PERIODIC SCHUR σ-GROUPS OF NON-ELEMENTARY BICYCLIC TYPE

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Abstract. Infinitely many large Schur σ-groups G with non-elementary bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$, $e \geq 2$, are constructed as periodic sequences of vertices in descendant trees of finite 3-groups. A single root gives rise to pairs of metabelian groups G with logarithmic order $\mathrm{lo}(G) = 4+e$ for $e \geq 3$. Three roots are ancestors of pairs of non-metabelian groups G with moderate rank distribution $\varrho(G) \sim (2, 2, 3; 3)$ and $\mathrm{lo}(G) = 7 + e$ for $e \geq 5$. Twentyseven roots produce sextets of non-metabelian groups G with elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$ and $\mathrm{lo}(G) = 19 + e$ for $e \geq 9$. The soluble length of non-metabelian groups is always $\mathrm{sl}(G) = 3$. The groups can be realized as 3-class field tower groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d < 0$.

1. Introduction

Why are Schur σ-groups of eminent algebraic and arithmetic relevance? For an assigned odd prime number $p \geq 3$, the automorphism group $G_\infty = \mathrm{Gal}(\mathrm{F}_p^\infty(K)/K)$ of the maximal unramified pro-p extension $\mathrm{F}_p^\infty(K)$ of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d < 0$ is a Schur σ-group [29, 14, 1, 9, 8], in fact, either a finite group with order a power of p or an infinite pro-p group. All Galois groups $G_n = \mathrm{Gal}(\mathrm{F}_p^n(K)/K)$ of the stages of the unramified Hilbert p-class field tower

$$K \leq \mathrm{F}_p^1(K) \leq \mathrm{F}_p^2(K) \leq \ldots \leq \mathrm{F}_p^n(K) \leq \ldots \leq \mathrm{F}_p^\infty(K)$$

of K are derived quotients $G_n \simeq G_\infty/G_\infty^{(n)}$ of the Schur σ-group G_∞. In turn, imaginary quadratic fields K are the simplest algebraic number fields with smallest possible degree 2 and trivial torsion-free unit group U_K/W_K. At least for the smallest odd prime number $p = 3$, imaginary quadratic fields are distinguished by the experimental constructibility of their unramified abelian extensions [11, 6, 7, 15] with relative degrees p and usually also p^2 in a few minutes of CPU-time.

In our most recent investigations, we succeeded in finding unexpected periodic sequences of Schur σ-groups G possessing a bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$ with one non-elementary component and logarithmic exponent $e \geq 2$. Periodicity sets in for a minimal exponent $e \geq e_0$ in dependence on the particular kind of the Schur σ-groups G. The value e_0 is given by the p-nilpotency class $\mathrm{cl}_p(G)$ of G. There exist periodic sequences of

- metabelian Schur σ-groups G with logarithmic order $\mathrm{lo}(G) = 4 + e$ for $e \geq e_0 = 3$ [24],
- non-metabelian Schur σ-groups G with moderate rank distribution $\varrho(G) \sim (2, 2, 3; 3)$ or $\varrho(G) \sim (2, 2, 2; 3)$ and logarithmic order $\mathrm{lo}(G) = 7 + e$ for $e \geq e_0 = 5$ [24, 26],
- non-metabelian Schur σ-groups G with elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$ and logarithmic order $\mathrm{lo}(G) = 19 + e$ for $e \geq e_0 = 9$ [25].

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2. Metabelian Schur-groups G with $G/G' \simeq (3^e,3)$, $e \ge 1$

A common feature of all periodic constructions in this article is the constitution of the resulting p-descendant tree by

- an *infinite main trunk* of vertices T with parametrized presentation,
- graph theoretically isomorphic *finite twigs* emanating from each vertex of the main trunk.

The leaves G of the twigs are usually Schur σ-groups. In order to avoid ambiguity, we use subscripts T_e, G_e in all the following theorems.

Since the simplest main trunk consists of all abelian groups of type $(3^e,3)$, we begin with this important instance, although the leaves of the twigs are only Schur-groups (with balanced presentation) but not σ-groups (without generator-inverting automorphism).

Theorem 1. *For each logarithmic exponent $e \ge 2$, the **unique** non-elementary bi-heterocyclic 3-group T_e of type $(3^e,3)\hat{=}(e1)$ is given by the **periodic sequence** of iterated p-descendants*

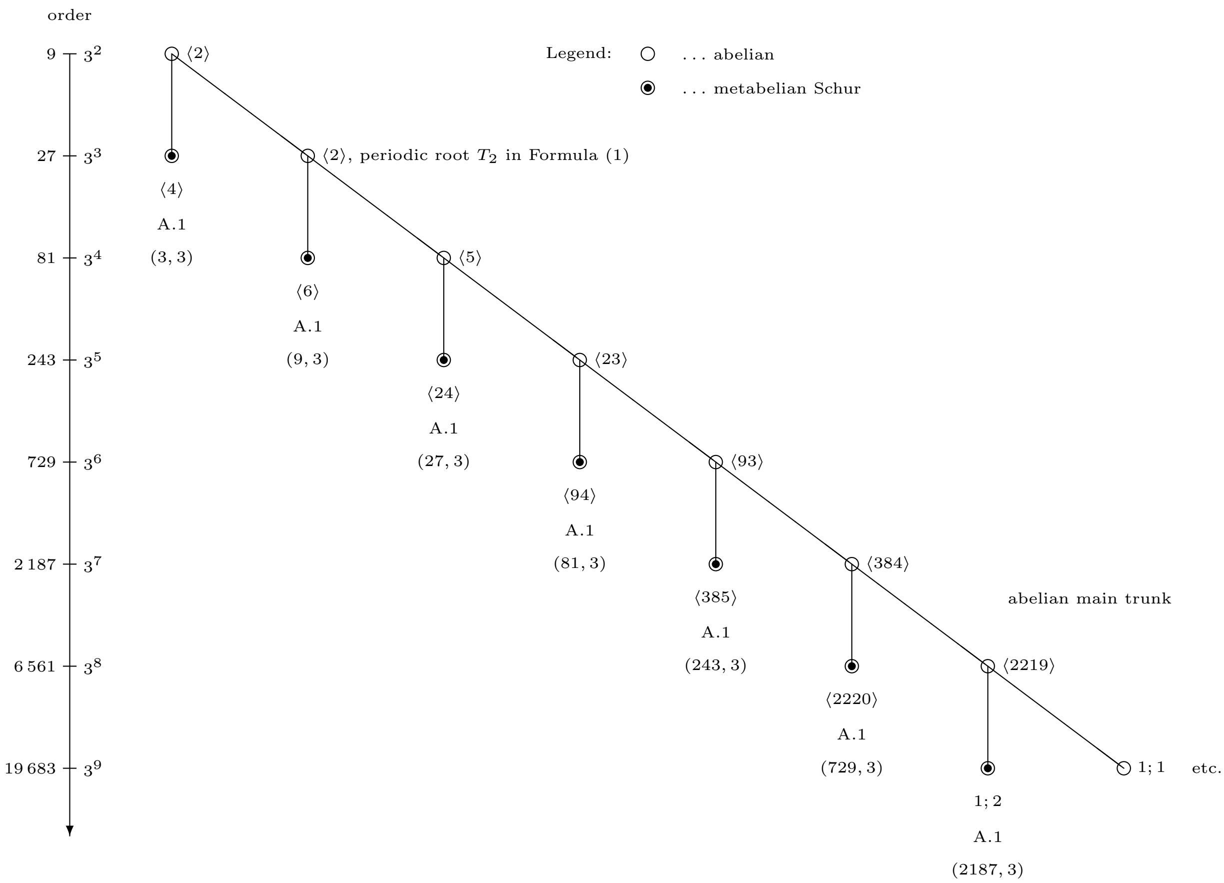
$$T_e \simeq \mathrm{SmallGroup}(27,2)(-\#1;1)^{e-2}. \tag{1}$$

These abelian groups have logarithmic order $\mathrm{lo}(T_e) = 1+e$, punctured transfer kernel type a.1*, $\varkappa(T_e) = (000;0)$, and first abelian quotient invariants $\alpha_1(T_e) \sim (e,e,e;(e-1)1)$. They form the infinite main trunk of a descendant tree with singlets as finite twigs. For each $e \ge 2$, the singlet*

$$G_e \simeq \mathrm{SmallGroup}(27,2)(-\#1;1)^{e-2} - \#1;2 \tag{2}$$

*is the **unique** metabelian Schur-group G_e with commutator quotient $G_e/G_e' \simeq (3^e,3)\hat{=}(e1)$ and punctured transfer kernel type* A.1*, $\varkappa(G_e) \sim (111;1)$. It has first abelian quotient invariants $\alpha_1(G_e) \sim (e+1,e+1,e+1;e1)$, $\mathrm{lo}(G_e) = 2+e$, and is not a σ-group. See Figure 1.*

Figure 1. Periodic metabelian Schur-groups G with $G/G' \simeq (3^e,3)$, $e \ge 2$



Proof. Let $s_2 = [y, x]$ denote the main commutator and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 1$, we have parametrized presentations

$$(3)\qquad \begin{aligned} T_{e+1} &= \langle x, y, w \mid x^{3^{e+1}} = w^3 = 1,\ y^3 = 1 \rangle, \\ G_e &= \langle x, y, s_2, w \mid x^{3^{e+1}} = w^3 = 1,\ y^3 = 1,\ s_2 = w \rangle. \end{aligned}$$

For $e \geq 1$, the last non-trivial lower p-central is given by $P_e(G_e) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence both groups share the common p-parent $G_e/P_e(G_e) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$. Observe that the p-nilpotency class is given by $\mathrm{cl}_p(G_e) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$.

The p-group generation algorithm [13] by Newman [27] and O'Brien [28] is implemented in the ANUPQ package [12] of the computational algebra system Magma [15, 7, 6]. This algorithm is used to construct all immediate p-descendants of an assigned finite p-group. Vertices are identified by absolute counters SmallGroup$(o, i) = \langle o, i \rangle = \langle i \rangle$, defined in the SmallGroups database [5] for orders $o \leq 3^8$, or by relative counters $P - \#s; i$ with respect to the p-parent P and the step size s, assigned by the ANUPQ package [12] for $o \geq 3^9$. Repeated recursive applications of the algorithm eventually produce Figure 1, and thus confirm Formula (1) and (2).

