

# THE GROUP $\text{Gal}(M_5^{(2)}|M)$ FOR $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ OF TYPE (5, 5)

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ABSTRACT. Let  $\zeta_5$  be a primitive fifth root of unity and  $d \neq 1$  be a quadratic fundamental discriminant not divisible by 5. For the 5-dual cyclic quartic field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  of the quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$  and  $k_2 = \mathbb{Q}(\sqrt{5d})$  in the sense of the quintic reflection theorem, the possibilities for the isomorphism type of the Galois group  $G_5^{(2)}M = \text{Gal}(M_5^{(2)}|M)$  of the second Hilbert 5-class field  $M_5^{(2)}$  of  $M$  are investigated, when the 5-class group  $\text{Cl}_5M$  has type invariants (5, 5). The precise length  $\ell_5M$  of the 5-class tower of  $M$  is determined, if  $G_5^{(2)}M$  is of order less than or equal  $3125 = 5^5$ . Theoretical results are underpinned by the actual computation of all 83, resp. 109, cases in the range  $0 < d < 10^4$ , resp.  $-2 \cdot 10^5 < d < 0$ .

## 1. INTRODUCTION

The present article arose from the desire to generalize our results [1] for the second 3-class group  $\text{Gal}(k_3^{(2)}|k)$  of the bicyclic biquadratic field  $k = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$  of type (3, 3), which is the compositum of 3-dual quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$  and  $k_2 = \mathbb{Q}(\sqrt{-3d})$  in the cubic reflection theorem, to the situation of the quintic reflection theorem. The precise statement of both reflection theorems requires the concept of *virtual units*. Let  $p$  be a prime number and  $K$  be a number field with multiplicative group  $K^\times = K \setminus \{0\}$ , maximal order  $\mathcal{O}$ , unit group  $U$ , fractional ideal group  $\mathcal{I}$ , and  $p$ -class rank  $\varrho_p$ . The quotient  $V_p = I_p / (K^\times)^p$ , where  $I_p = \{\alpha \in K^\times \mid \alpha\mathcal{O} = \mathfrak{a}^p \text{ for some } \mathfrak{a} \in \mathcal{I}\}$ , is an elementary abelian  $p$ -group of rank  $\sigma_p = \varrho_p + \dim_{\mathbb{F}_p}(U/U^p)$  and is called the  *$p$ -Selmer group* of non-trivial  *$p$ -virtual units*, i.e., generators of principal  $p$ th powers of ideals of  $K$ . We refer to  $\sigma_p$  as the  *$p$ -Selmer rank* of  $K$ .

**1.1. Cubic reflection theorem.** It is well known that the 3-Selmer ranks  $\sigma_3(k_1)$  and  $\sigma_3(k_2)$  of 3-dual quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$  and  $k_2 = \mathbb{Q}(\sqrt{-3d})$  ( $d > 0$  square free) with respect to the quadratic cyclotomic mirror field  $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$ ,  $\zeta_3 = \exp(2\pi i/3)$ , satisfy the *cubic reflection theorem*

$$(1.1) \quad \sigma_3(k_2) = \sigma_3(k_1) - \delta,$$

which is a consequence of comparing the numbers of cyclic cubic extensions of  $k_1$  and  $k_2$  which are unramified outside of 3 from the viewpoint of both, class field theory and Kummer theory. The invariant  $0 \leq \delta \leq 1$  depends on the 3-virtual units of  $k_1$  and  $k_2$ . More precisely, we have

$$(1.2) \quad \delta = \begin{cases} 0, & \text{if } V_3(k_2) \text{ (imaginary)} \\ 1, & \text{if } V_3(k_1) \text{ (real)} \end{cases} \text{ contains a 3-virtual unit which is not 3-primary.}$$

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*Date:* December 17, 2016.

*2000 Mathematics Subject Classification.* Primary 11R37, 11R29, 11R11, 11R16, 11R20, 11Y40; Secondary 20D15.

*Key words and phrases.* 5-class field tower, 5-principalization, quadratic fields, 5-dual cyclic quartic fields, Frobenius fields; finite 5-groups, Schur  $\sigma$ -groups.

Research of the third author supported by the Austrian Science Fund (FWF): P 26008-N25.

**1.2. Quintic reflection theorem.** If  $d \neq 1$  denotes a square free integer prime to 5, then the 5-Selmer ranks  $\sigma_5(k_1)$ ,  $\sigma_5(k_2)$  of associated quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$ ,  $k_2 = \mathbb{Q}(\sqrt{5d})$  and the 5-class rank  $\varrho_5(M)$  of their 5-dual cyclic quartic field  $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ ,  $\zeta_5 = \exp(2\pi i/5)$ , with respect to the quartic cyclotomic mirror field  $k_0 = \mathbb{Q}(\zeta_5)$  satisfy the *quintic reflection theorem*

$$(1.3) \quad \varrho_5(M) = \sigma_5(k_1) + \sigma_5(k_2) - \delta_1 - \delta_2,$$

where the invariants  $0 \leq \delta_1, \delta_2 \leq 1$  depend on the 5-virtual units of  $k_1$  and  $k_2$  [8, p. 2]. The formula is derived by comparing the numbers of cyclic quintic extensions of  $k_1$ ,  $k_2$  and  $M$  which are unramified outside of 5. The maximal real subfield of  $k_0 = \mathbb{Q}(\zeta_5)$  is the quadratic field  $k_0^+ = \mathbb{Q}(\sqrt{5})$ .

**1.3. Overview.** The layout of this article is as follows. In § 2, the action of the absolute Galois group  $\text{Gal}(M|\mathbb{Q})$  on the 5-class group  $\text{Cl}_5 M$  considerably reduces the possibilities for the metabelianization  $G_5^{(2)} M$  of the 5-tower group  $G_5^{(\infty)} M$  of  $M$ . In § 3, it is shown that the six unramified cyclic quintic relative extensions  $E_i/M$ ,  $1 \leq i \leq 6$ , give rise to absolute extensions  $E_i/\mathbb{Q}$  which are either Frobenius or non-Galois. Using class number relations for the dihedral subextensions  $E_i/k_0^+$  of  $E_i/\mathbb{Q}$ , we determine further constraints for the second 5-class group  $G_5^{(2)} M$  in § 4. The paper concludes with tables of concrete numerical realizations in § 5 which underpin all theoretical statements and additionally reveal the statistical distribution of possible cases.

## 2. $p$ -PRINCIPALIZATION ENFORCED BY GALOIS ACTION

The generating automorphism  $\sigma$  of a cyclic number field  $F/\mathbb{Q}$  of degree  $d$  with Galois group  $\text{Gal}(F|\mathbb{Q}) = \langle \sigma \rangle$  acts on the class group  $\text{Cl}(F)$  of  $F$  and thus also on the higher  $p$ -class groups  $G_p^{(n)} F$  with  $n \in \mathbb{N} \cup \{\infty\}$ , for a fixed prime number  $p$ . When  $d$  and  $p$  are coprime, a remarkable restriction of the possibilities for the metabelian second  $p$ -class group  $\mathfrak{M} = G_p^{(2)} F$  and consequently for the transfer kernel type  $\varkappa(F)$  of  $F$  is due to the fact that the trace  $T_\sigma = \sum_{i=0}^{d-1} \sigma^i$  of  $\sigma$  annihilates the commutator quotient of all the groups  $G_p^{(n)} F$ .

**Definition 2.1.** Let  $p$  be a prime number and  $G$  be a pro- $p$  group with finite abelianization  $G/G'$ . Suppose that  $d \geq 2$  is a fixed integer.  $G$  is said to be a  $\sigma$ -group of degree  $d$ , if  $G$  possesses an automorphism  $\sigma$  of order  $d$  whose trace  $T_\sigma = \sum_{j=0}^{d-1} \sigma^j \in \mathbb{Z}[\text{Aut}(G)]$  annihilates  $G$  modulo  $G'$ , that is, if there exists  $\sigma \in \text{Aut}(G)$  such that  $\text{ord}(\sigma) = d$  and  $x^{T_\sigma} = \prod_{j=0}^{d-1} \sigma^j(x) \in G'$  for all  $x \in G$ .

We show that an epimorphism with characteristic kernel preserves the property of being a  $\sigma$ -group of degree  $d$ .

**Theorem 2.1.** *Let  $\phi : G \rightarrow H$  be an epimorphism of groups, whose kernel  $\ker(\phi)$  is characteristic in  $G$ . If  $G$  is a  $\sigma$ -group of degree  $d$  then  $H$  is also a  $\sigma$ -group of degree  $d$ .*

*Proof.* If  $G$  is a  $\sigma$ -group of degree  $d$ , then there exists an automorphism  $\sigma \in \text{Aut}(G)$  of order  $\text{ord}(\sigma) = d$  such that  $x^{T_\sigma} = \prod_{i=0}^{d-1} \sigma^i(x) \in G'$ , for all  $x \in G$ . According to [14, Thm. 6.2], there exists an induced automorphism  $\hat{\sigma} \in \text{Aut}(H)$  such that  $\hat{\sigma} \circ \phi = \phi \circ \sigma$ . By induction we obtain  $\hat{\sigma}^n \circ \phi = \phi \circ \sigma^n$ , for all  $n \in \mathbb{Z}$ : let  $n \geq 2$  be an integer and assume that  $\hat{\sigma}^{n-1} \circ \phi = \phi \circ \sigma^{n-1}$ , then

$$\hat{\sigma}^n \circ \phi = \hat{\sigma}^{n-1} \circ \hat{\sigma} \circ \phi = \hat{\sigma}^{n-1} \circ \phi \circ \sigma = \phi \circ \sigma^{n-1} \circ \sigma = \phi \circ \sigma^n.$$

Furthermore,  $(\sigma^{-1})^\wedge = \hat{\sigma}^{-1}$ . Now let  $y \in H$ . Since  $\phi$  is surjective, there exists  $x \in G$  with  $\phi(x) = y$ , and we obtain, as required,

$$y^{T_\sigma} = \prod_{i=0}^{d-1} \hat{\sigma}^i(y) = \prod_{i=0}^{d-1} \hat{\sigma}^i(\phi(x)) = \prod_{i=0}^{d-1} \phi(\sigma^i(x)) = \phi\left(\prod_{i=0}^{d-1} \sigma^i(x)\right) = \phi(x^{T_\sigma}) \in \phi(G') = H'.$$

□

**Corollary 2.1.** *In a descendant tree  $\mathcal{T}$  of finite  $p$ -groups with edges  $\pi : G \rightarrow \pi G$ , the property of **not** being a  $\sigma$ -group of degree  $d$  is inherited from the parent  $\pi G$  by the immediate descendant  $G$ .*

*Proof.* The parent operator  $\pi : G \rightarrow \pi G$  is the canonical projection from  $G$  onto the quotient  $\pi G = G/\gamma_c G$  by the last non-trivial member  $\gamma_c G$ ,  $c = \text{cl}(G)$ , of the lower central series  $(\gamma_i G)_{i \geq 1}$  of  $G$ , and thus  $\pi$  is an epimorphism with characteristic kernel  $\ker(\pi) = \gamma_c G$ , whence Theorem 2.1 justifies the claim.  $\square$

**Remark 2.1.** A  $\sigma$ -group  $G$  in the classical sense is a  $\sigma$ -group of degree 2 in the new sense, since  $x\sigma(x) \in G'$  is equivalent with  $\sigma(x)G' = x^{-1}G'$ . Such a group is also referred to as a group with *generator inverting* automorphism or briefly GI-automorphism.

