1, p. 412] *shares all* $p+1$ *components of the TTT* $\tau(G)$ *and of the TKT* $\varkappa(G)$ *with its parent* $\pi(G) = G/\gamma_c(G)$.

Proof. Let $\pi : G \to \pi(G) = G/\gamma_c(G)$ denote the natural projection onto the parent. If $k = k(G) \geq 1$ then necessarily $p \geq 3$, $m - 1 = \mathrm{cl}(G) \geq 4$ and $\ker(\pi) = \gamma_{m-1}(G) \leq \gamma_{m-k}(G) \leq \gamma_2(U_i)$ for $1 \leq i \leq p+1$ [43, Cor. 3.1, p. 476]. Thus, the Theorems 3.1 and 3.2 ensure that $\tilde{\pi}(\tau_i(G)) = \tau_i(\pi(G))$ and $\pi(\varkappa_i(G)) = \varkappa_i(\pi(G))$ for $1 \leq i \leq p+1$. □

Remark 4.3. The p-groups G of coclass $\mathrm{cc}(G) = 1$ which possess an abelian maximal subgroup and are located at depth 1 outside of the mainline [46, Fig. 3.1–3.3] show that the condition $k(G) \geq 1$ in Theorem 4.3 is sharp. Their defect is $k(G) = 0$ and we have $\pi(\varkappa(G)_1) = \pi(U_i) < \pi(G) = \varkappa(\pi(G))_1$ with $1 \leq i \leq 2$.

Now assume that G is a metabelian 3-group of coclass $\mathrm{cc}(G) \geq 2$ with abelianization $G/\gamma_2(G)$ of type $(3,3)$, order $|G| = 3^n$, $n \geq 5$, and nilpotency class $\mathrm{cl}(G) = m-1 \geq 3$. Then G is generated by two elements x, y, and possesses 4 maximal subgroups $U_i = \langle g_i, \gamma_2(G)\rangle$ with generators $g_1 = y$ and $g_i = xy^{i-2}$ for $2 \leq i \leq 4$. According to [43, Cor. 3.2, p. 480], the commutator subgroups of the last 2 maximal subgroups have the property $\gamma_2(U_i) > \gamma_4(G)$ for $3 \leq i \leq 4$.

Theorem 4.4. *(Partial stabilization criterion for non-maximal class)*
A metabelian 3*-group G with abelianization* $G/G' \simeq (3,3)$*, coclass* $\mathrm{cc}(G) \geq 2$*, and class* $c = \mathrm{cl}(G) \geq 4$ *shares the last* 2 *components of the TTT* $\tau(G)$ *and of the TKT* $\varkappa(G)$ *with its parent* $\pi(G) = G/\gamma_c(G)$.

Proof. Let $\pi : G \to \pi(G) = G/\gamma_c(G)$ denote the canonical projection onto the parent. If $c = \mathrm{cl}(G) \geq 4$, then $\ker(\pi) = \gamma_c(G) \leq \gamma_4(G) < \gamma_2(U_i)$ for $3 \leq i \leq 4$ [43, Cor. 3.2, p. 480]. Thus, our Theorems 3.1 and 3.2 give a warranty that $\tilde{\pi}(\tau_i(G)) = \tau_i(\pi(G))$ and $\pi(\varkappa_i(G)) = \varkappa_i(\pi(G))$ for $3 \leq i \leq 4$. □

Remark 4.4. The last 2 components of the TTT $\tau(G)$ and of the TKT $\varkappa(G)$ of a metabelian 3-group G with abelianization $G/G' \simeq (3,3)$, of coclass $\mathrm{cc}(G) \geq 3$ and of class $\mathrm{cl}(G) \geq 4$ are uniformly given by $\tau(G)_i = U_i/\gamma_2(U_i) \simeq (3,3,3)$ for $3 \leq i \leq 4$ [45, Thm. 4.5], and $\varkappa(G)_3 = \ker(T_{G,U_3}) = U_4$, $\varkappa(G)_4 = \ker(T_{G,U_4}) = U_3$, briefly denoted by the transposition $\varkappa(G)_3 = 4$, $\varkappa(G)_4 = 3$ [44, Thm. 3.2, p. 487].