Special care is required for the tree root $T_1 = \langle 9, 2 \rangle$ only. It has exceptional nuclear rank $n(T_1) = 3$, and we can neglect step sizes $s \in \{2, 3\}$. Among the immediate p-descendants of step size $s = 1$, the abelian group $T_2 = \langle 27, 2 \rangle$ is exo-genetic with regular identifier $T_1 - \#1; 1$, but $G_1 = \langle 27, 4 \rangle$ is endo-genetic with irregular identifier $T_1 - \#1; 3$, since $T_1 - \#1; 2 \simeq \langle 27, 3 \rangle$ is the extra-special 3-group with punctured transfer kernel type a.1, $\varkappa = (000; 0)$. Therefore, it is not possible to extend Theorem 1 to all $e \geq 1$ in a uniform way. □

3. Metabelian Schur σ-groups G with $G/G' \simeq (3^e, 3)$, $e \geq 3$

Theorem 2. *For each logarithmic exponent $e \geq 3$, the* ***unique*** *metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e, 3) \widehat{=} (e1)$, punctured transfer kernel type* b.16*, $\varkappa(T_e) \sim (004; 0)$, and logarithmic order $\mathrm{lo}(T_e) = 3 + e$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$(4)\qquad T_e \simeq \mathrm{SmallGroup}(729, 8)(-\#1; 1)^{e-3}.$$

These groups have first abelian quotient invariants $\alpha_1(T_e) \sim (e1, e1, e11; (e-1)11)$. They form the infinite main trunk of a descendant tree with doublets as finite twigs. For each integer $e \geq 3$, the doublet

$$(5)\qquad G_{e,i} \simeq \mathrm{SmallGroup}(729, 8)(-\#1; 1)^{e-3} - \#1; i, \qquad i \in \{2, 3\},$$

is the ***unique pair*** *of* ***metabelian*** *Schur σ-groups $G_{e,i}$ with commutator quotient $G_{e,i}/G'_{e,i} \simeq (3^e, 3) \widehat{=} (e1)$. It has punctured transfer kernel type* D.11*, $\varkappa(G_{e,i}) \sim (124; 1)$, $\mathrm{lo}(G_{e,i}) = 4 + e$, and first abelian quotient invariants $\alpha_1(G_{e,i}) \sim ((e+1)1, (e+1)1, e11; e11)$. See Figure 2.*

Proof. Let $s_2 = [y, x]$ denote the main commutator, $s_3 = [s_2, x]$ and $t_3 = [s_2, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 3$, we have parametrized presentations

$$(6)\qquad \begin{aligned} T_{e+1} &= \langle x, y, s_2, s_3, t_3, w \mid x^{3^{e+1}} = w^3 = 1,\ y^3 = s_3,\ t_3 = s_3 \rangle, \\ G_{e,i} &= \langle x, y, s_2, s_3, t_3, w \mid x^{3^{e+1}} = w^3 = 1,\ y^3 = s_3,\ t_3 = s_3 w^{i-1} \rangle, \qquad i \in \{2, 3\}. \end{aligned}$$

The p-class is given by $\mathrm{cl}_p(G_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 3$, the last non-trivial lower p-central is given by $P_e(G_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $G_{e,i}/P_e(G_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

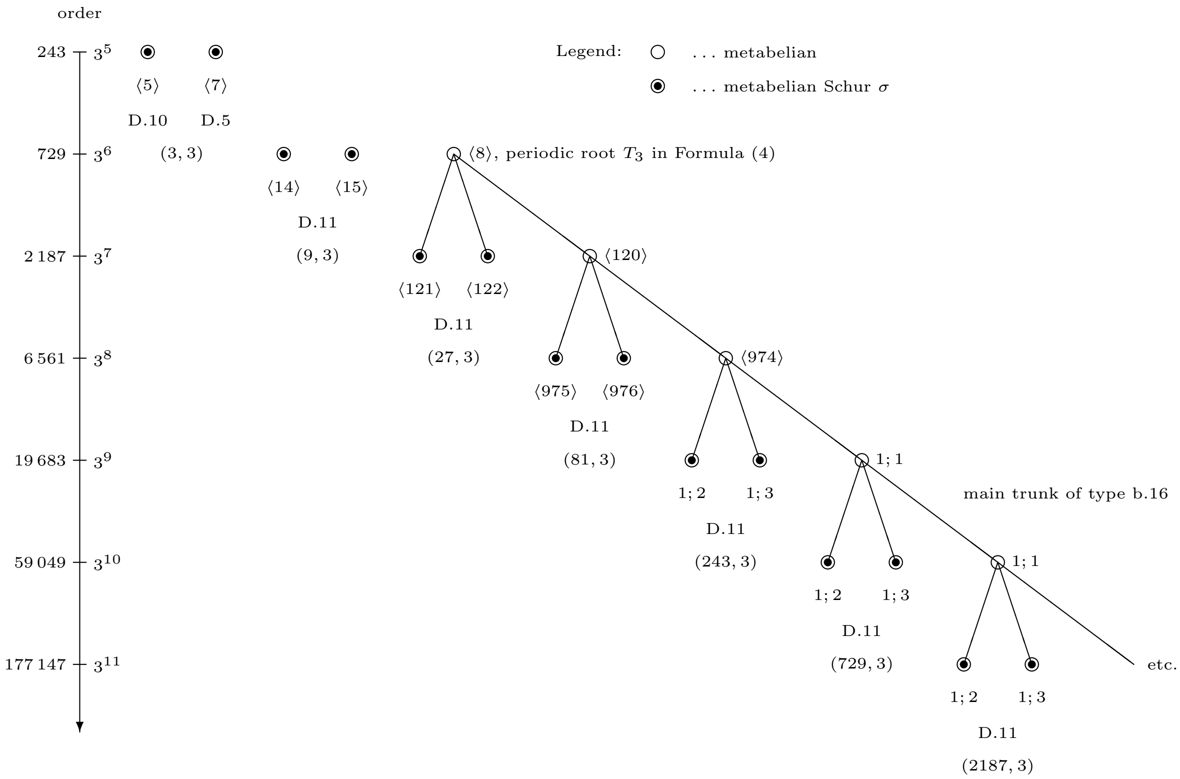
Repeated recursive applications of the p-group generation algorithm eventually produce Figure 2, and thus confirm Formula (4) and (5). All groups have $\varrho \sim (2, 2, 3; 3)$. Cf. [24, Thm. 4–7]. □

In particular, for $e = 3$ and $e = 4$, we have the pairs
SmallGroup$(2187, 121) \simeq G_{3,2} = \langle x, y, s_2, s_3, t_3, w \mid x^{27} = w, y^3 = s_3, t_3 = s_3 \cdot w \rangle$,
SmallGroup$(2187, 122) \simeq G_{3,3} = \langle x, y, s_2, s_3, t_3, w \mid x^{27} = w, y^3 = s_3, t_3 = s_3 \cdot w^2 \rangle$; and
SmallGroup$(6561, 975) \simeq G_{4,2} = \langle x, y, s_2, s_3, t_3, w \mid x^{81} = w, y^3 = s_3, t_3 = s_3 \cdot w \rangle$,
SmallGroup$(6561, 976) \simeq G_{4,3} = \langle x, y, s_2, s_3, t_3, w \mid x^{81} = w, y^3 = s_3, t_3 = s_3 \cdot w^2 \rangle$.

The smallest pair, for $e = 2$, however, is exceptional:
$\mathrm{SmallGroup}(729, 14) \simeq \langle x, y, s_2, s_3, t_3 \mid x^9 = s_3, y^3 = t_3 \rangle$,
$\mathrm{SmallGroup}(729, 15) \simeq \langle x, y, s_2, s_3, t_3 \mid x^9 = s_3, y^3 = t_3^2 \rangle$.

Theorem 2 is illustrated for $3 \le e \le 7$ by Figure 2, using the notation of [5, 12].

FIGURE 2. Periodic metabelian Schur σ-groups G with $G/G' \simeq (3^e, 3)$, $e \ge 3$



4. MODERATE SCHUR σ-GROUPS G WITH $G/G' \simeq (3^e, 3)$, $e \ge 5$, GROUND STATE

Figure 3, resp. 4, shows that the construction process for the two non-metabelian Schur σ-groups G with order $\#G = 3^{7+e}$ and punctured transfer kernel type D.10, resp. D.6, becomes increasingly difficult for the commutator quotients $G/G' \simeq (27, 3)$, $(81, 3)$, $(243, 3)$. For the commutator quotient $G/G' \simeq (729, 3)$, however, an **unexpected tranquilization occurs**, and the construction process becomes settled with a **simple step size one periodicity**.

Theorem 3. *The four pairs of Schur σ-groups G with soluble length $\mathrm{sl}(G) = 3$, commutator quotient $G/G' \simeq (3^e, 3) \hat{=} (e1)$, $e \ge 5$, punctured transfer kernel types* D.10, D.5, C.4, D.6, *and order $\#G = 3^{7+e}$ are given by the following* ***bottom up*** *construction process.*

- *For type* D.10, $\varkappa \sim (411; 3)$, $\alpha \sim (e22, (e+1)1, (e+1)1; e11)$, $\varrho \sim (3, 2, 2; 3)$, *let* $A_2 := \mathrm{SmallGroup}(6561, 93) - \#2; 2$, *then*

$$G \simeq A_2(-\#1; 1)^{e-5} - \#1; i - \#1; 1, \qquad i \in \{2, 3\}. \tag{7}$$

- *For type* D.5, $\varkappa \sim (211; 3)$, $\alpha \sim ((e+1)21, (e+1)1, (e+1)1; e11)$, $\varrho \sim (3, 2, 2; 3)$, *let* $A_4 := \mathrm{SmallGroup}(6561, 93) - \#2; 4$, *then*

$$G \simeq A_4(-\#1; 1)^{e-5} - \#1; i - \#1; 1, \qquad i \in \{2, 3\}. \tag{8}$$

- *For type* C.4, $\varkappa \sim (311; 3)$, $\alpha \sim ((e+1)21, (e+1)1, (e+1)1; e11)$, $\varrho \sim (3, 2, 2; 3)$, *let* $A_5 := \mathrm{SmallGroup}(6561, 93) - \#2; 5$, *then*