**Theorem 2.2.** *The  $p$ -class tower group  $G_p^{(\infty)}F$  and all higher  $p$ -class groups  $G_p^{(n)}F$  with  $n \geq 2$  of a cyclic quartic number field  $F$  are  $\sigma$ -groups of degree 4.*

*When the quadratic subfield  $k < F$  has a trivial  $p$ -class group, the groups  $G_p^{(\infty)}F$  and  $G_p^{(n)}F$  with  $n \geq 2$  are simultaneously  $\sigma$ -groups of degree 2.*

*Proof.* The generating automorphism  $\sigma$  of  $F/\mathbb{Q}$  annihilates the class group  $\text{Cl}_F$  when it acts by its trace  $T_\sigma = \sum_{i=0}^3 \sigma^i \in \mathbb{Z}[\langle \sigma \rangle]$ , since  $x^{T_\sigma} = \prod_{i=0}^3 \sigma^i(x) = \text{Norm}_{F/\mathbb{Q}}(x) \in \text{Cl}_{\mathbb{Q}} = 1$ , for all  $x \in \text{Cl}_F$ . Of course, the same is true for all  $p$ -class groups  $\text{Cl}_p F$  with primes  $p$ . Finally, we have isomorphisms  $G_p^n F / (G_p^n F)' \simeq \text{Cl}_p F$ , for any  $n \in \mathbb{N} \cup \{\infty\}$ .

When the unique (real) quadratic subfield  $k < F$  has trivial  $p$ -class group  $\text{Cl}_p k = 1$ , then the relative automorphism  $\tau = \sigma^2 \in \text{Gal}(F|k)$  with order 2 acts by inversion on  $\text{Cl}_p F$ , since  $x^{T_\tau} = x^{1+\tau} = x \cdot \tau(x) = \text{Norm}_{F/k}(x) \in \text{Cl}_p k = 1$ , and thus  $x^\tau = x^{-1}$ , for all  $x \in \text{Cl}_p F$ .  $\square$

**Remark 2.2.** A pro- $p$  group  $G$  with finite abelianization  $G/G'$  is called a *strong*  $\sigma$ -group if it possesses an automorphism  $\sigma$  of order 2 which acts as inversion on both cohomology groups  $H^1(G, \mathbb{F}_p)$  and  $H^2(G, \mathbb{F}_p)$ . We emphasize the following two facts:

- An epimorphism does not necessarily preserve the property of being a strong  $\sigma$ -group.
- Whereas the group  $G_p^{(\infty)}F$  of a quadratic field  $F$  is a strong  $\sigma$ -group, [16, Lem. (4.1), p. 217], this is not necessarily the case for a cyclic quartic field  $F$ . See for instance Theorem 4.5.

TABLE 1. The Artin pattern of the twelve 5-groups of order  $5^5$  in the stem of  $\Phi_6$

Identifier of the 5-Group		Flag	5-Principalization Type			
James	SmallGroup	$f$	$\varkappa$	Cycle Pattern	Property	
$\Phi_6(2^2 1)_a$	$\langle 3125, 14 \rangle^*$	1	(123456)	(1)(2)(3)(4)(5)(6)	identity	
$\Phi_6(2^2 1)_{b_1}$	$\langle 3125, 11 \rangle^*$	1	(125364)	(1)(2)(3564)	4-cycle	
$\Phi_6(2^2 1)_{b_2}$	$\langle 3125, 7 \rangle$	1	(126543)	(1)(2)(36)(45)	two 2-cycles	
$\Phi_6(2^2 1)_{c_1}$	$\langle 3125, 8 \rangle^*$	0	(612435)	(16532)(4)	5-cycle	
$\Phi_6(2^2 1)_{c_2}$	$\langle 3125, 13 \rangle^*$	0	(612435)	(16532)(4)	5-cycle	
$\Phi_6(2^2 1)_{d_0}$	$\langle 3125, 10 \rangle$	0	(214365)	(12)(34)(56)	three 2-cycles	
$\Phi_6(2^2 1)_{d_1}$	$\langle 3125, 12 \rangle^*$	0	(512643)	(154632)	6-cycle	
$\Phi_6(2^2 1)_{d_2}$	$\langle 3125, 9 \rangle^*$	0	(312564)	(132)(456)	two 3-cycles	
$\Phi_6(21^3)_a$	$\langle 3125, 4 \rangle$	1	(022222)		nrl. const. with fp.	
$\Phi_6(21^3)_{b_1}$	$\langle 3125, 5 \rangle$	1	(011111)		nearly constant	
$\Phi_6(21^3)_{b_2}$	$\langle 3125, 6 \rangle$	1	(011111)		nearly constant	
$\Phi_6(1^5)$	$\langle 3125, 3 \rangle$	1	(000000)		constant	

In view of our special situation with  $p = 5$ ,  $F = M$ ,  $\text{Cl}_5 M = (5, 5)$  and  $k = k_0^+$ , we tested finite metabelian 5-groups  $G$  with  $G/G' \simeq (5, 5)$  of order  $|G| = 3125 = 5^5$  and coclass  $\text{cc}(G) = 2$ , for the property of simultaneously being a  $\sigma$ -group of degree 4 and degree 2. These groups are crucial

contestants for second 5-class groups  $G_5^2 M$  and form the stem of Hall's isoclinism family  $\Phi_6$  [12, § 3.5, p. 445–448]. In Table 1, the groups are characterized by their identifiers according to James [7] and the SmallGroups Library [2]. An asterisk \* marks a Schur  $\sigma$ -group, and a flag  $f \in \{0, 1\}$  indicates a  $\sigma$ -group of simultaneous degrees 4 and 2.

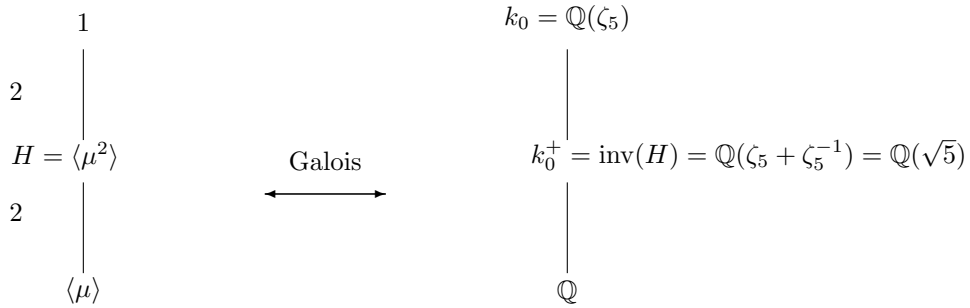
**Theorem 2.3.** *A finite 5-group  $G$  with  $G/G' \simeq (5, 5)$  which is a  $\sigma$ -group of degree 4 is either of coclass  $\text{cc}(G) = 1$  or isomorphic to one of the two Schur  $\sigma$ -groups  $\langle 3125, i \rangle$  with  $i \in \{11, 14\}$  or isomorphic to a descendant of one of the groups  $\langle 3125, i \rangle$  with  $i \in \{3, 4, 5, 6, 7\}$ .*

*Proof.* Using permutation representations, we compiled a program script in Magma [11] for testing whether an assigned 5-group  $G$  with  $G/G' \simeq (5, 5)$  is a  $\sigma$ -group of degree 4.  $\square$

### 3. FROBENIUS AND NON-GALOIS UNRAMIFIED 5-EXTENSIONS

**3.1. On the cyclic quartic fields  $M$ .** Let  $\zeta_5$  be a primitive 5th root of unity, then the irreducible polynomial of  $\zeta_5$  is given by  $\text{Irr}_{\mathbb{Q}}(\zeta_5) = X^4 + X^3 + X^2 + X + 1$ , and  $\text{Gal}(\mathbb{Q}(\zeta_5)|\mathbb{Q}) = \langle \mu \rangle$  is a cyclic group of order 4 which admits one subgroup  $\langle \mu^2 \rangle$  of order 2. By Galois correspondence, this subgroup corresponds to  $\mathbb{Q}(\zeta_5)^+ = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$ . (See Figure 1.)

FIGURE 1. Galois correspondence between  $\text{Gal}(k_0/\mathbb{Q})$  and  $k_0$



Let  $d$  be a square free integer prime to 5, then  $L = \mathbb{Q}(\sqrt{d}, \zeta_5)$  is a normal extension over  $\mathbb{Q}$  of degree 8, and the Galois group is

$$\text{Gal}(L|\mathbb{Q}) = \langle \tau, \mu \rangle = \langle 1, \tau, \mu, \mu^2, \mu^3, \tau\mu, \tau\mu^2, \tau\mu^3 \rangle, \text{ where } \tau(\sqrt{d}) = -\sqrt{d}.$$

This is an abelian group of type  $(2, 4)$  and has six proper subgroups ordered as follows :

$$H_1 = \langle \tau \rangle, H_2 = \langle \mu \rangle, H_3 = \langle \mu^2 \rangle, H_4 = \langle \tau\mu \rangle, H_5 = \langle \tau\mu^2 \rangle \text{ and } H_6 = \langle \tau, \mu^2 \rangle.$$

Note that the subgroups  $H_1, H_3, H_5$  are cyclic of order 2, the subgroups  $H_2, H_4$  are cyclic of order 4, and the group  $H_6$  is bicyclic of order 4. (See Figure 2.)

We consider the field  $M$  fixed by the subgroup  $\langle \tau\mu^2 \rangle$ . Then  $M$  is a cyclic quartic field and can be generated by adjunction  $M = \mathbb{Q}(\alpha)$  of  $\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{-\frac{5d}{2} - \frac{d}{2}\sqrt{5}}$  to  $\mathbb{Q}$ . With respect to the quartic cyclotomic mirror field  $k_0 = \mathbb{Q}(\zeta_5)$ ,  $M$  satisfies the *quintic reflection theorem* (Equation (1.3)).

**Lemma 3.1.** *Let  $K$  be a number field and  $F/K$  be a cyclic quartic extension, then there exist  $n, e$  and  $f \neq 0$  in  $K$  such that*

- (1)  $n$  is not a square in  $K$ ,
- (2)  $n(e^2 - f^2n)$  is a square in  $K$ ,
- (3)  $F = K(\alpha)$ , where  $\alpha = \sqrt{e + f\sqrt{n}}$ ,

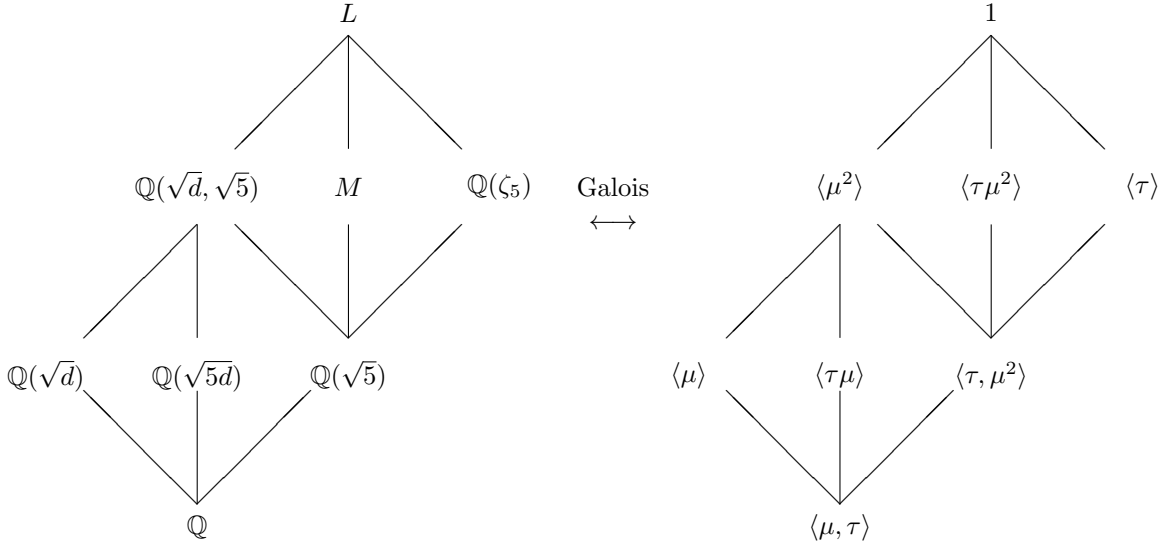
and the minimal polynomial of  $\alpha$  over  $K$  is given by  $\text{Irr}_K(\alpha) = X^4 - 2eX^2 + (e^2 - f^2n)$ .