5. Finding three-stage towers via pruned descendant trees

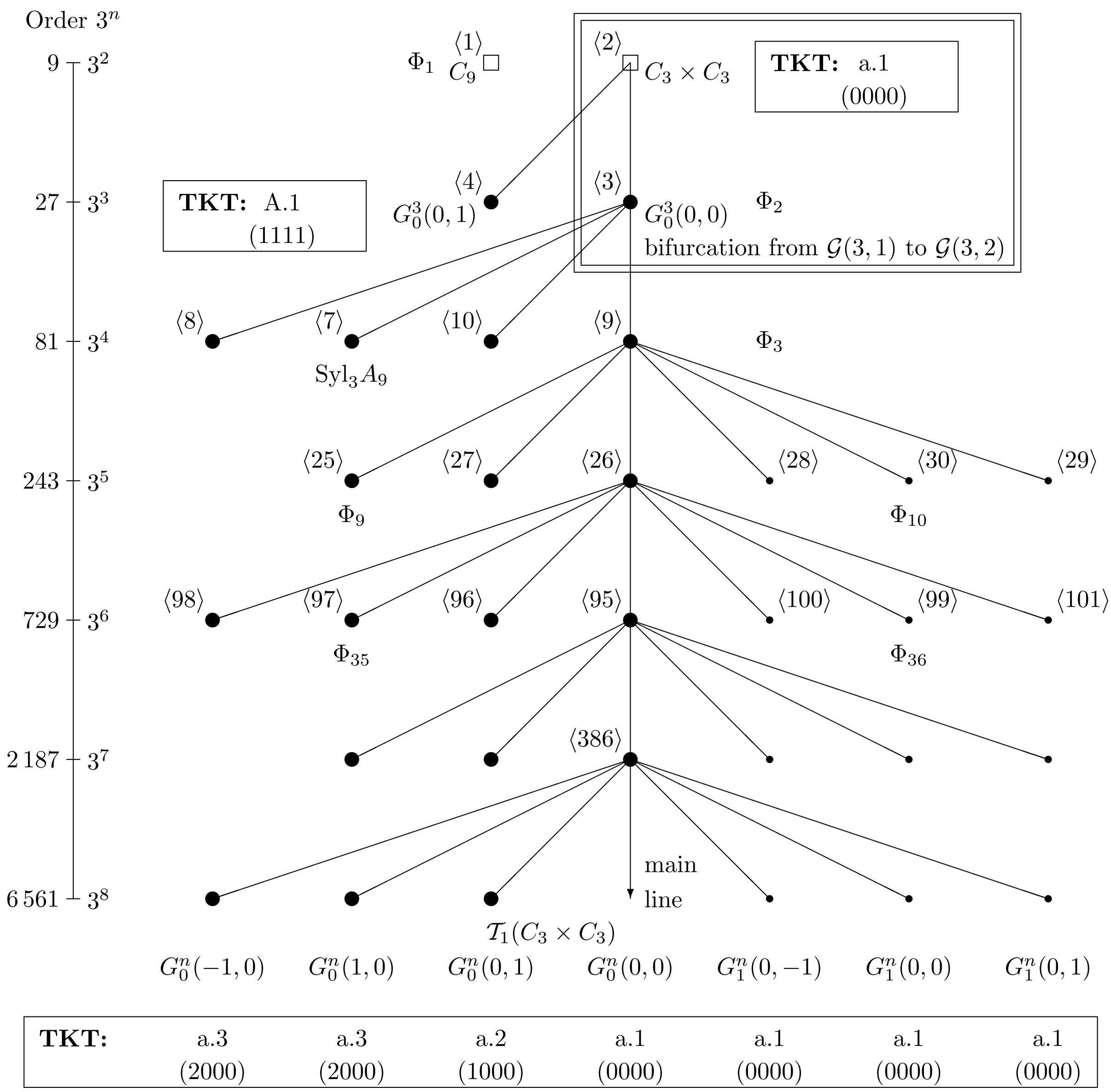
In this section, we demonstrate the entire process of pattern recognition via Artin transfers, as outlined in the Introduction § 1, by means of a series of complex quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$, whose 3-capitulation in the four unramified cyclic cubic extensions $L_i|K$, $1 \leq i \leq 4$, is of one of the closely related types E.6, $\varkappa(K) = (1313)$, or E.14, $\varkappa(K) = (2313)$, or E.8, $\varkappa(K) = (1231)$, or E.9, $\varkappa(K) = (2231)$ [44, Tbl. 6, p. 492], briefly and unitedly called *section* E. According to [43, Tbl. 3, p.497], these fields with TKT in section E form a considerable proportion of $186 + 197 + 15 + 13 = 411$ cases among a total of 2 020 complex quadratic fields with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and discriminants in the range $-10^6 < D < 0$. So their relative frequency in that range is 20.3%. The smallest absolute discriminant $|D| = 9\,748$ with TKT E.9 was discovered by Scholz and Taussky already [56], $|D| = 15\,544$ with TKT E.6 and $|D| = 16\,627$ with TKT E.14 are due to Heider and Schmithals [35], and $|D| = 34\,867$ with TKT E.8 was found in the year 2003 by ourselves [43].

5.1. Determining the patterns to search for.

Since the four unramified cyclic cubic extensions $L_i|K$, $1 \leq i \leq 4$, of a complex quadratic field $K = \mathbb{Q}(\sqrt{D})$ with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ can be constructed as composita of K with four non-Galois complex cubic fields K_i sharing the same discriminant $d(K_i) = D$, and thus using Fieker's sophisticated class field techniques in the MAGMA computer algebra system would be like taking a sledgehammer to crack a nut, we employed our own principalization algorithm via class group structure [45, § 5], which is based on

the system PARI/GP [55], to determine the patterns of TKTs and TTTs required for identifying the second 3-class group $G_3^2(K)$ and the 3-tower group $G_3^\infty(K)$ of K. The TKTs are those in section E, as given previously, and the TTTs turned out to be
$\tau(K) = [(9,27),(3,9),(3,3,3),(3,9)]$ for the ground state of TKTs E.6 and E.14,
$\tau(K) = [(27,81),(3,9),(3,3,3),(3,9)]$ for the first excited state of TKTs E.6 and E.14,
$\tau(K) = [(9,27),(3,9),(3,9),(3,9)]$ for the ground state of TKTs E.8 and E.9,
$\tau(K) = [(27,81),(3,9),(3,9),(3,9)]$ for the first excited state of TKTs E.8 and E.9,
which has also been proved theoretically in [45, Thm. 4.4].