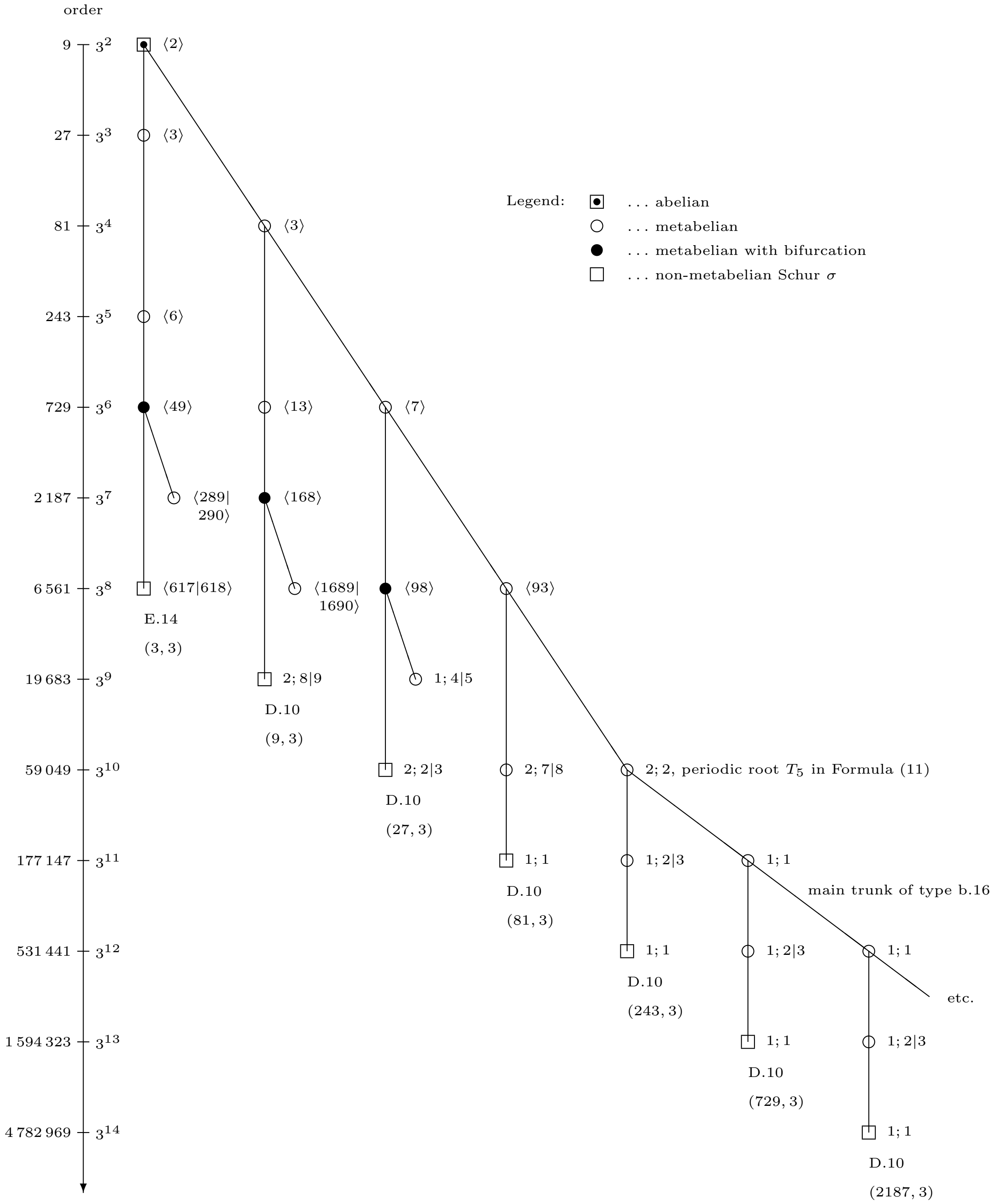
$$G \simeq A_5(-\#1; 1)^{e-5} - \#1; i - \#1; 1, \qquad i \in \{2, 3\}. \tag{9}$$

- *For type* D.6, $\varkappa \sim (123;1)$, $\alpha \sim ((e+1)1,(e+1)1,(e+1)1;e22)$, $\varrho \sim (2,2,2;3)$, *let* $A_0 := \mathrm{SmallGroup}(6561,85) - \#2;4$, *then*

$$G \simeq A_0(-\#1;1)^{e-5} - \#1;i - \#1;1, \qquad i \in \{2,3\}. \tag{10}$$

Proof. (Theorem 3) The proof consists of the construction of successive descendants of A_2, A_4, A_5, A_0 in the way indicated in Theorem 3 by means of the p-group generation algorithm [13] by Newman [27] and O'Brien [28], implemented in Magma [6, 7, 15]. Cf. [24, Thm. 12–13]. □

FIGURE 3. Schur σ-groups G with $\varrho(G) \sim (2,2,3;3)$, $G/G' \simeq (3^e,3)$, $2 \le e \le 7$



After Theorem 3, we begin with a thorough investigation of the ground state of type D.10.

Theorem 4. *For each logarithmic exponent* $e \ge 5$*, a metabelian* CF*-group* T_e *with commutator quotient* $T_e/T_e' \simeq (3^e,3)\hat{=}(e1)$*, punctured transfer kernel type* b.16*,* $\varkappa(T_e) \sim (004;0)$*, and rank distribution* $\varrho \sim (2,2,3;3)$ *is given by the* ***periodic sequence*** *of iterated* p*-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561,93) - \#2;2(-\#1;1)^{e-5}. \tag{11}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 5+e$ *and first abelian quotient invariants* $\alpha_1(T_e) \sim (e1,e1,e22;(e-1)11)$*. They form the infinite main trunk of a descendant tree with finite double-twigs of depth two. For each integer* $e \ge 5$*, the doublet*

$$M_{e,i} \simeq \mathrm{SmallGroup}(6561,93) - \#2;2(-\#1;1)^{e-5} - \#1;i, \qquad i \in \{2,3\}, \tag{12}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561,93) - \#2;2(-\#1;1)^{e-5} - \#1;i - \#1;1, \qquad i \in \{2,3\}, \tag{13}$$

is the ***unique pair*** *of* ***metabelian*** σ*-groups* $M_{e,i} \simeq G_{e,i}/G_{e,i}''$*, respectively* ***non-metabelian*** *Schur* σ*-groups* $G_{e,i}$*, with commutator quotient* $(3^e,3)\hat{=}(e1)$*, punctured transfer kernel type* D.10*,* $\varkappa \sim (114;3)$*,* $\mathrm{lo}(M_{e,i}) = 6+e$*, respectively* $\mathrm{lo}(G_{e,i}) = 7+e$*, and first abelian quotient invariants* $\alpha_1 \sim ((e+1)1,(e+1)1,e22;e11)$*. See Figure 3.*

Proof. Let $s_2 = [y,x]$ denote the main commutator, $s_3 = [s_2,x]$, $s_4 = [s_3,x]$, $s_5 = [s_4,x]$ and $t_3 = [s_2,y]$, $t_4 = [s_3,y]$, $t_5 = [s_4,y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \ge 5$, we have parametrized presentations with $i \in \{2,3\}$

$$\begin{aligned} T_{e+1} &= \langle x,y \mid w^3=1,\ y^3=s_5,\ s_2^3=s_4^2s_5,\ s_3^3=s_5^2,\ t_3=s_3,\ t_4=s_4,\ t_5=s_5\rangle, \\ M_{e,i} &= \langle x,y \mid w^3=1,\ y^3=s_5,\ s_2^3=s_4^2s_5,\ s_3^3=s_5^2,\ t_3=s_3w^{i-1},\ t_4=s_4,\ t_5=s_5\rangle. \end{aligned} \tag{14}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e+1$ respectively $\mathrm{cl}_p(T_{e+1}) = e+1$. For $e \ge 5$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w\rangle$ respectively $P_e(T_{e+1}) = \langle w\rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce Figure 3, and thus confirm Formulas (11), (12) and (13). All groups have $\varrho \sim (2,2,3;3)$. Cf. [24, Thm. 11–13]. □

We continue with the ground state of type C.4.

Theorem 5. *For each logarithmic exponent* $e \ge 5$*, a metabelian* CF*-group* T_e *with commutator quotient* $T_e/T_e' \simeq (3^e,3)\hat{=}(e1)$*, punctured transfer kernel type* a.1*,* $\varkappa(T_e) \sim (000;0)$*, and rank distribution* $\varrho \sim (2,2,3;3)$ *is given by the* ***periodic sequence*** *of iterated* p*-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561,93) - \#2;5(-\#1;1)^{e-5}. \tag{15}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 5+e$ *and first abelian quotient invariants* $\alpha_1(T_e) \sim (e1,e1,e21;(e-1)11)$*. They form the infinite main trunk of a descendant tree with finite double-twigs of depth two. For each integer* $e \ge 5$*, the doublet*

$$M_{e,i} \simeq \mathrm{SmallGroup}(6561,93) - \#2;5(-\#1;1)^{e-5} - \#1;i, \qquad i \in \{2,3\}, \tag{16}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561,93) - \#2;5(-\#1;1)^{e-5} - \#1;i - \#1;1, \qquad i \in \{2,3\}, \tag{17}$$

is the ***unique pair*** *of* ***metabelian*** σ*-groups* $M_{e,i} \simeq G_{e,i}/G_{e,i}''$*, respectively* ***non-metabelian*** *Schur* σ*-groups* $G_{e,i}$*, with commutator quotient* $(3^e,3)\hat{=}(e1)$*, punctured transfer kernel type* C.4*,* $\varkappa \sim (113;3)$*,* $\mathrm{lo}(M_{e,i}) = 6+e$*, respectively* $\mathrm{lo}(G_{e,i}) = 7+e$*, and first abelian quotient invariants* $\alpha_1 \sim ((e+1)1,(e+1)1,(e+1)21;e11)$*.*

Proof. Let $s_2 = [y, x]$ denote the main commutator, $s_3 = [s_2, x]$, $s_4 = [s_3, x]$, $s_5 = [s_4, x]$ and $t_3 = [s_2, y]$, $t_4 = [s_3, y]$, $t_5 = [s_4, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 5$, we have parametrized presentations with $i \in \{2, 3\}$

$$\begin{aligned} T_{e+1} &= \langle x, y \mid w^3 = 1,\ y^3 = s_5,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_3 s_5,\ t_4 = s_4,\ t_5 = s_5 \rangle, \\ M_{e,i} &= \langle x, y \mid w^3 = 1,\ y^3 = s_5,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_3 s_5 w^{i-1},\ t_4 = s_4,\ t_5 = s_5 \rangle. \end{aligned} \tag{18}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 5$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce a graph isomorphic to the tree in Figure 3, and thus confirm Formulas (15), (16) and (17). All groups have $\varrho \sim (2, 2, 3; 3)$. Cf. [24, Thm. 11–13]. □

Next, we look at the ground state of type D.5.