Conversely, if there exist numbers  $n, e$ , and  $f \neq 0$  in  $K$  which satisfy the conditions (1), (2) and (3), then  $F/K$  is a cyclic quartic extension and the polynomial  $P(X) = X^4 - 2eX^2 + (e^2 - f^2n)$  is irreducible over  $K$ . In fact,  $F$  is the splitting field of  $P(X)$ .

*Proof.* It is known that the group  $\mathbb{Z}/4\mathbb{Z}$  has a single subgroup of order 2. By Galois theory, there exists a corresponding intermediary field  $R$  of the cyclic quartic extension  $F/K$ . Thus we can find  $n \in K$ , no square in  $K$ , such that  $R = K(\sqrt{n})$ . Since  $F$  is a quadratic extension of  $R$ , there exists  $\alpha \in F$  such that  $\alpha^2 = e + f\sqrt{n} \in R$  with  $e, f \in K$ ,  $f \neq 0$ , and  $F = R(\alpha)$ . Thus we have  $K(\alpha) = F$ , because  $\alpha \notin R$ . Furthermore, it is obvious that the minimal polynomial of  $\alpha$  is  $P(X) = X^4 - 2eX^2 + (e^2 - f^2n)$  and the splitting field of  $P(X)$  over  $\mathbb{Q}$  is  $F$ .

The discriminant of  $P$  is given by  $D = 16(e^2 - f^2n)(2f\sqrt{n})^4 = 2^8 f^4 n^2 (e^2 - f^2n)$ . Therefore the Galois group  $\text{Gal}(F|K)$  can be seen as a subgroup of the permutation group of  $P(X)$ , which is isomorphic to  $S_4$ , and cannot be injected into  $\mathcal{A}_4$ , since the group  $\mathcal{A}_4$  does not have a subgroup of order 4. We conclude that the discriminant is not a square in  $K$ , whence  $K(\sqrt{e^2 - f^2n})/K$  is of degree 2 and is contained in  $F$ . It follows that  $R = K(\sqrt{n}) = K(\sqrt{e^2 - f^2n})$ , so  $\frac{e^2 - f^2n}{n}$  is a square in  $K$ . Consequently, we see that  $n(e^2 - f^2n) = n^2 \frac{e^2 - f^2n}{n}$  is a square in  $K$ .

Conversely, let  $P(X) = X^4 - 2eX^2 + (e^2 - f^2n)$  with  $n, e, f \in K$ ,  $f \neq 0$ , such that the conditions (1), (2) and (3) are satisfied. Since  $\alpha$  is a root of  $P(X)$ , the degree  $[F : K]$  must be a divisor of 4. Since  $K(\sqrt{n}) \subseteq F$ , we have either  $F = K(\sqrt{n})$  or  $[F : K] = 4$ . If we have  $F = K(\sqrt{n})$ , there exist  $u, v \in K$  such that  $\sqrt{e + f\sqrt{n}} = u + v\sqrt{n}$ . Thus  $e^2 - f^2n = (u^2 - v^2n)^2$ , and by (2) we conclude that  $n$  is a square in  $K$ , which is a contradiction. So  $[F : K] = 4$ , and this enforces that  $P(X)$  is the minimal polynomial of  $\alpha$ . From the fact  $e - f\sqrt{n} = \left(\frac{1}{\sqrt{n}}\right)^2 \frac{n(e^2 - f^2n)}{\alpha^2}$ , we conclude that  $F$  is the splitting field of  $P(X)$  over  $K$ . Moreover,  $F$  is normal and  $\#\text{Gal}(F|K) = 4$ . Now we prove that  $\text{Gal}(F|K)$  is cyclic of order 4. If the Galois group  $\text{Gal}(F|K)$  were isomorphic to  $V_4$ , then the discriminant would be a square in  $K$ . This would imply that  $n$  were a square in  $K$ , which is a contradiction.  $\square$

 FIGURE 2. Galois correspondence between  $L$  and  $\text{Gal}(L/\mathbb{Q})$ 


**Corollary 3.1.** *Let  $d$  be a square free integer prime to 5 and  $\zeta_5$  be a primitive 5th root of unity, then the mirror image  $M$  of  $k_1 = \mathbb{Q}(\sqrt{d})$  can always be generated by adjoining the algebraic number  $\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{\frac{-5d}{2} + \frac{-d}{2}\sqrt{5}}$  to the rational, whence  $M$  is complex for  $d > 0$  and  $M$  is real for  $d < 0$ . For the construction of  $M$  one can therefore use the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ , which is given by*

$$(3.1) \quad \text{Irr}_{\mathbb{Q}}(\alpha) = X^4 + 5dX^2 + 5d^2.$$

**Remark 3.1.** The conductor  $c(M)$  and the discriminant  $d(M)$  of  $M$  are given by

(3.2)

$$c(M) = \begin{cases} 20d & \text{if } d \equiv 1 \pmod{4}, \\ 5d & \text{if } d \equiv 2, 3 \pmod{4}, \end{cases} \quad d(M) = c(M)^2 d(k_0^+) = \begin{cases} 2000d^2 & \text{if } d \equiv 1 \pmod{4}, \\ 125d^2 & \text{if } d \equiv 2, 3 \pmod{4}, \end{cases}$$

where  $d(k_0^+) = 5$  is the discriminant of the quadratic subfield  $k_0^+ = \mathbb{Q}(\sqrt{5})$  of  $M$ . (See [4, 18].)

**3.2. Imaginary cyclic quartic fields  $M$  with  $d > 0$ .** In the following, the two Frobenius groups  $F_{5,w}$  of order 20 with primitive root  $2 \leq w \leq 3$  modulo 5 will be denoted by

$$(3.3) \quad \begin{aligned} F_{5,2} &= \langle \sigma, \iota \mid \sigma^5 = 1, \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^2 \rangle, \\ F_{5,3} &= \langle \sigma, \iota \mid \sigma^5 = 1, \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^3 \rangle, \end{aligned}$$

where  $\iota|_M = \mu|_M$ .

TABLE 2. All possible 5-class ranks  $r_1 := \varrho_5(k_1)$ ,  $r_2 := \varrho_5(k_2)$  and invariants  $\delta_1, \delta_2$  for the associated quadratic fields  $k_1, k_2$  which are 5-dual to an imaginary cyclic quartic field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ ,  $d > 0$ , with 5-class rank  $r := \varrho_5(M) = 2$

Case	$r_1$	$\delta_1$	$r_2$	$\delta_2$
(a)	1	0	0	1
(b)	0	1	1	0
(c)	1	1	1	1
(d)	0	0	0	0
(e)	1	1	0	0
(f)	0	0	1	1
(g)	2	1	0	1
(h)	0	1	2	1

**Proposition 3.1.** *Let  $E_1, \dots, E_6$  be the six unramified cyclic quintic extensions of the imaginary cyclic quartic field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ ,  $d > 0$ , with 5-class group  $\text{Cl}_5 M \simeq C_5 \times C_5$  or, more generally, of 5-class rank 2. The properties of these fields as absolute extensions  $E_i/\mathbb{Q}$ , in dependence on the eight cases in Table 2, are given as follows.*

- In case (a) and (g), all six fields  $E_1, \dots, E_6$  are normal and share isomorphic automorphism groups  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,2}$ , for  $1 \leq i \leq 6$ .
- In case (b) and (h), all six fields  $E_1, \dots, E_6$  are normal and share isomorphic automorphism groups  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,3}$ , for  $1 \leq i \leq 6$ .
- In all the other cases (c), (d), (e), (f), two extensions are normal with non-isomorphic automorphism groups, say  $\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}$  and  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ , but the other four extensions are non-Galois and form two conjugate pairs  $E_3 \simeq E_4$  and  $E_5 \simeq E_6$ .

*Proof.* According to the quintic reflection theorem [8], the assumption  $r = 2$  implies that one of the eight disjoint cases in Table 2 is satisfied.

- In case (a), the 5-Selmer group of  $k_1$  is given by  $V_5(k_1) = \langle \alpha_{11}, \varepsilon_1 \rangle$ . See [8, p.2, 1.3]. Let  $E_1 := \text{Spl}_{\mathbb{Q}} f(X, \alpha_{11})$  and  $E_2 := \text{Spl}_{\mathbb{Q}} f(X, \varepsilon_1)$ . In virtue of  $\delta_1 = 0$ ,  $E_1$  and  $E_2$  are unramified cyclic quintic extensions of  $M$ . According to [8, p. 17, 1.9–23],  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,2}$ , for  $1 \leq i \leq 2$ . Let  $L := E_1 \cdot E_2$  be the compositum. Then, by [8, Lem.2.5], all proper subextensions  $E$  of  $L|M$  have  $\text{Gal}(E|\mathbb{Q}) \simeq F_{5,2}$ .
- In case (b), the 5-Selmer group of  $k_2$  is given by  $V_5(k_2) = \langle \alpha_{21}, \varepsilon_2 \rangle$ . See [8, p. 2, 1.3]. Let  $E_1 := \text{Spl}_{\mathbb{Q}} f(X, \alpha_{21})$  and  $E_2 := \text{Spl}_{\mathbb{Q}} f(X, \varepsilon_2)$ . In virtue of  $\delta_2 = 0$ ,  $E_1$  and  $E_2$  are unramified cyclic quintic extensions of  $M$ . According to [8, p. 17, 1.9–23],  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,3}$ , for  $1 \leq i \leq 2$ . Let  $L := E_1 \cdot E_2$  be the compositum. Then, by [8, Lem.2.5], all proper subextensions  $E$  of  $L|M$  have  $\text{Gal}(E|\mathbb{Q}) \simeq F_{5,3}$ .