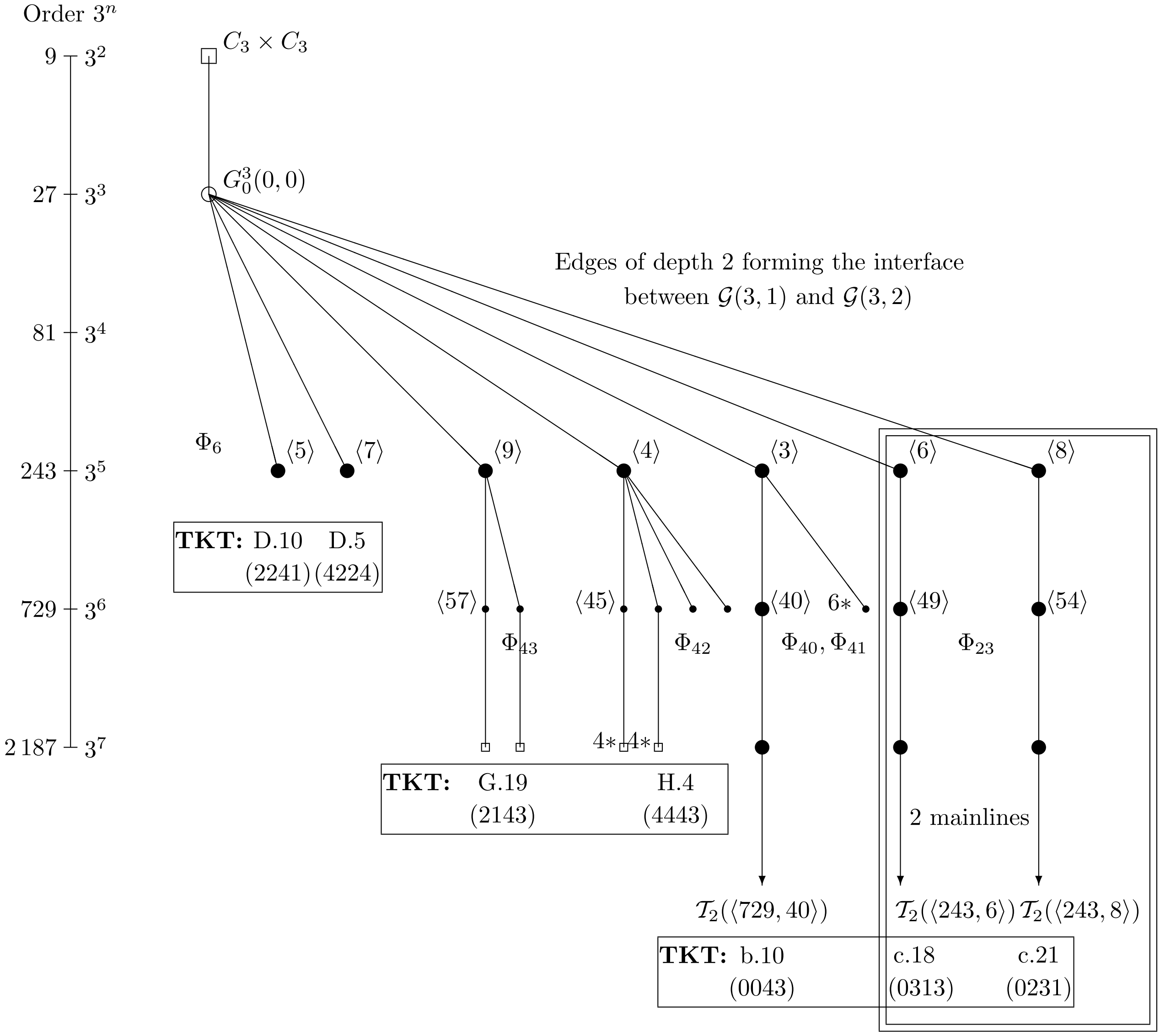
Figure 1. Starting 3-group generation at the top of coclass graph $\mathcal{G}(3,1)$



5.2. **Using O'Brien's p-group generation algorithm.** We start our search for 3-groups with TKT in section E at the abelian root $C_3 \times C_3 \simeq \langle 9,2\rangle$ of the unique coclass tree $\mathcal{T}_1$ in coclass graph $\mathcal{G}(3,1)$. However, we leave this graph very quickly, since all 3-groups of maximal class have TKTs in sections a,A.