Theorem 6. *For each logarithmic exponent $e \geq 5$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* a.1*, $\varkappa(T_e) \sim (000; 0)$, and rank distribution $\varrho \sim (2, 2, 3; 3)$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561, 93) - \#2; 4(-\#1; 1)^{e-5}. \tag{19}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 5 + e$ and first abelian quotient invariants $\alpha_1(T_e) \sim (e1, e1, e21; (e-1)11)$. They form the infinite main trunk of a descendant tree with finite double-twigs of depth two. For each integer $e \geq 5$, the doublet

$$M_{e,i} \simeq \mathrm{SmallGroup}(6561, 93) - \#2; 4(-\#1; 1)^{e-5} - \#1; i, \qquad i \in \{2, 3\}, \tag{20}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561, 93) - \#2; 4(-\#1; 1)^{e-5} - \#1; i - \#1; 1, \qquad i \in \{2, 3\}, \tag{21}$$

is the ***unique pair*** *of* ***metabelian*** *σ-groups $M_{e,i} \simeq G_{e,i}/G_{e,i}''$, respectively* ***non-metabelian*** *Schur σ-groups $G_{e,i}$, with commutator quotient $(3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* D.5*, $\varkappa \sim (112; 3)$, $\mathrm{lo}(M_{e,i}) = 6 + e$, respectively $\mathrm{lo}(G_{e,i}) = 7 + e$, and first abelian quotient invariants $\alpha_1 \sim ((e+1)1, (e+1)1, (e+1)21; e11)$.*

Proof. Let $s_2 = [y, x]$ denote the main commutator, $s_3 = [s_2, x]$, $s_4 = [s_3, x]$, $s_5 = [s_4, x]$ and $t_3 = [s_2, y]$, $t_4 = [s_3, y]$, $t_5 = [s_4, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 5$, we have parametrized presentations with $i \in \{2, 3\}$

$$\begin{aligned} T_{e+1} &= \langle x, y \mid w^3 = 1,\ y^3 = s_5^2,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_3 s_5,\ t_4 = s_4,\ t_5 = s_5 \rangle, \\ M_{e,i} &= \langle x, y \mid w^3 = 1,\ y^3 = s_5^2,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_3 s_5 w^{i-1},\ t_4 = s_4,\ t_5 = s_5 \rangle. \end{aligned} \tag{22}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 5$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce a graph isomorphic to the tree in Figure 3, and thus confirm Formulas (19), (20) and (21). All groups have $\varrho \sim (2, 2, 3; 3)$. Cf. [24, Thm. 11–13]. □

Finally, we come to the ground state of type D.6.

Theorem 7. *For each logarithmic exponent $e \geq 5$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* a.1*, $\varkappa(T_e) \sim (000; 0)$, and rank distribution $\varrho \sim (2, 2, 2; 3)$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561, 85) - \#2; 4(-\#1; 1)^{e-5}. \tag{23}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 5 + e$ and first abelian quotient invariants $\alpha_1(T_e) \sim (e1, e1, e1; (e-1)22)$. They form the infinite main trunk of a descendant tree with finite double-twigs of depth two. For each integer $e \geq 5$, the doublet

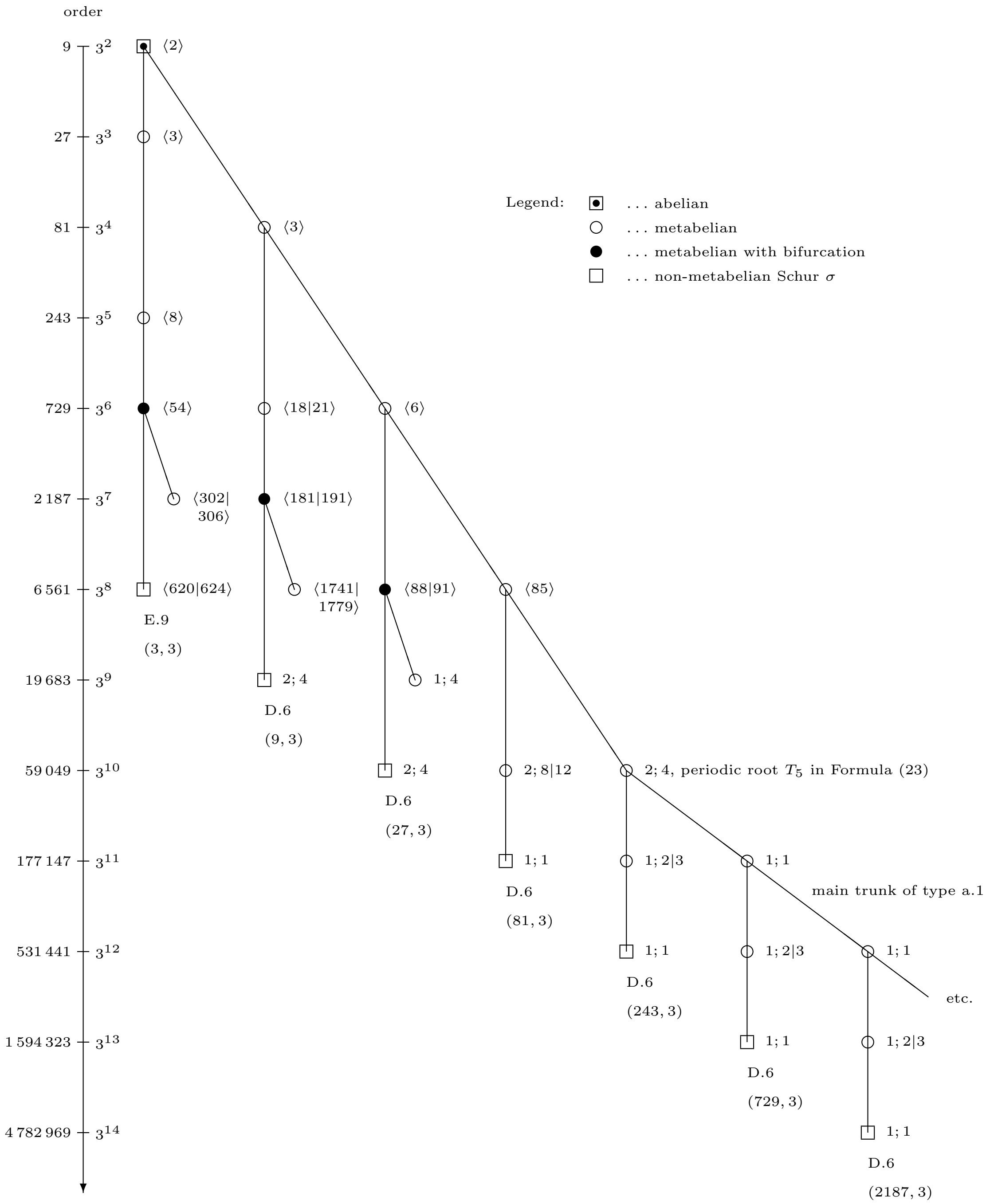
$$M_{e,i} \simeq \mathrm{SmallGroup}(6561, 85) - \#2; 4(-\#1; 1)^{e-5} - \#1; i, \qquad i \in \{2, 3\}, \tag{24}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561,85) - \#2;4(-\#1;1)^{e-5} - \#1;i - \#1;1, \qquad i \in \{2,3\}, \tag{25}$$

is the ***unique pair*** *of* ***metabelian*** *σ-groups $M_{e,i} \simeq G_{e,i}/G''_{e,i}$, respectively* ***non-metabelian*** *Schur σ-groups $G_{e,i}$, with commutator quotient $(3^e,3)\hat{=}(e1)$, punctured transfer kernel type* D.6*, $\varkappa \sim (123;1)$,* $\mathrm{lo}(M_{e,i}) = 6+e$*, respectively* $\mathrm{lo}(G_{e,i}) = 7+e$*, and first abelian quotient invariants $\alpha_1 \sim ((e+1)1,(e+1)1,(e+1)1;e22)$. See Figure 4.*

FIGURE 4. Schur σ-groups G with $\varrho(G) \sim (2,2,2;3)$, $G/G' \simeq (3^e,3)$, $2 \le e \le 7$



Proof. Let $s_2 = [y, x]$ denote the main commutator, $s_3 = [s_2, x]$, $s_4 = [s_3, x]$, $s_5 = [s_4, x]$ and $t_3 = [s_2, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 5$, we have parametrized presentations

$$\begin{aligned} T_{e+1} &= \langle x, y \mid w^3 = 1,\ y^3 = s_3^2 s_4,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_5 \rangle, \\ M_{e,i} &= \langle x, y \mid w^3 = 1,\ y^3 = s_3^2 s_4,\ s_2^3 = s_4^2 s_5,\ s_3^3 = s_5^2,\ t_3 = s_5 w^{i-1} \rangle, \qquad i \in \{2, 3\}. \end{aligned} \tag{26}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 5$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce Figure 4, and thus confirm Formulas (23), (24) and (25). All groups have $\varrho \sim (2, 2, 2; 3)$. □

5. Moderate Schur σ-groups G with $G/G' \simeq (3^e, 3)$, $e \geq 7$, excited state

We begin with a thorough investigation of the first excited state of type D.10.