- Exemplarily, we consider case (d). Then the 5-Selmer groups of  $k_1$  and  $k_2$  are given by  $V_5(k_1) = \langle \varepsilon_1 \rangle$ ,  $V_5(k_2) = \langle \varepsilon_2 \rangle$ . Let  $E_1 := \text{Spl}_{\mathbb{Q}} f(X, \varepsilon_1)$  and  $E_2 := \text{Spl}_{\mathbb{Q}} f(X, \varepsilon_2)$ . Then, in virtue of  $\delta_1 = \delta_2 = 0$ ,  $E_1$  and  $E_2$  are unramified cyclic quintic extensions of  $M$ . According to [8, p. 17, 1.9–23],  $\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}$  and  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ . Let  $L := E_1 \cdot E_2$  be the compositum. Then  $E|M$  is also an unramified cyclic quintic extension, for any proper subextension  $E$  of  $L|M$  distinct from  $E_1$  and  $E_2$ . Assume that  $\text{Gal}(E|\mathbb{Q}) \simeq F_{5,2}$ . Since  $L = E_1 \cdot E$ , all proper subextensions  $E'$  of  $L|M$  have  $\text{Gal}(E'|\mathbb{Q}) \simeq F_{5,2}$ , by [8, Lem.2.5]. This is a contradiction to  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ . In the same manner, the assumption that  $\text{Gal}(E|\mathbb{Q}) \simeq F_{5,3}$  leads to a contradiction. Therefore  $E|\mathbb{Q}$  must be a non-Galois extension.  $\square$

**3.3. Infinite family of imaginary cyclic quartic fields  $M$  whose 5-rank is at least 2.** As before, let  $d \neq 1$  be a square free integer prime to 5, and let  $k_1 = \mathbb{Q}(\sqrt{d})$  and  $k_2 = \mathbb{Q}(\sqrt{5d})$  be the associated quadratic fields. For  $\gamma \in k = k_i$ ,  $i \in \{1, 2\}$ , Y. Kishi [8, p. 6] has defined the polynomial

$$f(X, \gamma) = X^5 - 5N_k(\gamma)X^3 + 5N_k(\gamma)^2X - N_k(\gamma)\text{Tr}_k(\gamma),$$

where  $N_k, \text{Tr}_k$  are the norm map and the trace map of  $k/\mathbb{Q}$ . The minimal splitting field of  $f(X, \gamma)$  is noted by  $K_\gamma$ . Furthermore, M. Imaoka and Y. Kishi [6], have characterized all  $F_{5,w}$ -extensions with  $w \in \{2, 3\}$  as  $K_\gamma$  for a suitable elements  $\gamma \in k_i$  with  $i \in \{1, 2\}$ . If  $\gamma \in k_1$ , then  $\text{Gal}(K_\gamma|\mathbb{Q}) \simeq F_{5,2}$ , and if  $\gamma \in k_2$ , then  $\text{Gal}(K_\gamma|\mathbb{Q}) \simeq F_{5,3}$ . Now, We consider the real quadratic fields  $k_1 = \mathbb{Q}(\sqrt{(\alpha + \beta)^2 - 4})$  given by Kishi in [9, Exm. 3.5, p. 489] for  $p = 5$ , where the pair of integers  $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$  such that  $\alpha \geq 2$ ,  $\beta \geq 2$  satisfies the simultaneous conditions

$$(3.4) \quad \begin{cases} \alpha^2 - 5^3\beta^2 = 4 \\ \alpha + \beta \equiv 0 \pmod{5^2}. \end{cases}$$

**Remark 3.2.** The Pellian equation  $\alpha^2 - 5^3\beta^2 = 4$  has infinitely many solutions  $(\alpha, \beta)$ , which correspond to the powers  $\eta^n = \frac{\alpha + \beta\sqrt{5^3}}{2}$  of the normpositive fundamental unit  $\eta = \frac{123 + 11\sqrt{5^3}}{2} = \frac{123 + 11 \cdot 5\sqrt{5}}{2}$  of the suborder with conductor  $f = 5$  of  $\mathbb{Q}(\sqrt{5})$ . The solution  $(\alpha, \beta)$  satisfies the additional constraint  $\alpha + \beta \equiv 0 \pmod{5^2}$  in (3.4) if and only if  $n = 7 + 5^2k$  with an integer  $k \geq 0$ .

**Proposition 3.2.** *Let  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha + \beta)^2 - 4})$ , where  $\zeta_5$  is a primitive 5th root of unity and  $\alpha, \beta$  satisfy the conditions (3.4). Then the 5-rank of the class group of  $M$  is greater than or equal to 2.*

*Proof.* Let  $\epsilon_1 = \frac{\alpha + \beta + \sqrt{d}}{2}$ , resp.  $\epsilon_2 = \frac{\alpha + 5^3\beta + 5\sqrt{5d}}{2}$ , be an element of  $k_1 = \mathbb{Q}(\sqrt{(\alpha + \beta)^2 - 4})$ , resp.  $k_2 = \mathbb{Q}(\sqrt{5((\alpha + \beta)^2 - 4)})$ . According to [9, Exm. 3.5, p. 489],  $\epsilon_1$  and  $\epsilon_2$  are units of  $k_1$  and  $k_2$ , respectively. They satisfy the conditions

$$(3.5) \quad \begin{cases} N_{\mathbb{Q}(\sqrt{d})}(\epsilon_1^2) = N_{\mathbb{Q}(\sqrt{5d})}(\epsilon_2) = 1, \\ \text{Tr}_{\mathbb{Q}(\sqrt{d})}(\epsilon_1^2) \equiv \text{Tr}_{\mathbb{Q}(\sqrt{5d})}(\epsilon_2) \equiv \pm 2 \pmod{5^3}. \end{cases}$$

By applying [9, Thm 1.1, p. 482, Prp 3.1, p. 487], we prove that  $K_{\epsilon_1^2}$  and  $K_{\epsilon_2}$  are two different absolute Galois  $F_5$ -extensions, unramified over  $M$ , it suffices to show that  $\epsilon_1^2$ , resp.  $\epsilon_2$ , cannot be the fifth power of an element of  $k_1$ , resp.  $k_2$ .

According to [15, Lem. 1, p. 16], we have the following general fact: Let  $p$  be a prime number and  $\xi$  be an element of  $\mathbb{Q}(\sqrt{\delta})$  such that  $\xi = \frac{u+v\sqrt{\delta}}{2}$ . If  $0 < |v| < \frac{\delta^{p-1/2}}{2^{p-1}}$ , then  $\xi \notin \mathbb{Q}(\sqrt{\delta})^p$ .

Let us apply this result to  $\epsilon_2$  and  $\epsilon_1^2$ . By the assumptions (3.4),  $\alpha + \beta = 5^2c$ , for some  $c \geq 1$ . Hence  $(\alpha + \beta)^2 = 5^4c^2$  and  $5(\alpha + \beta)^2 = 5^5c^2$ . Furthermore,  $5^5c^2 \geq 5^5 > 36$  and thus  $5(\alpha + \beta)^2 - 20 > 16$ , whence  $5d > 16$ ,  $(5d)^2 > 16^2$ , and  $\frac{(5d)^2}{16} > 16$ . Finally  $5 < 16 < \frac{(5d)^2}{2^4}$ , and if we put  $v := 5$  and  $\delta := 5d$ , then  $v < \frac{\delta^2}{2^4}$ , whence  $\epsilon_2$  cannot be the fifth power of an element in  $k_2$ .

For  $\epsilon_1^2$ , we express the square in the form  $\epsilon_1^2 = \frac{(\alpha+\beta)^2+d+(\alpha+\beta)\sqrt{d}}{2}$ . Moreover we have,

$$\alpha + \beta < \frac{d^2}{16} \iff 16(\alpha + \beta) < (\alpha + \beta)^4 - 8(\alpha + \beta)^2 + 16.$$

Put  $u := \alpha + \beta$ , then  $\alpha + \beta < \frac{d^2}{16} \iff u^4 - 8u^2 - 16u + 16 > 0$ . Since  $\alpha \geq 2$  and  $\beta \geq 2$ , it follows that  $u \geq 3$ , whence  $\phi(u) = u^4 - 8u^2 - 16u + 16$  is positive. Thus we get  $\alpha + \beta < \frac{d^2}{24}$ , and putting  $v := \alpha + \beta$  and  $\delta := d$  we conclude that  $\epsilon_1^2$  cannot be a fifth power in  $k_1$  either.  $\square$

**Corollary 3.2.** *Let  $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha + \beta)^2 - 4}\right)$ , where  $\zeta_5$  is a primitive 5th root of unity and the integers  $\alpha, \beta$  satisfy the conditions (3.4). Assume that the 5-class group  $\text{Cl}_5 M$  of  $M$  is of type (5, 5). Then  $M_5^{(1)}/M$  contains six unramified cyclic quintic extensions  $E_i/M$ , which give rise to absolute extensions of degree 20 over  $\mathbb{Q}$ , ordered as*

- $E_1 = K_{\epsilon_1^2}$  of Type (I) with  $\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}$ , the splitting field of the polynomial  $f(X, \epsilon_1^2) = X^5 - 5X^3 + 5X - (d + 2)$ , and
- $E_2 = K_{\epsilon_2}$  of Type (II) with  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ , the splitting field of the polynomial  $f(X, \epsilon_2) = X^5 - 5X^3 + 5X - (\alpha + 5^3\beta)$ , and
- the other four extensions  $E_3, E_4 = E_3^\varphi, E_5, E_6 = E_5^\varphi$ , which are non-Galois of Type (III) over  $\mathbb{Q}$  and form two conjugate pairs.

*Proof.* The claims are a consequence of Proposition 3.2, the Formulas (3.5) and the fact that  $\text{Tr}_{\mathbb{Q}(\sqrt{d})}(\epsilon_1^2) = d + 2$  and  $\text{Tr}_{\mathbb{Q}(\sqrt{5d})}(\epsilon_2) = \alpha + 5^3\beta$ . By  $\varphi$  we denote the generator of  $\text{Gal}(\mathbb{Q}(\sqrt{5})|\mathbb{Q})$ .  $\square$

**Remark 3.3.** For  $d > 0$ , if the fundamental units of the real quadratic fields  $k_i$ ,  $i = 1, 2$ , are 5-primary, then the field  $M$  has a non-Galois unramified cyclic quintic extension of Type (III) [8]. In this case, there are four pairwise conjugate extensions of Type (III), and among the remaining two Frobenius-extensions one is of Type (I) and one is of Type (II) [8]. Note that in the case  $d > 0$  the cyclic quartic field  $M$  is imaginary. Also, if 5 divides the class number of  $k_i$ ,  $i = 1, 2$ , there exists at most one 5-primary element of  $k_i$ ,  $i = 1, 2$ , which gives rise to the Frobenius-extensions of Type (I) and Type (II).

**3.4. Real cyclic quartic fields  $M$  with  $d < 0$ .** As before, the two Frobenius groups  $F_{5,w}$  of order 20 with primitive root  $2 \leq w \leq 3$  modulo 5 will be denoted as in formula (3.3).

TABLE 3. Some possible 5-class ranks  $r_1 := \varrho_5(k_1)$ ,  $r_2 := \varrho_5(k_2)$  and invariants  $\delta_1, \delta_2$  for the associated quadratic fields  $k_1, k_2$  which are 5-dual to a real cyclic quartic field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ ,  $d < 0$ , with 5-class rank  $r := \varrho_5(M) = 2$

Case	$r_1$	$\delta_1$	$r_2$	$\delta_2$
(a)	2	0	0	0
(b)	0	0	2	0
(c)	1	0	1	0
(d)	2	1	1	0
(e)	1	0	2	1

**Proposition 3.3.** *Let  $E_1, \dots, E_6$  be the six unramified cyclic quintic extensions of the real cyclic quartic field  $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ ,  $d < 0$ , with 5-class group  $\text{Cl}_5 M \simeq C_5 \times C_5$  or, more generally, of 5-class rank 2. The properties of these fields as absolute extensions  $E_i|\mathbb{Q}$ , in dependence on the five cases in Table 3, are given as follows.*

- In case (a), all six fields  $E_1, \dots, E_6$  are normal and share isomorphic automorphism groups  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,2}$ , for  $1 \leq i \leq 6$ .