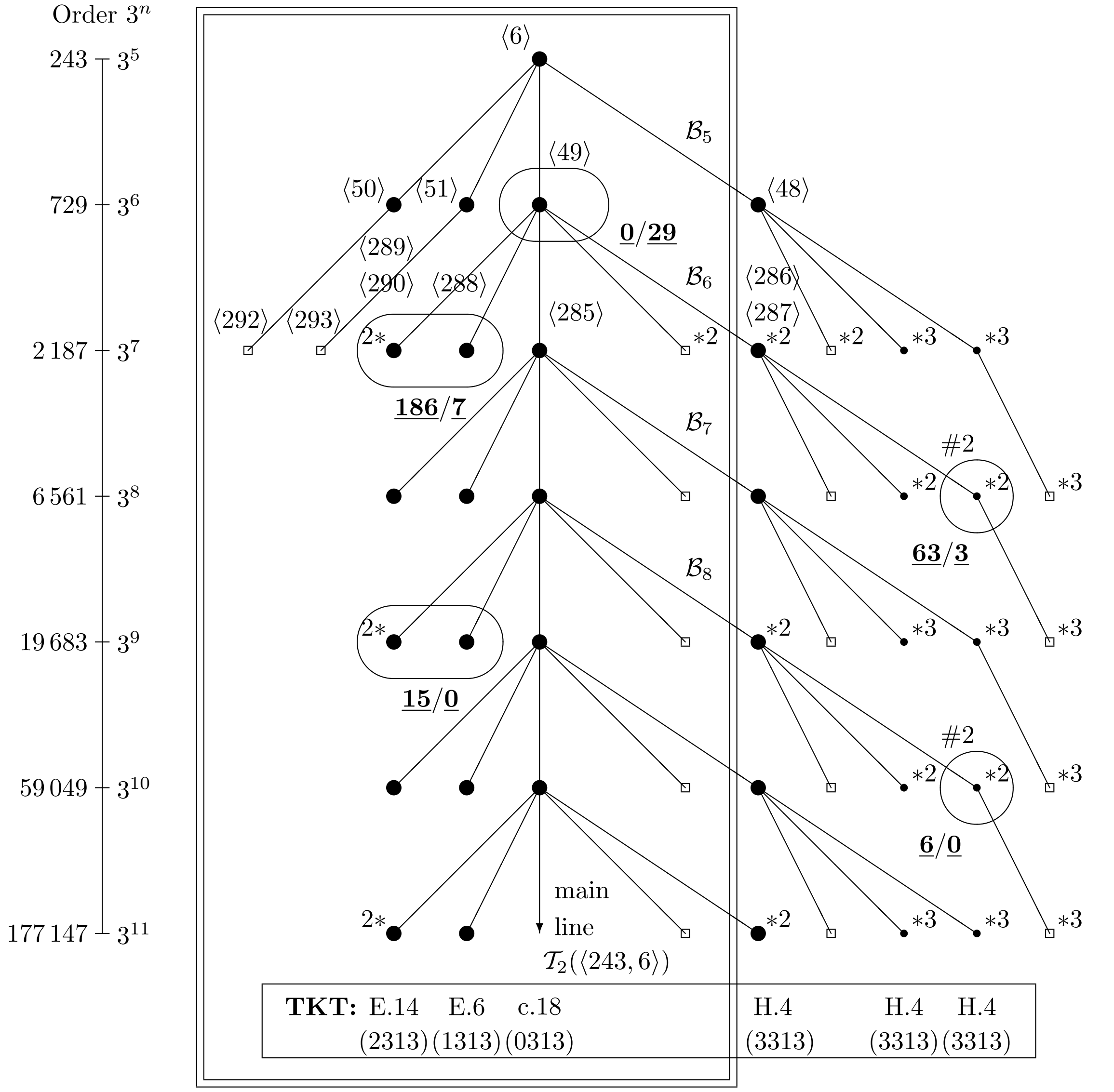
The immediate descendant $G_0^3(0,0) \simeq \langle 27,3\rangle$ gives rise to a bifurcation from $\mathcal{G}(3,1)$ to $\mathcal{G}(3,2)$, but the following mainline vertex $G_0^4(0,0) \simeq \langle 81,9\rangle$ is coclass-settled and no further bifurcations can occur.

FIGURE 2. TKT-pruning the top of coclass graph $\mathcal{G}(3,2)$



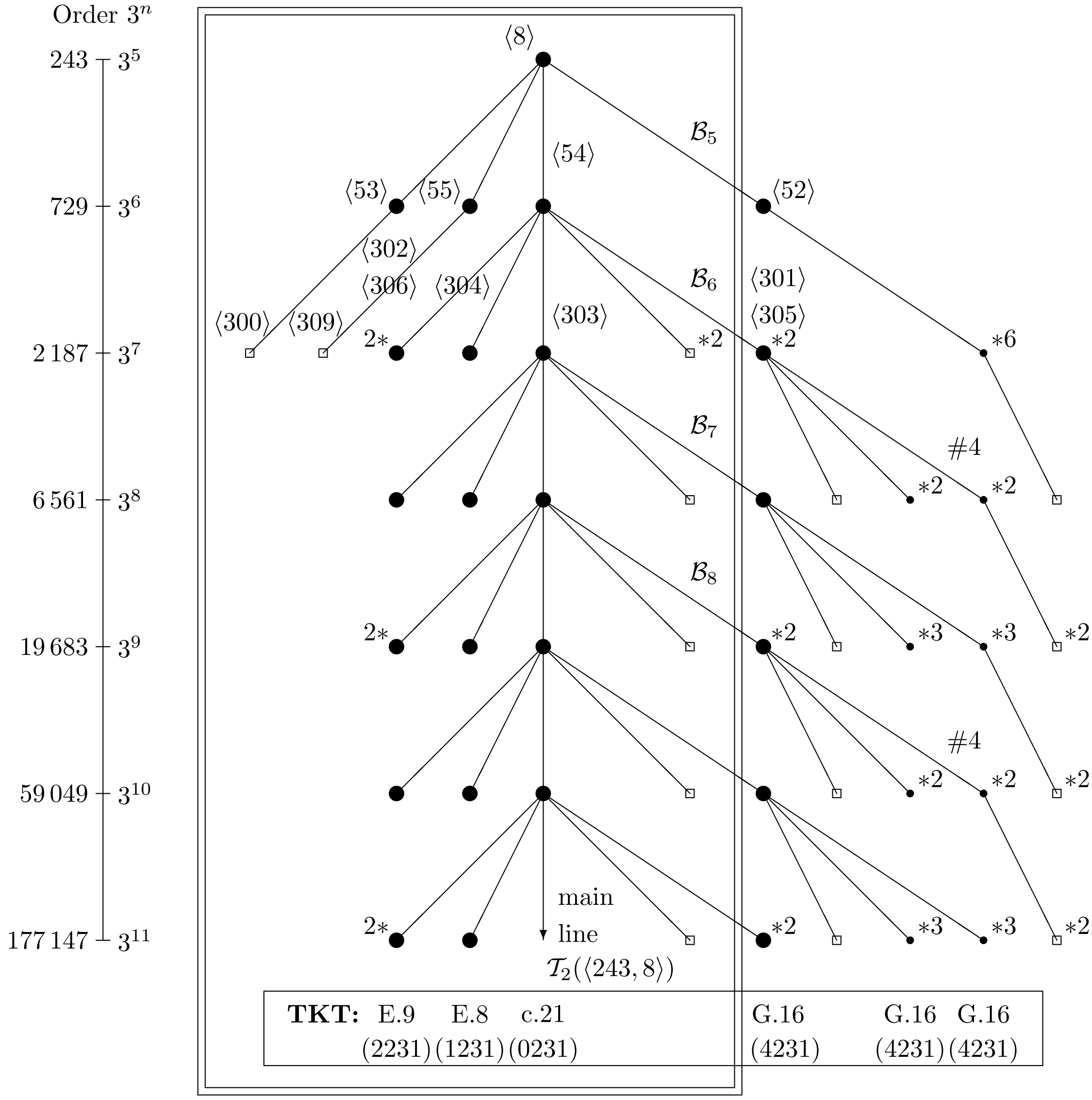
The top vertices $\langle 243,5\rangle$ and $\langle 243,7\rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243,9\rangle$, resp. $\langle 243,4\rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243,3\rangle$ must contain a transposition, which is not the case for TKTs in sections c and E. Therefore, only descendants of $\langle 243,6\rangle$ and $\langle 243,8\rangle$ can have TKTs in sections c and E.