Theorem 8. *For each logarithmic exponent $e \geq 7$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* b.16*, $\varkappa(T_e) \sim (004; 0)$, and rank distribution $\varrho \sim (2, 2, 3; 3)$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 2(-\#1; 1)^{e-7}. \tag{27}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 7 + e$ and first abelian quotient invariants $\alpha_1(T_e) \sim (e1, e1, e33; (e-1)11)$. They form the infinite main trunk of a descendant tree with finite double-twigs of depth three. For each integer $e \geq 7$, the doublet

$$M_{e,i} \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 2(-\#1; 1)^{e-7} - \#1; i, \qquad i \in \{2, 3\}, \tag{28}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 2(-\#1; 1)^{e-7} - \#1; i(-\#1; 1)^2, \qquad i \in \{2, 3\}, \tag{29}$$

is the ***unique pair*** *of* ***metabelian*** *σ-groups $M_{e,i} \simeq G_{e,i}/G_{e,i}''$, respectively* ***non-metabelian*** *Schur σ-groups $G_{e,i}$, with commutator quotient $(3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* D.10*, $\varkappa \sim (114; 3)$, $\mathrm{lo}(M_{e,i}) = 8 + e$, respectively $\mathrm{lo}(G_{e,i}) = 10 + e$, and first abelian quotient invariants $\alpha_1 \sim ((e+1)1, (e+1)1, e33; e11)$. See Figure 5.*

Proof. Let $s_2 = [y, x]$ denote the main commutator, $\forall_{j=3}^{7}\ s_j = [s_{j-1}, x]$ and $t_j = [s_{j-1}, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 7$, we have parametrized presentations with $i \in \{2, 3\}$

$$\begin{aligned} T_{e+1} &= \langle x, y \mid w^3 = 1,\ y^3 = s_7,\ \forall_{j=2}^{4}\, s_j^3 = s_{j+2}^2 s_{j+3},\ s_5^3 = s_7^2,\ \forall_{j=3}^{7}\, t_j = s_j \rangle, \\ M_{e,i} &= \langle x, y \mid w^3 = 1,\ y^3 = s_7,\ \forall_{j=2}^{4}\, s_j^3 = s_{j+2}^2 s_{j+3},\ s_5^3 = s_7^2,\ t_3 = s_3 w^{i-1},\ \forall_{j=4}^{7}\, t_j = s_j \rangle. \end{aligned} \tag{30}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 7$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce Figure 5, and thus confirm Formulas (27), (28) and (29). All groups have $\varrho \sim (2, 2, 3; 3)$. Cf. [26, Thm. 1–3]. □

We continue with the first excited state of type C.4.

Theorem 9. *For each logarithmic exponent $e \geq 7$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* a.1*, $\varkappa(T_e) \sim (000; 0)$, and rank distribution $\varrho \sim (2, 2, 3; 3)$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 5(-\#1; 1)^{e-7}. \tag{31}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 7 + e$ *and first abelian quotient invariants* $\alpha_1(T_e) \sim (e1, e1, e32; (e-1)11)$. *They form the infinite main trunk of a descendant tree with finite double-twigs of depth three. For each integer* $e \geq 7$, *the doublet*

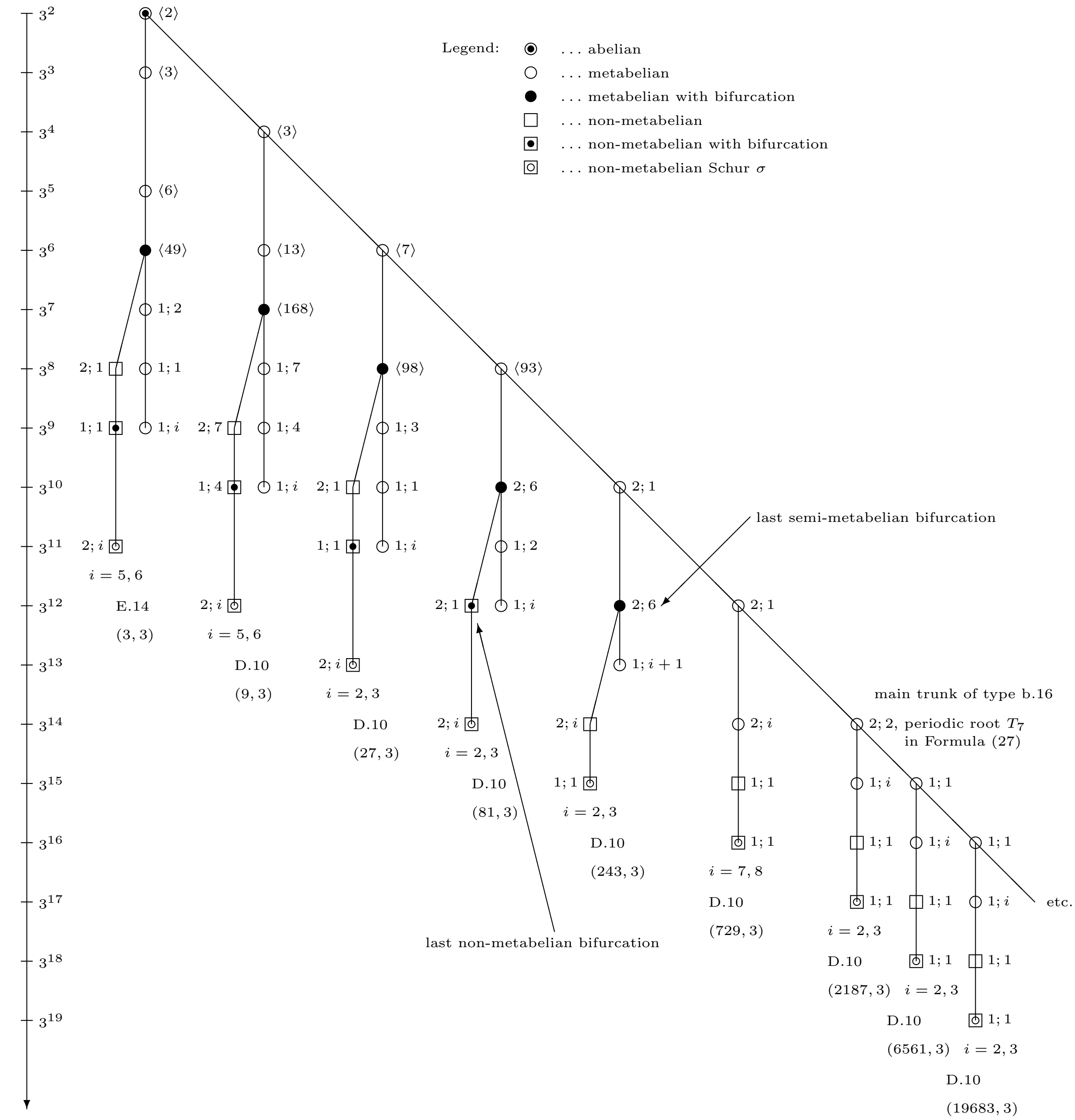
$$M_{e,i} \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 5(-\#1; 1)^{e-7} - \#1; i, \qquad i \in \{2, 3\}, \tag{32}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561, 93)(-\#2; 1)^2 - \#2; 5(-\#1; 1)^{e-7} - \#1; i(-\#1; 1)^2, \qquad i \in \{2, 3\}, \tag{33}$$

is the ***unique pair*** *of* ***metabelian*** σ*-groups* $M_{e,i} \simeq G_{e,i}/G''_{e,i}$, *respectively* ***non-metabelian*** *Schur* σ*-groups* $G_{e,i}$, *with commutator quotient* $(3^e, 3) \hat{=} (e1)$, *punctured transfer kernel type* C.4,

FIGURE 5. Schur σ-groups G with $\varrho(G) \sim (2, 2, 3; 3)$, $G/G' \simeq (3^e, 3)$, $2 \leq e \leq 9$



$\varkappa \sim (113;3)$, $\mathrm{lo}(M_{e,i}) = 8+e$, *respectively* $\mathrm{lo}(G_{e,i}) = 10+e$, *and first abelian quotient invariants* $\alpha_1 \sim ((e+1)1,(e+1)1,(e+1)32;e11)$.

Proof. Let $s_2 = [y,x]$ denote the main commutator, $\forall_{j=3}^{7}\ s_j = [s_{j-1},x]$ and $t_j = [s_{j-1},y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \ge 7$, we have parametrized presentations with $i \in \{2,3\}$

$$\begin{aligned} T_{e+1} &= \langle x,y \mid w^3=1,\ y^3=s_7,\ \forall_{j=2}^{4} s_j^3 = s_{j+2}^2 s_{j+3}, s_5^3 = s_7^2,\ t_3 = s_3 s_7,\ \forall_{j=4}^{7} t_j = s_j\rangle, \\ M_{e,i} &= \langle x,y \mid w^3=1,\ y^3=s_7,\ \forall_{j=2}^{4} s_j^3 = s_{j+2}^2 s_{j+3}, s_5^3 = s_7^2,\ t_3 = s_3 s_7 w^{i-1},\ \forall_{j=4}^{7} t_j = s_j\rangle. \end{aligned} \tag{34}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e+1$ respectively $\mathrm{cl}_p(T_{e+1}) = e+1$. For $e \ge 7$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w\rangle$ respectively $P_e(T_{e+1}) = \langle w\rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce a graph isomorphic to the tree in Figure 5, and thus confirm Formulas (31), (32) and (33). All groups have $\varrho \sim (2,2,3;3)$. Cf. [26, Thm. 1–3]. □

Next, we look at the first excited state of type D.5.

Theorem 10. *For each logarithmic exponent $e \ge 7$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e,3) \hat{=} (e1)$, punctured transfer kernel type* a.1, $\varkappa(T_e) \sim (000;0)$, *and rank distribution $\varrho \sim (2,2,3;3)$ is given by the **periodic sequence** of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561,93)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7}. \tag{35}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 7+e$ and first abelian quotient invariants $\alpha_1(T_e) \sim (e1,e1,e32;(e-1)11)$. They form the infinite main trunk of a descendant tree with finite double-twigs of depth three. For each integer $e \ge 7$, the doublet

$$M_{e,i} \simeq \mathrm{SmallGroup}(6561,93)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7} - \#1;i, \qquad i \in \{2,3\}, \tag{36}$$

respectively

$$G_{e,i} \simeq \mathrm{SmallGroup}(6561,93)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7} - \#1;i(-\#1;1)^2, \qquad i \in \{2,3\}, \tag{37}$$

*is the **unique pair** of **metabelian** σ-groups $M_{e,i} \simeq G_{e,i}/G_{e,i}''$, respectively **non-metabelian** Schur σ-groups $G_{e,i}$, with commutator quotient $(3^e,3) \hat{=} (e1)$, punctured transfer kernel type* D.5, $\varkappa \sim (112;3)$, $\mathrm{lo}(M_{e,i}) = 8+e$, *respectively* $\mathrm{lo}(G_{e,i}) = 10+e$, *and first abelian quotient invariants* $\alpha_1 \sim ((e+1)1,(e+1)1,(e+1)32;e11)$.