- In case (b), all six fields  $E_1, \dots, E_6$  are normal and share isomorphic automorphism groups  $\text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,3}$ , for  $1 \leq i \leq 6$ .
- In all the other cases (c), (d), (e), two extensions are normal with non-isomorphic automorphism groups, say  $\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}$  and  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ , but the other four extensions are non-Galois and form two conjugate pairs  $E_3 \simeq E_4$  and  $E_5 \simeq E_6$ .

*Proof.* Similar to the proof of Proposition 3.1.  $\square$

#### 4. THE SECOND 5-CLASS GROUP $G_5^{(2)}(M)$ OF $M$

Based on class number formula [10] for dihedral relative extensions  $E$  of degree 10 over a base field  $F$  with class number coprime to 5, we are now in the position to determine the isomorphism type of the Galois group  $G_5^{(2)}M = \text{Gal}(M_5^{(2)}|M)$  of the second Hilbert 5-class field  $M_5^{(2)}$  of a cyclic quartic field  $M = \mathbb{Q}\left(\left(\zeta_5 - \zeta_5^{-1}\right)\sqrt{d}\right)$  with 5-class group of type  $(5, 5)$ , because its unramified cyclic quintic extensions  $E_i$ ,  $1 \leq i \leq 6$ , turn out to be relatively dihedral over the quadratic subfield  $k_0^+ = \mathbb{Q}(\sqrt{5})$  of  $M$ , which has class number 1.

**Theorem 4.1.** *The relation between the 5-class numbers  $h_5(E_i)$  of the six unramified cyclic quintic extensions  $E_i$  of  $M$  and the 5-class numbers  $h_5(L_i)$  of their non-Galois subfields  $L_i$ , which are of relative degree 5 over the field  $k_0^+ = \mathbb{Q}(\sqrt{5})$ , is given by*

$$(4.1) \quad h_5(E_i) = \begin{cases} h_5(L_i)^2 & \text{if } \#\ker(j_{E_i|M}) = 25, (U_{k_0^+} : N_{L_i|k_0^+}(U_{L_i})) = 1, \\ 5 \cdot h_5(L_i)^2 & \text{if } \#\ker(j_{E_i|M}) = 25, (U_{k_0^+} : N_{L_i|k_0^+}(U_{L_i})) = 5, \\ 5 \cdot h_5(L_i)^2 & \text{if } \#\ker(j_{E_i|M}) = 5, (U_{k_0^+} : N_{L_i|k_0^+}(U_{L_i})) = 1, \\ 25 \cdot h_5(L_i)^2 & \text{if } \#\ker(j_{E_i|M}) = 5, (U_{k_0^+} : N_{L_i|k_0^+}(U_{L_i})) = 5, \end{cases}$$

where  $U_F$  is the unit group of the field  $F$ .

*Proof.* According to Lemmermeyer [10, eq. (5.2), p. 685], we have the class number relation

$$h_5(E_i) = \frac{(U_{k_0^+} : N_{L_i|k_0^+}(U_{L_i}))}{\#\ker(j_{E_i|M})} \cdot h_5(M) \cdot h_5(L_i)^2,$$

where  $h_5(M) = 25$ , due to our general assumption on  $M$ . Distinction between total principalization,  $\#\ker(j_{E_i|M}) = 25$ , and partial principalization,  $\#\ker(j_{E_i|M}) = 5$ , immediately yields the four claimed cases, in dependence on the unit norm indices  $u_i := (U_{k_0^+} : N_{E_i|k_0^+}(U_{L_i}))$ .  $\square$

**Remark 4.1.** To prove Theorem 4.1 in a different manner we can use the class number formula, due to Lemmermeyer [10, Thm. 2.4, p. 681], and the following Lemma 4.1.

**Lemma 4.1.** *Let  $p$  be an odd prime and let  $F$  be a number field with class number coprime to  $p$ . Let  $k$  be a quadratic extension of  $F$ . Assume that  $L$  is an unramified cyclic extension of  $k$  of degree  $p$ , then the extension  $L/F$  is Galois, dihedral of degree  $2p$ , and we have the formula*

$$a := (U_L : U_K U_{K'} U_k) = \frac{(U_k : U_k^p)(U_F : N_{K/F}(U_K))}{(U_F : U_F^p)(U_k : N_{L/k}(U_L))}$$

for the subfield unit index  $a$ , where  $K \neq K'$  denote two conjugate non-Galois subfields of  $L$ .

*Proof.* Since  $p \geq 3$  is an odd prime and the existence of an unramified cyclic extension  $L/k$  of degree  $p$  excludes the irregular case  $p = 3$ ,  $F = \mathbb{Q}$ ,  $k = \mathbb{Q}(\sqrt{-3})$  with  $h_k = 1$ , either both fields  $k$  and  $F$  contain the  $p$ th roots of unity or both not. Therefore,  $\frac{(U_k : U_k^p)}{(U_F : U_F^p)} = p^{r(k) - r(F)}$  with the torsionfree Dirichlet unit ranks  $r(k)$  of  $k$  and  $r(F)$  of  $F$ . For an unramified extension  $L/k$ , the Theorem on the Herbrand quotient of  $U_L$  is equivalent with  $\#\ker(j_{L/k}) = p \cdot b$  with  $b := (U_k : N_{L/k}(U_L))$ . Using Lemma 4.1, which can be found in [10, p. 686], we can express the factor on the right hand side of the class number relation [10, Thm. 2.4, p. 681],

$$h_p(L) = \frac{a}{p^{1+r(k)-r(F)}} \cdot h_p(k) \cdot h_p(K)^2,$$

in the form  $\frac{a}{p^{1+r(k)-r(F)}} = \frac{a \cdot (U_F : U_F^p)}{p \cdot (U_k : U_k^p)} = \frac{(U_F : N_{K/F}(U_K))}{\# \ker(j_{L/k})}$ , which we have used for  $p = 5$  in the proof of Theorem 4.1.  $\square$

#### 4.1. Imaginary cyclic quartic fields $M$ with $d > 0$ .

**Theorem 4.2.** *The 5-class field tower of  $M$  has length  $\ell_5(M) = 1$  if and only if the second 5-class group  $G_5^2(M)$  of  $M$  is the abelian 5-group  $\langle 25, 2 \rangle$  of type  $(5, 5)$ . In this case,*

- (1) *the 5-class groups  $\text{Cl}_5(E_i)$  are cyclic of order 5, for  $1 \leq i \leq 6$ ,*
- (2) *the 5-class groups  $\text{Cl}_5(L_i)$  are trivial, for  $1 \leq i \leq 6$ ,*
- (3) *the principalization of  $M$  is of type a.1,  $\varkappa(M) = (000000)$ .*

*Proof.* For  $G_5^2(M) \simeq \langle 25, 2 \rangle$ , we have cyclic 5-class groups  $\text{Cl}_5(E_i) \simeq C_5$  and six total principalizations  $\# \ker(j_{E_i/M}) = 25$ . According to Theorem 4.1, we obtain  $h_5(E_i) = 5 = u_i \cdot h_5(L_i)^2$ , which enforces  $h_5(L_i) = 1$  and  $u_i = 5$ , for all  $1 \leq i \leq 6$ .  $\square$

**Example 4.1.** The values  $d = 4357$  and  $d = 4444$  give rise to fields  $M = \mathbb{Q} \left( (\zeta_5 - \zeta_5^{-1})\sqrt{d} \right)$  with 5-class group of type  $(5, 5)$  having a single-stage 5-class tower. Fields of this type are extremely rare, since they form a fraction of  $\frac{2}{83}$  among the fields with  $0 < d < 10000$ . Therefore, only 2% of the cases possess a single-stage tower.

**Proposition 4.1.** *Let  $M = \mathbb{Q} \left( (\zeta_5 - \zeta_5^{-1})\sqrt{d} \right)$  with  $d > 0$  be an imaginary cyclic quartic field with 5-class group of type  $(5, 5)$ . Let  $E_i$ ,  $1 \leq i \leq 6$ , be the six unramified cyclic quintic extensions of  $M$  and  $L_i$  their non-Galois subfields of relative degree 5 over the field  $k_0^+ = \mathbb{Q}(\sqrt{5})$ . Then,*

- (1) *the subfield unit indices  $a_i := (U_{E_i} : U_{L_i} U_{L_i'} U_M)$  are equal to 1,*
- (2) *the unit norm indices  $u_i$  satisfy the equivalence  $u_i = 1 \iff \# \ker(j_{E_i/M}) = 5$ , and*
- (3) *the relations between the 5-class numbers  $h_5(E_i)$  and  $h_5(L_i)$  are given by*

$$h_5(E_i) = 5 \cdot h_5(L_i)^2.$$

*Proof.* According to Lemma 4.1, we can deduce that

$$a_i b_i = \frac{(U_M : U_M^5) (U_{k_0^+} : N_{L_i/k_0^+}(U_{L_i}))}{(U_{k_0^+} : U_{k_0^+}^5)},$$

where  $b_i$  denotes the unit norm index  $(U_M : N_{E_i/M}(U_{E_i}))$ . Since  $d > 0$ , the field  $M$  is imaginary and it is a CM-field with maximal real subfield  $M^+ = k_0^+$ . Hence, the torsionfree Dirichlet unit rank of  $M$  is  $r(M) = 1$ , and  $U_M = \langle -1, \epsilon_5 \rangle$ , where  $\epsilon_5$  denotes the fundamental unit of the quadratic field  $k_0^+ = \mathbb{Q}(\sqrt{5})$ . This implies that

$$(U_M : U_M^5) = (U_{k_0^+} : U_{k_0^+}^5) \quad \text{and} \quad a_i b_i = u_i.$$

- (1) To prove the assertion (1) it suffices to show the following equivalence

$$u_i = 1 \quad \text{if and only if} \quad b_i = 1.$$

So it suffices to show that the fundamental unit  $\epsilon_5$  of  $k_0^+$ , which is also the fundamental unit of  $M$ , is norm of a unit of  $E_i$  if and only if it is norm of a unit of  $L_i$ .

If  $u_i = 1$ , for  $i \in \{1, \dots, 6\}$ , then  $\epsilon_5$  is norm of a unit of  $L_i$  (a non-Galois subfield of  $E_i$ ), hence it is also norm of the same unit in  $E_i$ , and  $b_i = 1$ .

Now suppose that  $b_i = 1$ , for  $1 \leq i \leq 6$ , then there exists a unit  $\xi \in U_{E_i}$  such that  $\epsilon_5 = N_{E_i/M}(\xi)$ , and we obtain the following chain of implications:

$$\begin{aligned} N_{M/\mathbb{Q}(\sqrt{5})}(\epsilon_5) &= N_{M/\mathbb{Q}(\sqrt{5})}(N_{E_i/M}(\xi)) \\ \Rightarrow \epsilon_5^2 &= N_{M/\mathbb{Q}(\sqrt{5})}(N_{E_i/M}(\xi)) = N_{L_i/\mathbb{Q}(\sqrt{5})}(N_{E_i/L_i}(\xi)) \\ \Rightarrow \epsilon_5^6 &= N_{L_i/\mathbb{Q}(\sqrt{5})}(N_{E_i/L_i}(\xi^3)) \\ \Rightarrow \epsilon_5 \cdot N_{L_i/\mathbb{Q}(\sqrt{5})}(\epsilon_5) &= N_{L_i/\mathbb{Q}(\sqrt{5})}(N_{E_i/L_i}(\xi^3)), \end{aligned}$$

whence

$$\epsilon_5 = N_{L_i/\mathbb{Q}(\sqrt{5})}(\epsilon_5^{-1} \cdot N_{E_i/L_i}(\xi^3)).$$

Since the element  $\epsilon_5^{-1} \cdot N_{E_i/L_i}(\xi^3)$  is a unit of  $L_i$ , we obtain the index  $u_i = 1$ . On the other hand, the possible values of  $b_i$  and  $u_i$  are  $\{1, 5\}$ , and we can deduce that  $u_i = b_i$ . Finally, it follows from the equation  $a_i b_i = u_i$  that  $a_i = 1$ .