FIGURE 3. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$



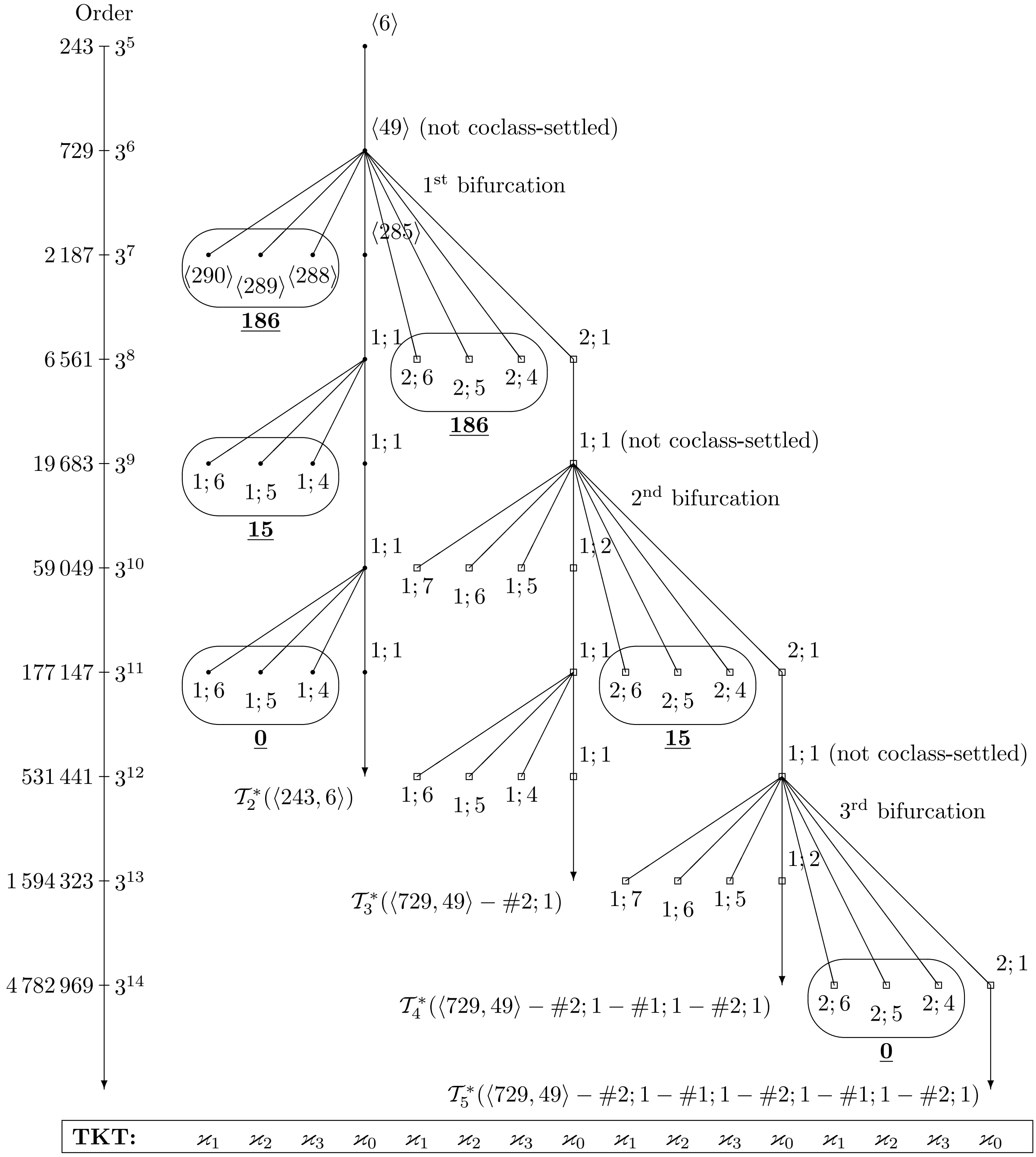
The bifurcation at $\langle 729, 49\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