Proof. Let $s_2 = [y,x]$ denote the main commutator, $\forall_{j=3}^{7}\ s_j = [s_{j-1},x]$ and $t_j = [s_{j-1},y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \ge 7$, we have parametrized presentations with $i \in \{2,3\}$

$$\begin{aligned} T_{e+1} &= \langle x,y \mid w^3=1,\ y^3=s_7^2,\ \forall_{j=2}^{4} s_j^3 = s_{j+2}^2 s_{j+3}, s_5^3 = s_7^2,\ t_3 = s_3 s_7,\ \forall_{j=4}^{7} t_j = s_j\rangle, \\ M_{e,i} &= \langle x,y \mid w^3=1,\ y^3=s_7^2,\ \forall_{j=2}^{4} s_j^3 = s_{j+2}^2 s_{j+3}, s_5^3 = s_7^2,\ t_3 = s_3 s_7 w^{i-1},\ \forall_{j=4}^{7} t_j = s_j\rangle. \end{aligned} \tag{38}$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e+1$ respectively $\mathrm{cl}_p(T_{e+1}) = e+1$. For $e \ge 7$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w\rangle$ respectively $P_e(T_{e+1}) = \langle w\rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce a graph isomorphic to the tree in Figure 5, and thus confirm Formulas (35), (36) and (37). All groups have $\varrho \sim (2,2,3;3)$. Cf. [26, Thm. 1–3]. □

Finally, we come to the first excited state of type D.6.

Theorem 11. *For each logarithmic exponent $e \ge 7$, a metabelian* CF*-group T_e with commutator quotient $T_e/T_e' \simeq (3^e,3) \hat{=} (e1)$, punctured transfer kernel type* a.1, $\varkappa(T_e) \sim (000;0)$, *and rank distribution $\varrho \sim (2,2,2;3)$ is given by the **periodic sequence** of iterated p-descendants*

$$T_e \simeq \mathrm{SmallGroup}(6561,85)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7}. \tag{39}$$

These groups have logarithmic order $\mathrm{lo}(T_e) = 7 + e$ *and first abelian quotient invariants* $\alpha_1(T_e) \sim (e1, e1, e1; (e-1)33)$. *They form the infinite main trunk of a descendant tree with finite double-twigs of depth three. For each integer* $e \geq 7$*, the doublet*

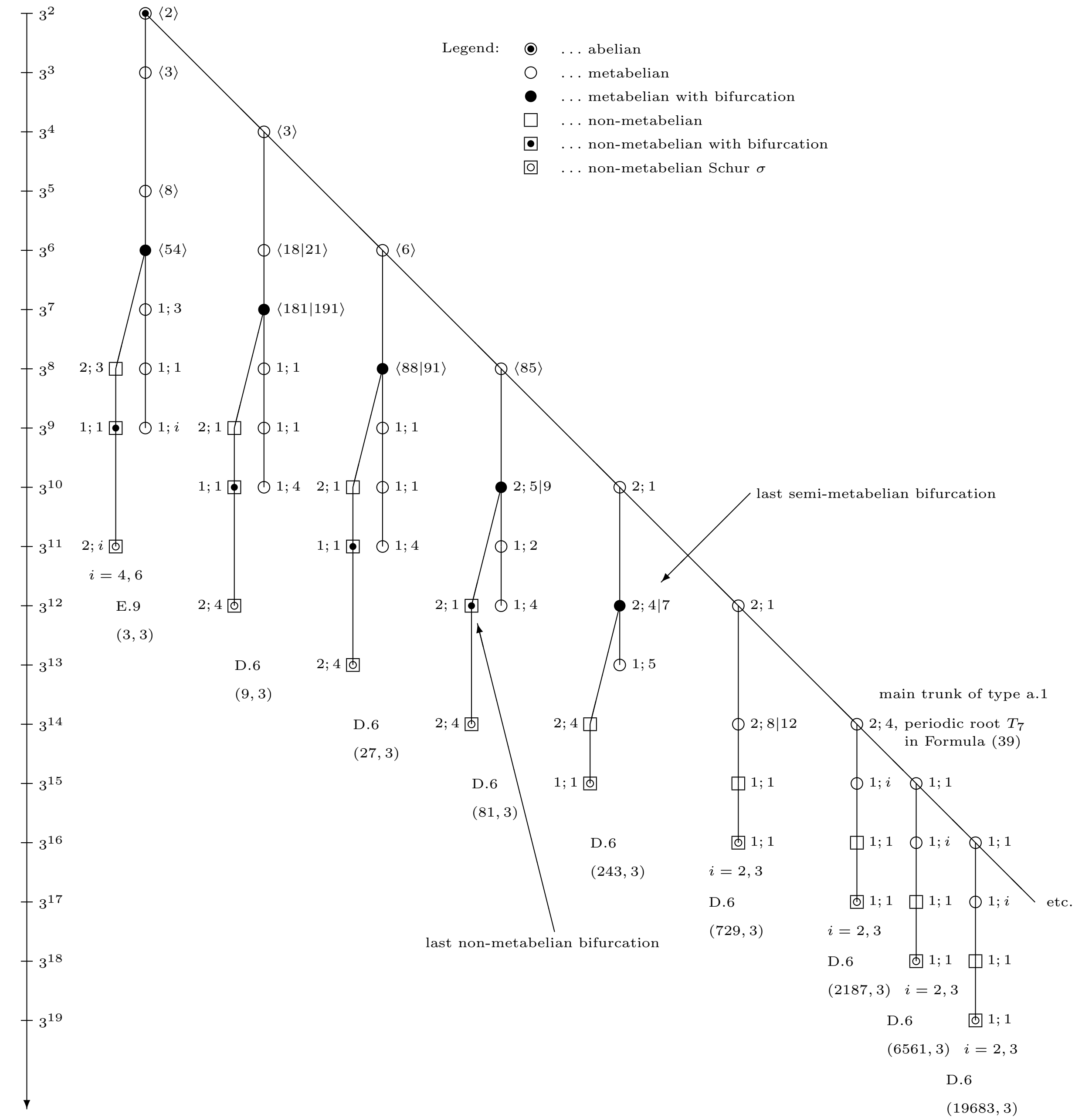
(40) $$M_{e,i} \simeq \mathrm{SmallGroup}(6561, 85)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7} - \#1;i, \qquad i \in \{2,3\},$$

respectively

(41) $$G_{e,i} \simeq \mathrm{SmallGroup}(6561, 85)(-\#2;1)^2 - \#2;4(-\#1;1)^{e-7} - \#1;i(-\#1;1)^2, \qquad i \in \{2,3\},$$

is the ***unique pair*** *of* ***metabelian*** σ*-groups* $M_{e,i} \simeq G_{e,i}/G''_{e,i}$*, respectively* ***non-metabelian*** *Schur* σ*-groups* $G_{e,i}$*, with commutator quotient* $(3^e, 3) \hat{=} (e1)$*, punctured transfer kernel type* D.6*,*

FIGURE 6. Schur σ-groups G with $\varrho(G) \sim (2,2,2;3)$, $G/G' \simeq (3^e, 3)$, $2 \leq e \leq 9$



$\varkappa \sim (123;1)$, $\mathrm{lo}(M_{e,i}) = 8 + e$, *respectively* $\mathrm{lo}(G_{e,i}) = 10 + e$, *and first abelian quotient invariants* $\alpha_1 \sim ((e+1)1,(e+1)1,(e+1)1;e33)$. *See Figure 6.*

Proof. Let $s_2 = [y,x]$ denote the main commutator, $\forall_{j=3}^{7}\ s_j = [s_{j-1},x]$ and $t_3 = [s_2,y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 7$, we have parametrized presentations with $i \in \{2,3\}$

$$
\begin{aligned}
T_{e+1} &= \langle x, y \mid w^3 = 1,\ y^3 = s_3^2 s_4,\ \forall_{j=2}^{4}\, s_j^3 = s_{j+2}^2 s_{j+3},\ s_5^3 = s_7^2,\ t_3 = s_7 \rangle, \\
M_{e,i} &= \langle x, y \mid w^3 = 1,\ y^3 = s_3^2 s_4,\ \forall_{j=2}^{4}\, s_j^3 = s_{j+2}^2 s_{j+3},\ s_5^3 = s_7^2,\ t_3 = s_7 w^{i-1} \rangle.
\end{aligned} \tag{42}
$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1}) = e + 1$. For $e \geq 7$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq T_{e+1}/P_e(T_{e+1}) \simeq T_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce Figure 6, and thus confirm Formulas (39), (40) and (41). All groups have $\varrho \sim (2,2,2;3)$. □

6. Elevated Schur σ-groups G with $G/G' \simeq (3^e,3)$, $e \geq 9$

Investigation of Schur σ-groups G with non-elementary bicyclic commutator quotient $G/G' \simeq (3^e,3)$, $e \geq 4$, punctured transfer kernel type B.18, $\varkappa(G) \sim (144;4)$, and lowest possible logarithmic order $\mathrm{lo}(G) = 19+e$ is firmly based on a crucial vertex T_4 of the main trunk with associated scaffold type b.31, $\varkappa(T_4) \sim (044;4)$. The identifier of this fork between non-metabelian Schur σ-groups G and their metabelianizations $M = G/G''$ is $T_4 = \langle 2187, 3\rangle - \#3; 2$. It may be called a *bifurcation of infinite order*, since it is responsible for all values $e \geq 4$ of the logarithmic exponent. All the groups involved have elevated rank distribution $\varrho \sim (3,3,3;3)$ [25].

Periodicity of the metabelianizations M sets in for $e \geq 5$ already.