- (2) The result follows immediately from the fact that  $\#\ker(j_{E_i/M}) = 5 \cdot b_i$ .  
(3) According to Theorem 4.1, we have two possible cases

$$u_i = 1 \text{ and } \#\ker j_{E_i/M} = 5$$

and

$$u_i = 5 \text{ and } \#\ker j_{E_i/M} = 25.$$

In both cases, the class number formula is given by  $h_5(E_i) = 5 \cdot h_5(L_i)^2$ .

□

**Theorem 4.3.** *Let  $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$  with  $d > 0$  be a cyclic quartic field with 5-class group,  $\text{Cl}_5 M \simeq C_5 \times C_5$ . If the second 5-class group  $G := G_5^2(M)$  of the imaginary cyclic quartic field  $M$  is non-abelian, then the coclass  $\text{cc}(G)$  of  $G$  is greater than or equal to 2,  $\text{cc}(G) \geq 2$ .*

*Proof.* Assume that  $G$  is non-abelian of coclass  $\text{cc}(G) = 1$ , then the possible capitulation types of  $M$  in the six intermediary cyclic quintic extensions of  $M_5^{(2)}/M$ , noted by  $E_6 = E_5^\varphi$ ,  $E_5$ ,  $E_4 = E_3^\varphi$ ,  $E_3$ ,  $E_2$ ,  $E_1$ , are given by  $\varkappa(G) = (111111)$  or  $\varkappa(G) = (00000\ell)$ ,  $\ell \in \{0, 1, 2\}$ .

First we consider the type  $\varkappa(G) = (111111)$ . In this case the group  $G$  is the extra special 5-group of order  $5^3$  and exponent  $5^2$ , whose maximal normal subgroups are of order  $5^2$ . This implies that the 5-class number of all  $E_i$  is equal to  $5^2$ . Using Proposition 4.1, we conclude that the valuation  $v_5(h_5(E_i))$  of the 5-class number of  $E_i$  must be odd, which is a contradiction. Thus the type  $\varkappa(G) = (111111)$  can not occur.

For the three other types, we have total capitulation in the four extensions  $E_6 = E_5^\varphi$ ,  $E_5$ ,  $E_4 = E_3^\varphi$ ,  $E_3$ ,  $E_2$ , so the value of the index  $b_i$ ,  $2 \leq i \leq 6$ , is  $b_i = 5$ , whence  $u_i = 5$ . On the other hand, for  $2 \leq i \leq 6$  we have  $h_5(E_i) = 5^2$ , and, by Proposition 4.1, we have  $u_i = 1$ , which is a contradiction. □

**Proposition 4.2** (Number of fields). *In the range  $0 < d < 10000$  of fundamental discriminants  $d$  of real quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$  with  $\gcd(5, d) = 1$  there exist precisely **83** cases such that the 5-dual field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  of  $k_1$  has a 5-class group  $\text{Cl}_5(M)$  of type  $(5, 5)$ .*

*Proof.* See Tables 4 and 5. □

**Theorem 4.4** (Schur- $\sigma$  two-stage towers of 5-class fields).

- (1) *For the **23** (**28%**) real quadratic fields  $k_1$  starting with the following discriminants*

$$d \in \{457, 501, 1996, 2573, 3253, 4189, 4957, 5129, 5233, 5308, 5361, \dots\}$$

*the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (125643)$ , two fixed points and a 4-cycle, IPAD  $\tau^{(1)}(M) = [1^2; (1^3)^2, (21)^4]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^5, \mathbf{11} \rangle$ .*

- (2) *For the **11** (**13%**) real quadratic fields  $k_1$  with the following discriminants*

$$d \in \{581, 753, 2296, 2829, 4553, 5116, 5736, 6761, 7489, 9013, 9829\}$$

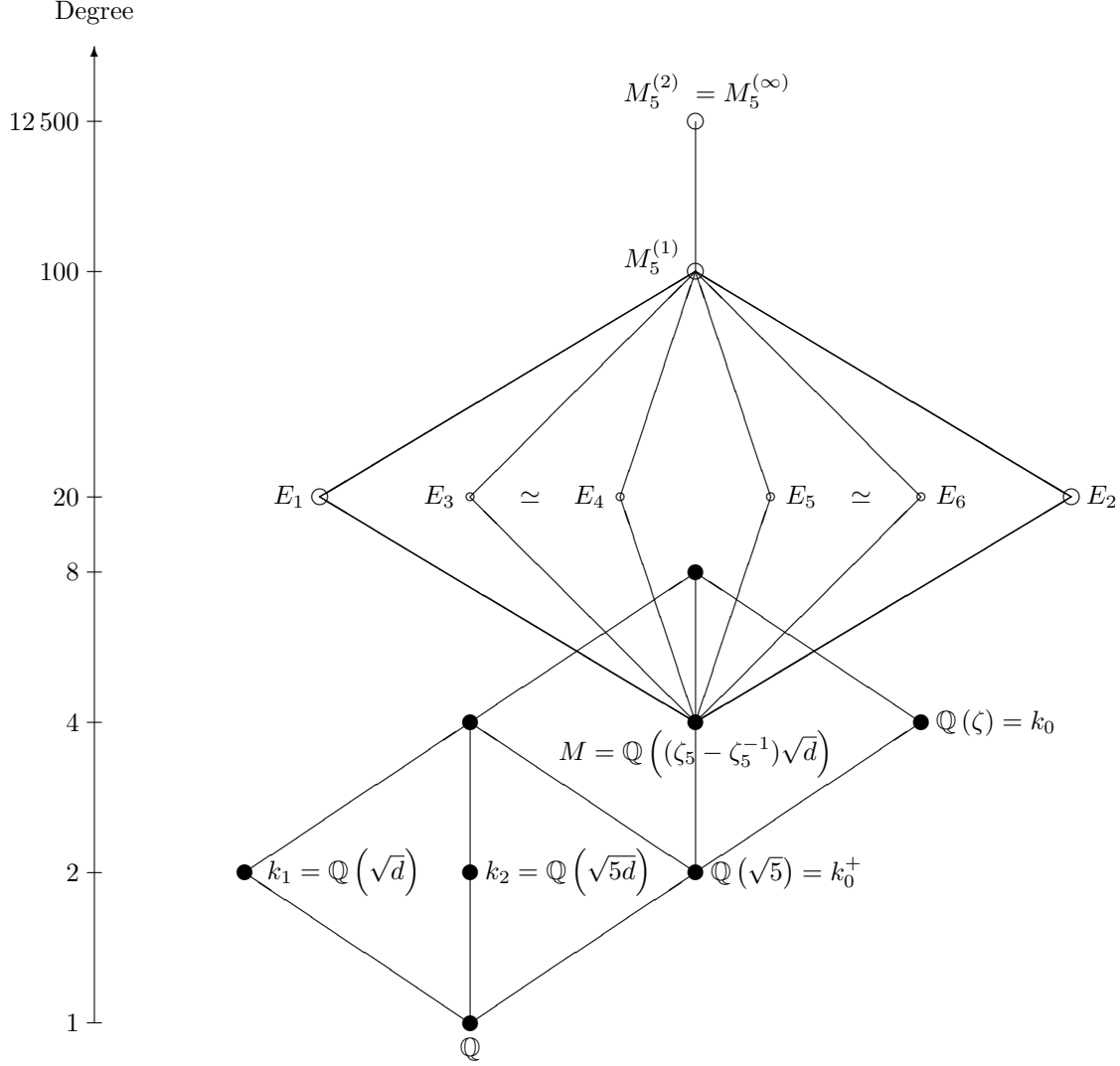
*the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (123456)$ , the identity permutation, IPAD  $\tau^{(1)}(M) = [1^2; (1^3)^6]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^5, \mathbf{14} \rangle$  (predicted by O. Taussky, 1970 [12, § 3.5.2, p. 448]).*

*Proof.* In each case, the length of the 5-class tower of  $M$  is given by  $\ell_5(M) = 2$ , since  $G := G_5^{(2)}M$  is a Schur  $\sigma$ -group with balanced presentation, i.e., relation rank  $d_2(G) = d_1(G) = \rho_5(M) = 2$ . □

**Remark 4.2.** The pairs of conjugate non-Galois extensions  $E_3 \simeq E_4$  and  $E_5 \simeq E_6$  of  $M$  are not adjacent in the factor (3546) of the cycle pattern (1)(2)(3546) of the 4-cycle  $\varkappa(M) = (125643)$ , and the Frobenius extensions  $E_1, E_2$  correspond to the fixed points (1), (2). The identity  $\varkappa(M) = (123456)$ , which does not have two distinguished fixed points a priori, is endowed with a random arithmetical bipolarization by the two Frobenius extensions  $E_1, E_2$ .

Figure 3 visualizes the situation of a two-stage 5-class tower in the Theorems 4.4, 4.5, 4.9.

FIGURE 3. 5-class tower  $M_5^{(\infty)}$  of the field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  if  $\#G_5^{(2)}M = 5^5$



**Theorem 4.5** (Unusual two-stage towers of 5-class fields).

- (1) For the **22** (**27%**) real quadratic fields  $k_1$  starting with the following discriminants

$$d \in \{257, 764, 1708, 1853, 2008, 2189, 3129, 4504, 4861, 5241, 5269, \dots\}$$

the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (022222)$ , nearly constant with a single total capitulation and a single fixed point, IPAD  $\tau^{(1)}(M) = [1^2; (1^3)^2, (21)^4]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^5, 4 \rangle$ .

- (2) For the **16** (**19%**) real quadratic fields  $k_1$  starting with the following discriminants

$$d \in \{508, 509, 629, 881, 1113, 1192, 1704, 1829, 3121, 4461, 7032, \dots\}$$

the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (124365)$ , two fixed points and two disjoint 2-cycles, IPAD  $\tau^{(1)}(M) = [1^2; (1^3)^2, (21)^4]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^5, 7 \rangle$ .

*Proof.* In each case, the length of the 5-class tower of  $M$  is given by  $\ell_5(M) = 2$ , since  $G := G_5^{(2)}M$  is a metabelian  $\sigma$ -group with trivial cover [13, Dfn. 5.1, p. 30], according to Heider and Schmithals [5, p. 20]. The presentation of  $G$  is not balanced, since the relation rank  $d_2(G) = 3$  is too big. However, the Shafarevich Theorem [17] in its corrected version [13, Thm. 5.1, p. 28] ensures that  $d_2(G) \leq d_1(G) + r = 3$  just reaches the admissible upper bound, since the generator rank of  $G$  and the torsionfree Dirichlet unit rank of  $M$  with signature  $(0, 2)$  are given by  $d_1(G) = \varrho_5(M) = 2$  and  $r = 0 + 2 - 1 = 1$ . The 5-tower groups  $\langle 5^5, 4 \rangle$  and  $\langle 5^5, 7 \rangle$  are unusual, because they are not strong  $\sigma$ -groups and thus are forbidden for quadratic base fields.  $\square$

**Remark 4.3.** The pairs of conjugate non-Galois extensions  $E_3 \simeq E_4$  and  $E_5 \simeq E_6$  of  $M$  correspond to the factors (34) and (56) of the cycle pattern (1)(2)(34)(56) of the two disjoint 2-cycles  $\varkappa(M) = (124365)$ , and the Frobenius extensions  $E_1, E_2$  correspond to the fixed points (1), (2). For the nearly constant type  $\varkappa(M) = (022222)$ , the first, resp. second, Frobenius extension  $E_1$ , resp.  $E_2$ , corresponds to the single total capitulation, resp. the single fixed point.