FIGURE 4. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$



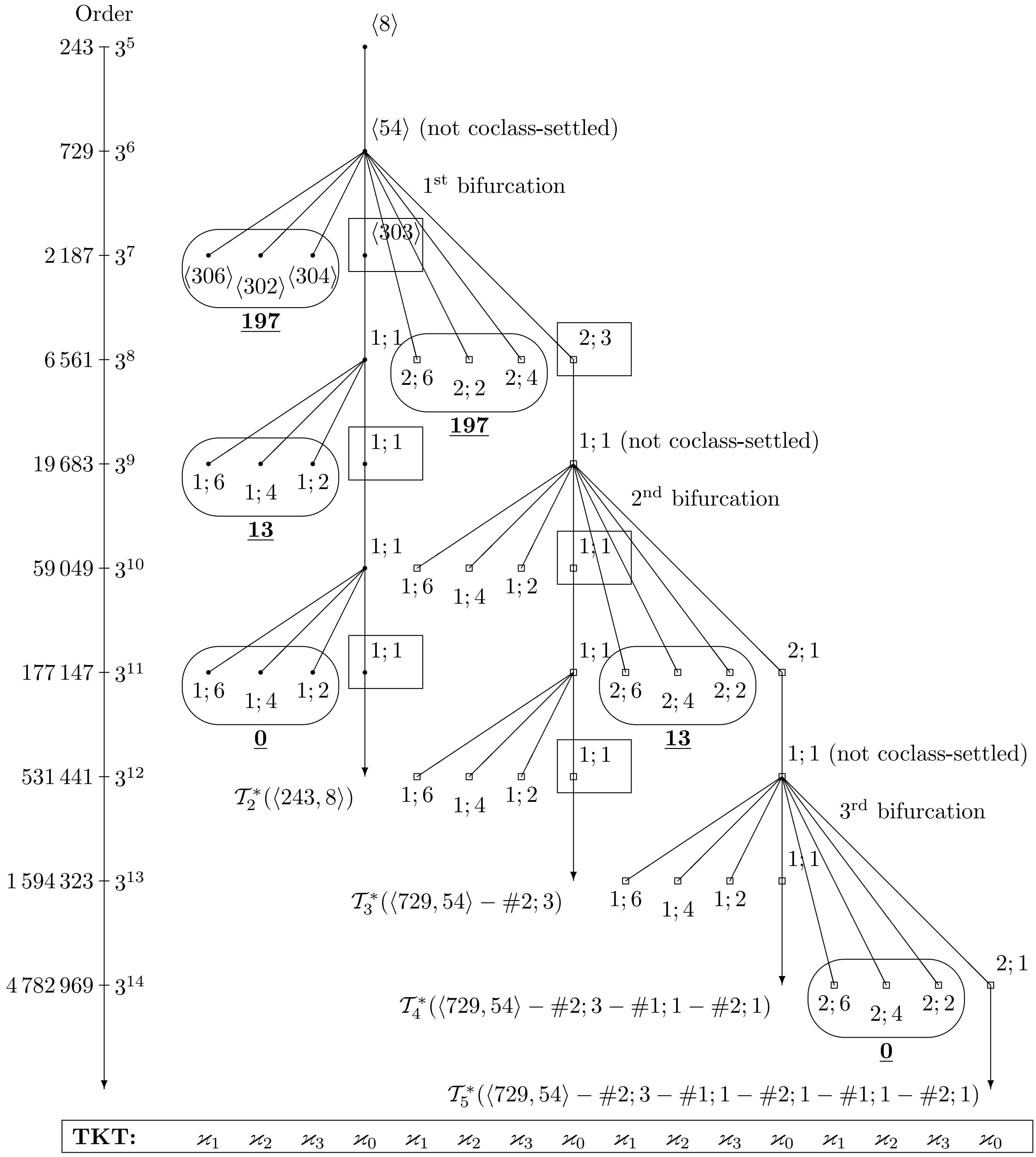
The bifurcation at $\langle 729, 54\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

FIGURE 5. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 6\rangle)$ restricted to σ-groups



Here we also prune the tree from vertices with TKT c.18 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (3122) \sim \varkappa_2 = (4122)$ E.14,
$\varkappa_3 = (1122)$ E.6,
$\varkappa_0 = (0122)$ c.18.