Theorem 12. *For each logarithmic exponent* $e \geq 5$, *a metabelian* BCF*-group* U_e *with commutator quotient* $U_e/U_e' \simeq (3^e,3) \hat{=} (e1)$, *punctured transfer kernel type* b.31, $\varkappa(U_e) \sim (044;4)$, *and elevated rank distribution* $\varrho \sim (3,3,3;3)$ *is given by the* ***periodic sequence*** *of iterated* p*-descendants*

$$U_e \simeq \mathrm{SmallGroup}(2187,3) - \#3;2 - \#2;93(-\#1;1)^{e-5}. \tag{43}$$

These groups have logarithmic order $\mathrm{lo}(U_e) = 7 + e$ *and first abelian quotient invariants* $\alpha_1(U_e) \sim (e21, e11, e11; (e-1)21)$. *They form the infinite main trunk of a descendant tree with finite twigs of depth one, consisting of a metabelian doublet and* 24 *non-metabelian vertices. For each integer* $e \geq 5$, *the doublet*

$$M_{e,i} \simeq \mathrm{SmallGroup}(2187,3) - \#3;2 - \#2;93(-\#1;1)^{e-5} - \#1;i, \qquad i \in \{2,3\}, \tag{44}$$

is the ***unique pair*** *of* ***metabelian*** σ*-groups with commutator quotient* $(3^e,3) \hat{=} (e1)$, *punctured transfer kernel type* B.18, $\varkappa \sim (144;4)$, $\mathrm{lo}(M_{e,i}) = 8 + e$, *and first abelian quotient invariants* $\alpha_1(M_{e,i}) \sim ((e+1)21, e11, e11; (e-1)21)$. *See Figure 8.*

Proof. Let $s_2 = t_2 = [y,x]$ denote the main commutator, $\forall_{j=3}^{5}\ s_j = [s_{j-1},x]$ and $t_j = [t_{j-1},y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. For each $e \geq 5$, we have parametrized presentations with $i \in \{2,3\}$

$$
\begin{aligned}
U_{e+1} = \langle x, y \mid & w^3 = 1,\ y^3 = s_3 s_4^2,\ s_2^3 = s_4 t_4^2,\ s_3^3 = s_5,\ t_3^3 = s_5^2, \\
& [x^3, y] = s_4 t_4 s_5^2,\ [x^3, s_2] = s_5,\ t_5 = s_5 \rangle, \\
M_{e,i} = \langle x, y \mid & w^3 = 1,\ y^3 = s_3 s_4^2,\ s_2^3 = s_4 t_4^2 w^{i-1},\ s_3^3 = s_5,\ t_3^3 = s_5^2 (w^{i-1})^2, \\
& [x^3, y] = s_4 t_4 s_5^2 (w^{i-1})^2,\ [x^3, s_2] = s_5,\ t_5 = s_5 w^{i-1} \rangle.
\end{aligned} \tag{45}
$$

The p-class is given by $\mathrm{cl}_p(M_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(U_{e+1}) = e + 1$. For $e \geq 5$, the last non-trivial lower p-central is given by $P_e(M_{e,i}) = \langle w \rangle$ respectively $P_e(U_{e+1}) = \langle w \rangle$, whence all three groups share the common p-parent $M_{e,i}/P_e(M_{e,i}) \simeq U_{e+1}/P_e(U_{e+1}) \simeq U_e$.

Repeated recursive applications of the p-group generation algorithm eventually produce the upper part of Figure 8, and thus confirm Formulas (43) and (44). Cf. [25, Cor. 1]. □

Periodicity of the Schur σ-groups G sets in for $e \geq 9$. According to [25, Thm. 1–7], the groups with minimal logarithmic order $\mathrm{lo}(G) = 19 + e$ are p-descendants of the seven roots $\mathrm{SmallGroup}(2187, 3) - \#3; 2 - \#4; \ell$ with $\ell \in \{24, 26, 28, 30, 31, 33, 37\}$. We abstain from complete generality and take $\ell = \mathbf{37}$, exemplarily.

Theorem 13. *A total of* 162 *Schur* σ*-groups* G *with commutator quotient* $G/G' \simeq (3^e, 3)$*, punctured transfer kernel type* B.18*,* $\varkappa(G) \sim (144; 4)$*, elevated rank distribution* $\varrho(G) = (3, 3, 3; 3)$*, first abelian quotient invariants* $\alpha_1(G) \sim [(e+1)21, e11, e11; (e-1)21]$*, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[(e+1)21; e2111, ((e+1)211)^3, ((e+1)2)^9],\\ &[e11; e2111, ((e+1)21)^3, ((e+1)2)^9],\\ &[e11; e2111, ((e+1)21)^3, ((e+1)2)^9];\\ &[(e-1)21; e2111, (e211)^3, (e21)^8, (e-1)22])\end{aligned} \tag{46}$$

and (minimal) logarithmic order $\mathrm{lo}(G) = 19 + e$ *is given for each* $e \geq 9$ *by the term*

$$G = T_{9,a,b}[-\#1; 1]^{e-9} - \#1; i - \#1; k - \#1; 1 \text{ with } i \in \{2, 3\} \text{ and } k \in \{1, 2, 3\}, \tag{47}$$

where 27 *periodic roots with* $1 \leq a \leq 9$*,* $\tilde{a} = 1$ *for* $a \in \{2, 6, 7\}$*,* $\tilde{a} = 2$ *otherwise, and* $1 \leq b \leq 3$ *are*

$$T_{9,a,b} := \mathrm{SmallGroup}(2187, 3) - \#3; 2 - \#\mathbf{4}; \mathbf{37} - \#3; 32 - \#4; a - \#2; \tilde{a} - \#2; b. \tag{48}$$

Proof. For a fixed step size $s \geq 1$, we denote by N the number of all immediate descendants of a 3-group, and by C the number of capable immediate descendants with positive nuclear rank $\nu \geq 1$. Generally, let $X := \langle 2187, 3\rangle - \#3; 2 - \#4; 37 - \#3; 32$. This is a non-metabelian 3-group of type $(729, 3)$. We consider a chain of exo-genetic propagations [24]:

- X has $N = C = 27$ for $s = \nu = 4$ but only the first 9 descendants are of type $(2187, 3)$.
- Each $X - \#4; a$ with $1 \leq a \leq 9$ has $N = C = 6$ for $s = \nu = 2$ but only the first, resp. second, descendant, indicated by $\tilde{a} \in \{1, 2\}$, is of type $(6561, 3)$.
- Each $X - \#4; a - \#2, \tilde{a}$ with $1 \leq a \leq 9$ has $N = C = 9$ for $s = \nu = 2$ but only the first 3 descendants are of type $(19683, 3)$.
- Each $T_{9,a,b} := X - \#4; a - \#2, \tilde{a} - \#2; b$ with $1 \leq a \leq 9$ and $1 \leq b \leq 3$ has 6 Schur σ-descendants $T_{9,a,b}[-\#1; 1]^{e-9} - \#1; i - \#1; k - \#1; 1$ with $i \in \{2, 3\}$ and $k \in \{1, 2, 3\}$, for each $e \geq 9$.

Together this census yields $9 \cdot 3 \cdot 6 = 162$ Schur σ-groups, for each $e \geq 9$. Cf. [25, Thm. 6–7]. $\square$

We expand the proof with more details for the particular instance $a = 1$, $\tilde{a} = 2$ and $b = 1$.

Theorem 14. *For each logarithmic exponent $e \ge 9$, a metabelian CF-group $T_{e,1,1}$ with commutator quotient $T_{e,1,1}/T'_{e,1,1} \simeq (3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* b.31, $\varkappa(T_{e,1,1}) \sim (044; 4)$, *and rank distribution $\varrho(T_{e,1,1}) \sim (3, 3, 3; 3)$ is given by the* ***periodic sequence*** *of iterated p-descendants*

$$T_{e,1,1} \simeq T_{9,1,1}(-\#1; 1)^{e-9}. \tag{49}$$

These groups have logarithmic order $\mathrm{lo}(T_{e,1,1}) = 16 + e$ *and first abelian quotient invariants $\alpha_1(T_{e,1,1}) \sim (e21, e11, e11; (e-1)21)$. They form the infinite main trunk of a descendant tree with finite double-twigs of depth three. For each integer $e \ge 9$, the multiplet (doublet or sextet)*

$$X_{e,i} \simeq T_{9,1,1}(-\#1; 1)^{e-9} - \#1; i, \qquad i \in \{2, 3\}, \tag{50}$$

respectively

$$Y_{e,i,k} \simeq T_{9,1,1}(-\#1; 1)^{e-9} - \#1; i - \#1; k, \qquad i \in \{2, 3\},\ k \in \{1, 2, 3\}, \tag{51}$$

respectively

$$G_{e,i,k} \simeq T_{9,1,1}(-\#1; 1)^{e-9} - \#1; i - \#1; k - \#1; 1, \qquad i \in \{2, 3\},\ k \in \{1, 2, 3\}, \tag{52}$$

is a ***multiplet*** *of* ***non-metabelian*** *σ-groups $(X_{e,i})_{i=2,3}$, $(Y_{e,i,k})_{i=2,3;k=1,2,3}$, respectively* ***non-metabelian Schur*** *σ-groups $(G_{e,i,k})_{i=2,3;k=1,2,3}$, with commutator quotient $(3^e, 3) \hat{=} (e1)$, punctured transfer kernel type* B.18, $\varkappa \sim (144; 4)$, $\mathrm{lo}(X_{e,i}) = 17 + e$, $\mathrm{lo}(Y_{e,i,k}) = 18 + e$, *respectively* $\mathrm{lo}(G_{e,i,k}) = 19 + e$, *and first abelian quotient invariants $\alpha_1 \sim ((e+1)21, e11, e11; (e-1)21)$. For $e \ge 9$, the metabelianizations are isomorphic to the groups in Formula (44),*

$$X_{e,i}/X''_{e,i} \simeq Y_{e,i,k}/Y''_{e,i,k} \simeq G_{e,i,k}/G''_{e,i,k} \simeq M_{e,i}, \qquad \text{for all } i \in \{2, 3\},\ k \in \{1, 2, 3\}. \tag{53}$$

See Figure 8. The second derived subgroups are of constant logarithmic order $\mathrm{lo} = 9, 10, 11$, *in fact, they are abelian of constant type*

$$T''_{e,1,1} \simeq X''_{e,i} \simeq (32211), \qquad Y''_{e,i,k} \simeq (322111), \qquad G''_{e,i,k} \simeq (332111). \tag{54}$$

Proof. Let $s_2 = t_2 = [y, x]$ denote the main commutator, $\forall_{j=3}^{9}\ s_j = [s_{j-1}, x]$ and $\forall_{j=3}^{5}\ t_j = [t_{j-1}, y]$ higher commutators, and $w = x^{3^e}$ the last non-trivial power. Since the groups are non-metabelian, additional commutators in the second derived subgroup are required: $\forall_{j=5}^{7}\ u_j = [s_{j-1}, y]$, $v_5 = [t_4, x]$, and $v_7 = [u_6, x]$. The second derived subgroup also contains $s_6, \ldots, s_9$, **but not** w. These facts, together with all relations (which are only given partially below) immediately lead to the metabelianizations

$$T_{e+1,1,1}/T''_{e+1,1,1} \simeq U_{e+1}, \qquad X_{e,i}/X''_{e,i} \simeq M_{e,i}, \tag{55}$$

for $e \ge 9$ and $i \in \{2, 3\}$.