**Theorem 4.6** (Single-stage towers of 5-class fields).

For the 2 (2%) real quadratic fields  $k_1$  with discriminants  $d \in \{4357, 4444\}$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (000000)$ , a constant with six total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; (1)^6]$ , and abelian 5-class tower with group  $G_5^{(\infty)}M = G_5^{(1)}M \simeq \langle 5^2, 2 \rangle$  and length  $\ell_5(M) = 1$ .

*Proof.* Here, the 5-class tower is abelian with length  $\ell_5(M) = 1$ .  $\square$

**Remark 4.4.** Outside of the range  $0 < d < 10^4$  of our systematic investigations, we have discovered three occurrences of case (g) in Table 2. For the real quadratic fields  $k_1$  with discriminants  $d \in \{244641, 1277996, 1915448\}$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (000000)$ , a constant with six total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; (1)^6]$ , and abelian 5-class tower with group  $G_5^{(\infty)}M = G_5^{(1)}M \simeq \langle 5^2, 2 \rangle$  and length  $\ell_5(M) = 1$ . The invariants are given by  $(r_1, r_2, \delta_1, \delta_2) = (2, 0, 1, 1)$ .

**Theorem 4.7** (Frobenius and non-Galois extensions).

The properties of the absolute extensions  $E_i|\mathbb{Q}$  and the values of the invariants in the Quintic Reflection Theorem, Table 2, and Proposition 3.1, for the 83 cases in Proposition 4.2 are:

- (1) For the 2 cases with  $\ell_5(M) = 1$  in Theorem 4.6, we have

$$(r_1, r_2, \delta_1, \delta_2) = (1, 0, 0, 1), \text{ and } \text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,2} \text{ for } 1 \leq i \leq 6 \text{ (Case (a))}.$$

- (2) For the other 81 cases, including the 34 cases of  $\ell_5(M) = 2$  in Theorem 4.4 and the 38 cases of  $\ell_5(M) = 2$  in Theorem 4.5, we have pairwise conjugate non-Galois extensions  $E_3 \simeq E_4, E_5 \simeq E_6$ ,

$$\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}, \text{ Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}, \text{ and}$$

$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (1, 0, 1, 0), \text{ for } d \in \{1996, 3121, 3129, 3253, 5241, 5269, 5308, \\ 6113, 8309, 8689, 9829\} \text{ (Case (e))}, \\ (r_1, r_2, \delta_1, \delta_2) = (0, 1, 0, 1), \text{ for } d \in \{5116, 8972, 9013\} \text{ (Case (f))}, \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 1, 1), \text{ for } d \in \{4504, 6949, 7221, 7229, 9669\} \text{ (Case (c))}, \\ (r_1, r_2, \delta_1, \delta_2) = (0, 0, 0, 0), \text{ otherwise (Case (d))}. \end{cases}$$

*Proof.* See Tables 4 and 5.  $\square$

#### 4.2. Real cyclic quartic fields $M$ with $d < 0$ .

**Proposition 4.3.** Let  $M = \mathbb{Q}\left(\left(\zeta_5 - \zeta_5^{-1}\right)\sqrt{d}\right)$  with  $d < 0$  be a real cyclic quartic field with 5-class group of type (5, 5). Denote by  $E_i$ ,  $1 \leq i \leq 6$ , the six unramified cyclic quintic extensions of  $M$  and by  $L_i$  their non-Galois subfields, which are of relative degree 5 over the field  $k_0^+ = \mathbb{Q}(\sqrt{5})$ .

- (1) If  $\#\ker(j_{E_i/M}) = 5$ , then the unit norm index is  $u_i = (U_{k_0^+} : N_{L_i/k_0^+}(U_{L_i})) = 1$ , and in this case the subfield unit index  $a_i = (U_{E_i} : U_{L_i}U_{L_i}U_M)$  is equal to 25.

(2) The relation between the 5-class numbers  $h_5(E_i)$  and  $h_5(L_i)$  is given by

$$h_5(E_i) = \begin{cases} 5 \cdot h_5(L_i)^2 & \text{if } b_i = u_i, \\ h_5(L_i)^2 & \text{if } b_i \neq u_i. \end{cases}$$

*Proof.* Since  $d < 0$ , the field  $M$  is totally real, and the Dirichlet rank of its torsionfree unit group is given by  $r(M) = 3$ .

(1) Denote by  $U_{M/\mathbb{Q}(\sqrt{5})}$  the group of relative units, that is  $\{\epsilon \in U_M \mid N_{M/\mathbb{Q}(\sqrt{5})}(\epsilon) = 1\}$ .

For a cyclic quartic field  $K/\mathbb{Q}$  with real quadratic subfield  $k$ , Hasse showed that the group  $U_k U_{K/k}$  has index at most 2 in the full group of units  $U_K$ .

In our case,  $U_{\mathbb{Q}(\sqrt{5})} U_{M/\mathbb{Q}(\sqrt{5})}$  has index at most 2 in  $U_M$ , and  $U_M = \langle -1, \epsilon_5, \eta, \eta^{\mu\tau} \rangle$ , where  $\eta$  satisfies  $\eta^{1+\mu\tau} = \pm 1$ .

If  $\#\ker(j_{E_i/M}) = 5$ , which means that the unit norm index  $b_i = (U_M : N_{E_i/M}(U_{E_i}))$  is equal to 1, then all units of  $M$  are norm of a unit of  $E_i$ , in particular  $\epsilon_5$ . In the same manner as in the proof of claim 1 of Proposition 4.1, we deduce that  $\epsilon_5$  is also norm of a unit of  $L_i$ , whence  $u_i = 1$ . On the other hand, by applying Lemma 4.1, we deduce that  $a_i \cdot b_i = 25 \cdot u_i$ , and consequently  $a_i = 25$ .

(2) According to Theorem 4.1 or the Lemmermeyer class number formula [10, Thm. 4.1, p. 456], we conclude that  $h_5(E_i) = 5 \cdot h_5(L_i)^2$  if  $b_i = 1$  or ( $b_i = 5$  and  $u_i = 1$ ). But if  $b_i = 5$  and  $u_i = 5$ , we have  $h_5(E_i) = h_5(L_i)^2$ , which completes the proof.  $\square$

**Remark 4.5.** For totally real or imaginary cyclic quartic fields  $M$ , the last case  $h_5(E_i) = 25 \cdot h_5(L_i)^2$ , is impossible for any  $1 \leq i \leq 6$ .

**Proposition 4.4.** Let  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $d < 0$  be a real cyclic quartic field with 5-class group of type  $(5, 5)$ . Let  $E_i$ ,  $1 \leq i \leq 6$ , be the six unramified cyclic quintic extensions of  $M$ . Denote by  $G := G_5^{(2)}(M)$  the second 5-class group of  $M$  and assume that the order of  $G$  is equal to  $|G| = 5^3$ , then the transfer kernel type of  $G$  is  $\varkappa(G) = (000000)$  (capitulation type of  $M$  in the six unramified extensions  $E_i$ ) and the transfer target type of  $G$  is  $\tau(G) = ((1^2)^6)$ .

*Proof.* In this case, the group  $G$  is extra special of maximal class. Thus, the possible types of capitulation are  $(111111)$  and  $(000000)$ . First, we know that the type  $(111111)$  is not possible, because in this case  $h_5(E_i) = 5^2$  and  $b_i = 1$ , which contradicts claim (2) of Proposition 4.3. Thus, the transfer kernel type of  $G$  is  $\varkappa(G) = (000000)$ .

On the other hand, for all  $1 \leq i \leq 6$ , the unit norm index is  $b_i = 5$ , and  $u_i$  must be equal to 1. Otherwise, the Lemmermeyer class number formula [10, Thm. 4.1, p. 456] implies  $|G| \geq 5^4$ . Thus, for all  $1 \leq i \leq 6$ , we have  $h_5(E_i) = h_5(L_i)^2$  and  $h_5(L_i) = 5$ . Since the six extensions  $E_i$  are of type  $A$  in the sense of Taussky and  $h_5(E_i) = 5^2$ , we deduce that  $\text{Cl}_5(E_i)$  is of type  $(5, 5)$ . Thus  $\tau(G) = ((1^2)^6)$ .  $\square$

**Remark 4.6.** Assume that the group  $G$  is not abelian and  $\varkappa(G) = (000000)$ , then the prime 5 must divide the class number of the fields  $L_i$ . Because, in this case  $b_i = 5$  and  $u_i = 1$  or 5. The case  $u_i = 1$  is obvious. Now suppose that  $u_i = 5$ . If 5 does not divide  $h(L_i)$ , then  $h_5(E_i) = 5$ . Hence  $E_i$  is an unramified extension of  $M$  and  $h_5(E_i) = \frac{h_5(M)}{5}$ . Then  $M_5^{(2)} = M_5^{(1)}$  and the group  $G$  is abelian, which is a contradiction.

**Proposition 4.5** (Number of fields). In the range  $-200000 < d < 0$  of fundamental discriminants  $d$  of imaginary quadratic fields  $k_1 = \mathbb{Q}(\sqrt{d})$  with  $\gcd(5, d) = 1$  there exist precisely **109** cases such that the 5-dual field  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  of  $k_1$  has a 5-class group  $\text{Cl}_5(M)$  of type  $(5, 5)$ .

*Proof.* See Tables 6, 7 and 8.  $\square$

**Theorem 4.8** (Low order two-stage towers of 5-class fields).

- (1) For the **66** (**61%**) imaginary quadratic fields  $k_1$  starting with the following discriminants  $d \in \{-12883, -13147, -14339, -23336, -23732, -26743, -28696, -35067, -35839, -38984, -47172, \dots\}$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type a.1,  $\varkappa(M) = (000000)$ , a constant with six total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; (1^2)^6]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^3, \mathbf{3} \rangle$ .
- (2) For the **26** (**24%**) imaginary quadratic fields  $k_1$  starting with the following discriminants  $d \in \{-27528, -27939, -39947, -40823, -54347, -75892, -91127, -99428, -101784, -105431, -114679, \dots\}$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type a.2,  $\varkappa(M) = (100000)$ , with a fixed point and five total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; 1^3, (1^2)^5]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^4, \mathbf{8} \rangle$ .
- (3) For the **11** (**10%**) imaginary quadratic fields  $k_1$  with the following discriminants  $d \in \{-15419, -16724, -31103, -42899, -67128, -70763, -77095, -83620, -105784, -115515, -194487\}$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type a.1,  $\varkappa(M) = (000000)$ , a constant with six total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; 1^3, (1^2)^5]$ , and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^4, \mathbf{7} \rangle$ .