Figure 6. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, and identifiers of SmallGroups and ANUPQ



Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

FIGURE 7. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, projections to the metabelianizations, and formal identifiers

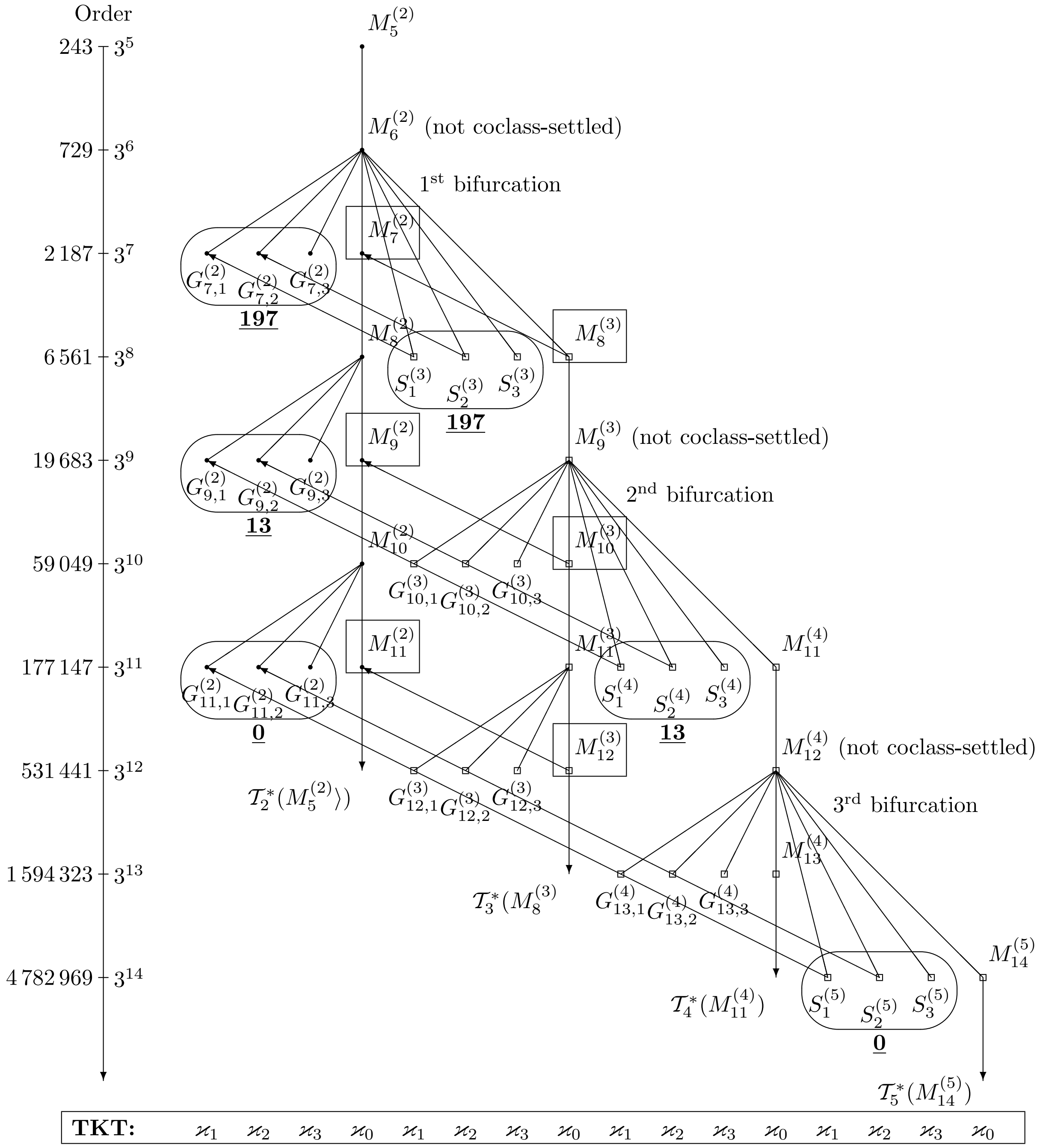


FIGURE 8. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(3)}$, with TKT E, class 5.