For each $e \ge 9$, we have extensive parametrized presentations with $i \in \{2, 3\}$, which we reduce to the decisive relations **distinct** for $T_{e+1,1,1}$ and $X_{e,i}$.

$$\begin{aligned} T_{e+1,1,1} = \langle x, y \mid\, & s_2^3 = s_4 t_4^2 s_5 u_5 v_5^2 s_7^2 v_7^2 s_8^2,\ t_3^3 = s_5 u_5^2 v_5^2 s_6 v_7^2 s_8^2 s_9^2, \\ & [x^3, y] = s_4 t_4 s_5 v_5^2 s_6 s_7 u_7 v_7 s_9,\ t_5 = s_5^2 u_5^2 v_5 u_6 s_8 \rangle, \\ X_{e,i} = \langle x, y \mid\, & s_2^3 = s_4 t_4^2 s_5 u_5 v_5^2 s_7^2 v_7^2 s_8^2 w^{i-1},\ t_3^3 = s_5 u_5^2 v_5^2 s_6 v_7^2 s_8^2 s_9^2 (w^{i-1})^2, \\ & [x^3, y] = s_4 t_4 s_5 v_5^2 s_6 s_7 u_7 v_7 s_9 (w^{i-1})^2,\ t_5 = s_5^2 u_5^2 v_5 u_6 s_8 w^{i-1} \rangle. \end{aligned} \tag{56}$$

The p-class is given by $\mathrm{cl}_p(X_{e,i}) = e + 1$ respectively $\mathrm{cl}_p(T_{e+1,1,1}) = e + 1$. For $e \ge 9$, the last non-trivial lower p-central is given by $P_e(X_{e,i}) = \langle w \rangle$ respectively $P_e(T_{e+1,1,1}) = \langle w \rangle$, whence all three groups share the common p-parent $X_{e,i}/P_e(X_{e,i}) \simeq T_{e+1,1,1}/P_e(T_{e+1,1,1}) \simeq T_{e,1,1}$.

Repeated recursive applications of the p-group generation algorithm eventually produce the lower part of Figure 8, and thus confirm Formulas (49), (50), (51) and (52). □

FIGURE 7. Schur σ-groups G with $\varrho(G) \sim (3,3,3;3)$, $G/G' \simeq (3^e,3)$, $2 \le e \le 4$

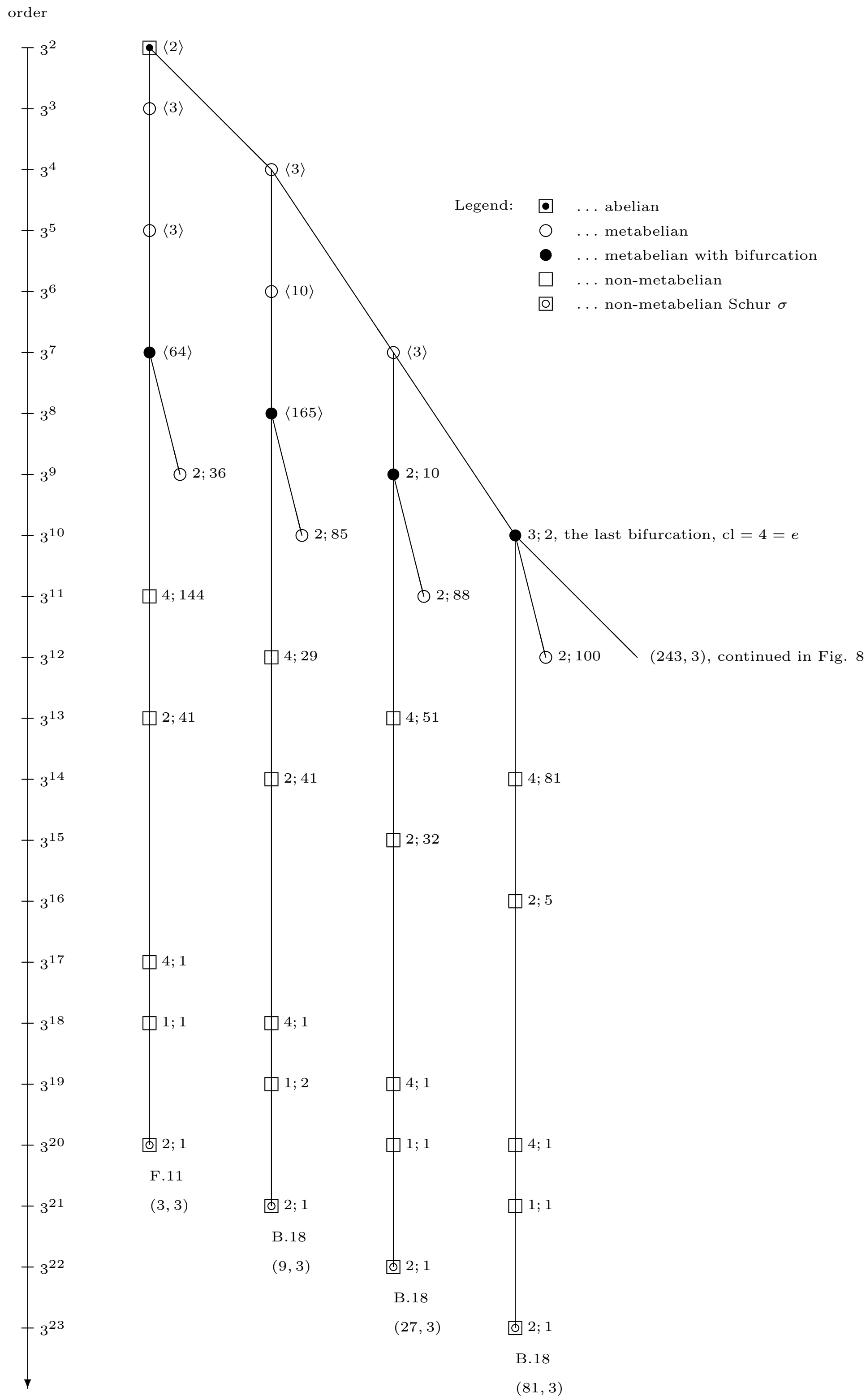
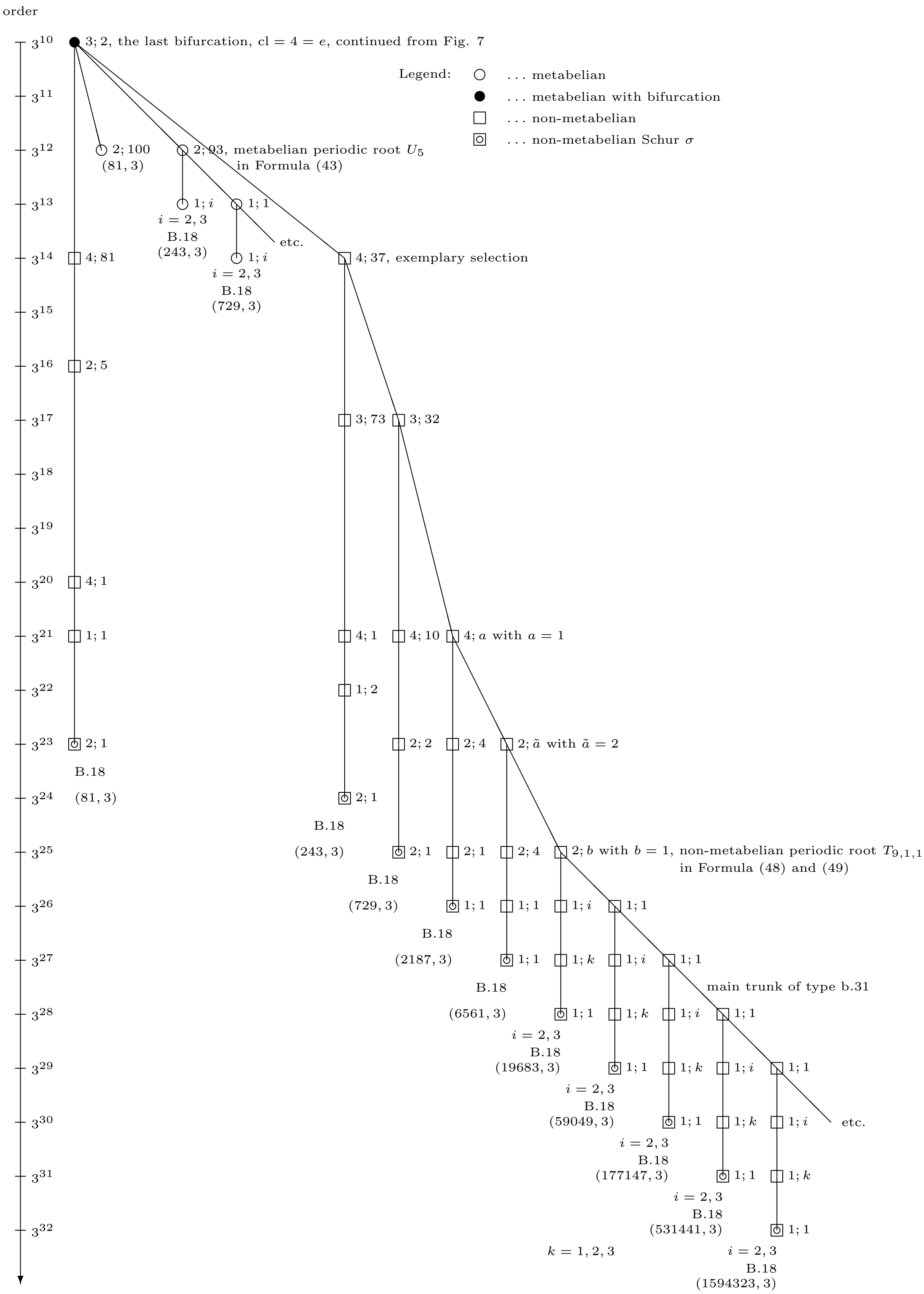


FIGURE 8. Schur σ-groups G with $\varrho(G) \sim (3,3,3;3)$, $G/G' \simeq (3^e,3)$, $4 \le e \le 13$



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