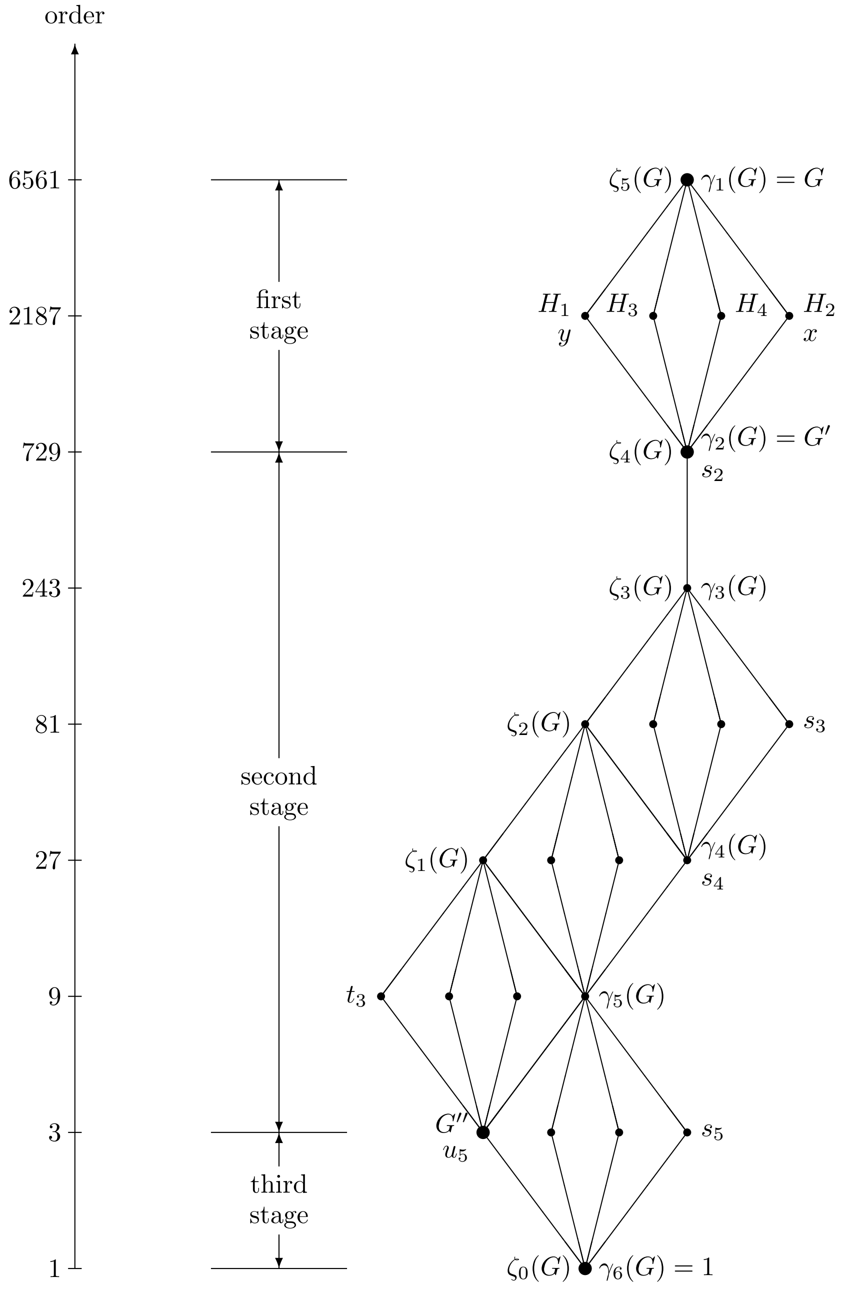
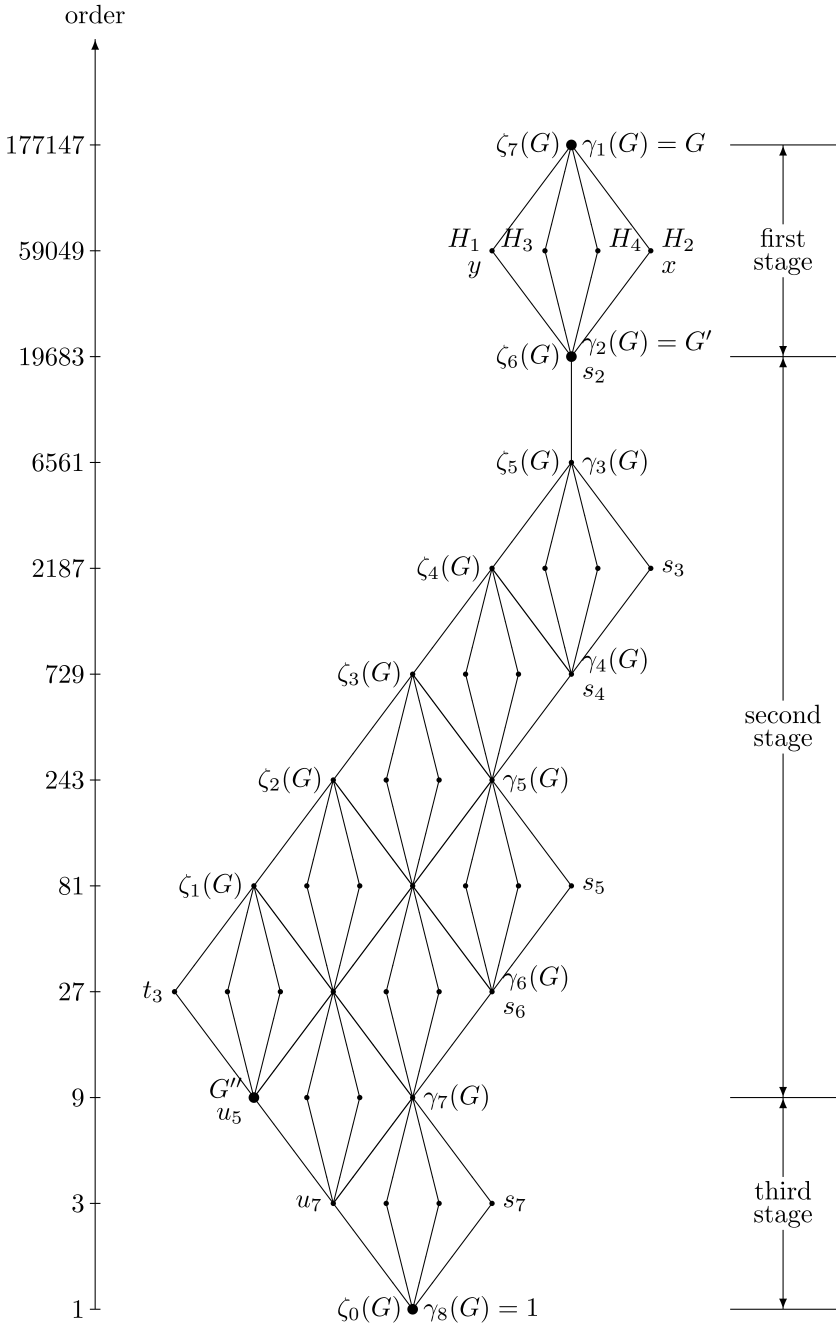


Figure 9. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(4)}$, with TKT E, class 7.



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