*Proof.* In each case, the length of the 5-class tower of  $M$  is given by  $\ell_5(M) = 2$ , according to Blackburn [3], since  $G := G_5^{(2)}M$  is a  $\sigma$ -group with at most two-generated commutator subgroup  $G' \in \{1, 1^2\}$ . The presentation of  $G$  is not balanced, since the relation rank  $d_2(G) \in \{3, 4\}$  is too big. However, the Shafarevich Theorem [17] in its corrected version [13, Thm. 5.1, p. 28] ensures that  $d_2(G) \leq d_1(G) + r = 5$  does not exceed the admissible upper bound, since the generator rank of  $G$  and the torsionfree Dirichlet unit rank of  $M$  with signature  $(4, 0)$  are given by  $d_1(G) = \rho_5(M) = 2$  and  $r = 4 + 0 - 1 = 3$ .  $\square$

**Theorem 4.9** (Exotic two-stage tower of 5-class fields).

For the **unique** imaginary quadratic field  $k_1$  with discriminant  $d = -114303$  the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (124563)$ , a 4-cycle and two fixed points, and 5-class tower group  $G_5^{(\infty)}M = G_5^{(2)}M \simeq \langle 5^5, \mathbf{11} \rangle$ . In this case, the length of the 5-class tower of  $M$  is given by  $\ell_5(M) = 2$ , and  $G := G_5^{(2)}M$  is a Schur  $\sigma$ -group with balanced presentation, that is, relation rank  $d_2(G) = d_1(G) = \rho_5(M) = 2$ .

*Proof.* Similar to the proof of Theorem 4.4.  $\square$

**Theorem 4.10** (Single-stage towers of 5-class fields).

For the **5** (**5%**) imaginary quadratic fields  $k_1$  with discriminants

$$d \in \{-58424, -115912, -148507, -151879, -154408\}$$

the 5-dual field  $M$  of  $k_1$  has 5-capitulation type  $\varkappa(M) = (000000)$ , a constant with six total capitulations, IPAD  $\tau^{(1)}(M) = [1^2; (1)^6]$ , and abelian 5-class tower with group  $G_5^{(\infty)}M = G_5^{(1)}M \simeq \langle 5^2, \mathbf{2} \rangle$  and length  $\ell_5(M) = 1$ .

*Proof.* Similar to the proof of Theorem 4.6.  $\square$

**Theorem 4.11** (Frobenius and non-Galois extensions).

The properties of the absolute extensions  $E_i|\mathbb{Q}$  and the values of the invariants in the Quintic Reflection Theorem, Table 3, and Proposition 3.3, for the **109** cases in Proposition 4.5 are:

- (1) For the **5** cases with  $\ell_5(M) = 1$  in Theorem 4.10, we have

$$(r_1, r_2, \delta_1, \delta_2) = (2, 0, 0, 0), \text{ and } \text{Gal}(E_i|\mathbb{Q}) \simeq F_{5,2} \text{ for } 1 \leq i \leq 6 \text{ (Case (a))}.$$

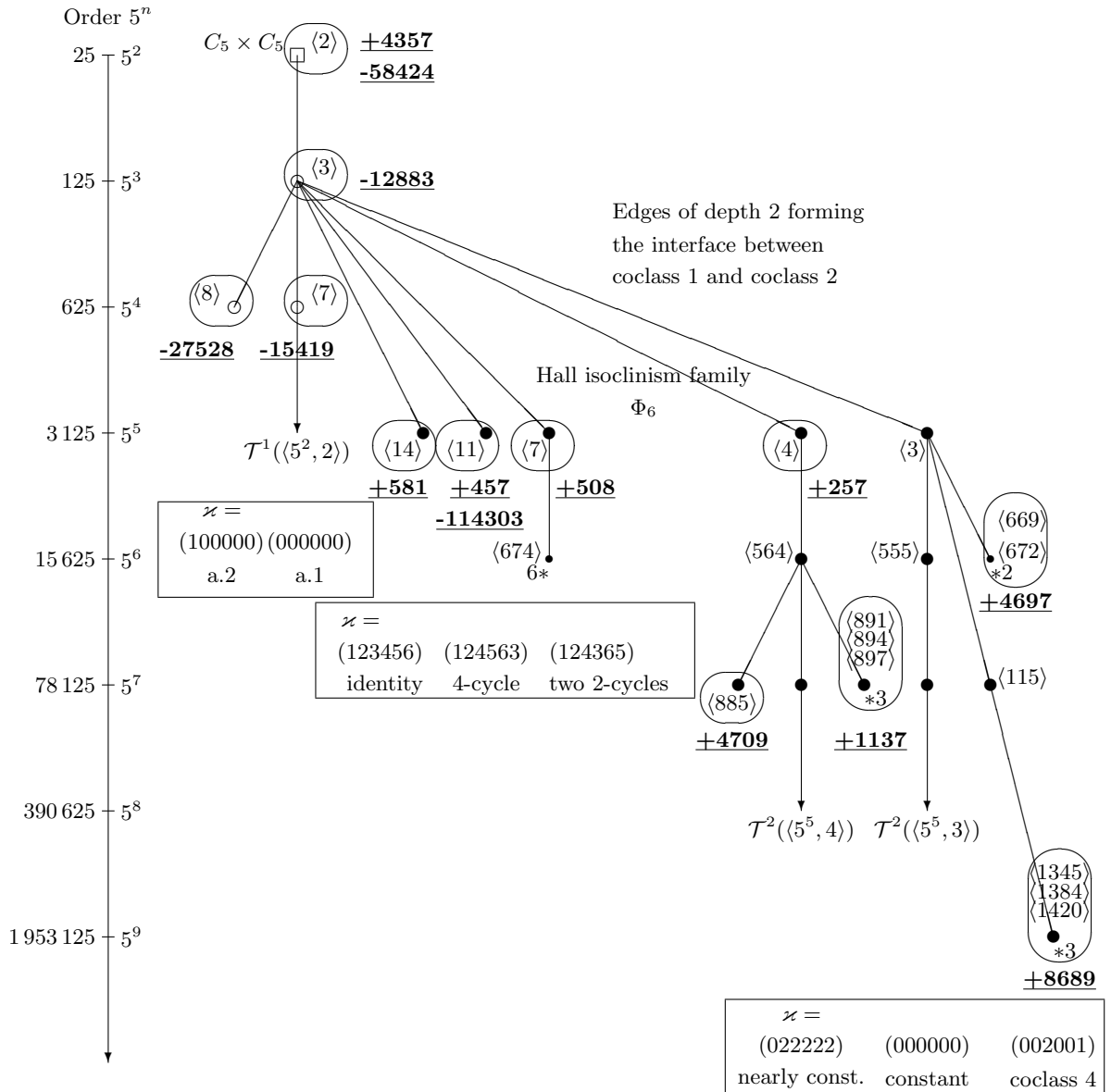
- (2) For the other **104** cases, including the **103** cases of  $\ell_5(M) = 2$  in Theorem 4.8, and the **unique** case of  $\ell_5(M) = 2$  in Theorem 4.9, we have pairwise conjugate non-Galois extensions  $E_3 \simeq E_4$ ,  $E_5 \simeq E_6$ ,  $\text{Gal}(E_1|\mathbb{Q}) \simeq F_{5,2}$ ,  $\text{Gal}(E_2|\mathbb{Q}) \simeq F_{5,3}$ , and

$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (2, 1, 1, 0), \text{ for } d \in \{-39947, -64103, -67128, -104503, -119191\} \text{ (Case (d))}, \\ (r_1, r_2, \delta_1, \delta_2) = (1, 2, 0, 1), \text{ for } d \in \{-110479, -199735\} \text{ (Case (e))}, \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 0, 0), \text{ otherwise (Case (c))}. \end{cases}$$

*Proof.* See Tables 6, 7 and 8. □

Figure 4 visualizes the relevant part of the descendant tree of finite 5-groups, beginning at the abelian root  $C_5 \times C_5 = \langle 5^2, 2 \rangle$ , on which the second 5-class groups  $G_5^{(2)}M$  of the fields  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  are located as vertices. The minimal positive, resp. maximal negative, discriminants  $d$  are indicated by underlined boldface integers adjacent to the oval surrounding the realized vertex [2, 11, 14].

FIGURE 4. Tree position of second 5-class groups  $G_5^{(2)}M$  of the fields  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$



### 5. TABLES OF SECOND 5-CLASS GROUPS $G_5^{(2)}M$

5.1. **Imaginary cyclic quartic fields  $M$  with  $d > 0$ .** Table 4, resp. Table 5, shows the factorized fundamental discriminant  $d$  of the dual quadratic field  $k_1$ , the 5-principalization type  $\varkappa = \varkappa(M)$ , the second 5-class group  $G_5^{(2)}M$ , the 5-class ranks  $r_1 := \varrho_5(k_1)$ ,  $r_2 := \varrho_5(k_2)$ , the



invariants  $\delta_1$ ,  $\delta_2$ , and the case in Proposition 3.1 for the 37, resp. 46, cyclic quartic fields  $M = \mathbb{Q}\left(\left(\zeta_5 - \zeta_5^{-1}\right)\sqrt{d}\right)$  with  $0 < d < 5000$ , resp.  $5000 < d < 10000$ .

For the fields with constant 5-principalization type, consisting of partial kernels, we have a *polarization* of the target type whose abelian invariants can be either homogeneous ( $1^5$ ) or inhomogeneous ( $21^3$ ). In the inhomogeneous case, there are three possibilities for the second 5-class group, namely  $\langle 5^7, 891 \rangle$ ,  $\langle 5^7, 894 \rangle$  and  $\langle 5^7, 897 \rangle$ . In the homogeneous case, the second 5-class group  $\langle 5^7, 885 \rangle$  is unique.

The *complete statistics* of the 83 *imaginary* cyclic quartic fields  $M$  with  $0 < d < 10^4$  is as follows. There are 23 (28%) cases with  $G_5^{(2)}M \simeq \langle 5^5, 11 \rangle$ , 22 (27%) cases with  $G_5^{(2)}M \simeq \langle 5^5, 4 \rangle$ , 16 (19%) cases with  $G_5^{(2)}M \simeq \langle 5^5, 7 \rangle$ , and 11 (13%) cases with  $G_5^{(2)}M \simeq \langle 5^5, 14 \rangle$ . For only 4 cases we have  $G_5^{(2)}M \simeq \langle 5^7, 885 \rangle$ , for 2 cases  $G_5^{(2)}M \simeq \langle 5^7, 891|894|897 \rangle$ , for 2 cases  $G_5^{(2)}M \simeq \langle 5^2, 2 \rangle$ , for 2 cases  $G_5^{(2)}M$  is descendant of  $\langle 5^5, 3 \rangle$ , and for a unique case  $G_5^{(2)}M \simeq \langle 5^7, 115 \rangle - \#2; 1345|1384|1420$ .

**5.2. Real cyclic quartic fields  $M$  with  $d < 0$ .** Table 6, resp. Table 7, resp. Table 8, shows the factorized fundamental discriminant  $d$  of the dual quadratic field  $k_1$ , the 5-principalization type  $\varkappa = \varkappa(M)$ , the second 5-class group  $G_5^{(2)}M$ , the 5-class ranks  $r_1 := \varrho_5(k_1)$ ,  $r_2 := \varrho_5(k_2)$ , the invariants  $\delta_1$ ,  $\delta_2$ , and the case in Proposition 3.3 for the 43, resp. 45, resp. 21, cyc. quart. fields  $M = \mathbb{Q}\left(\left(\zeta_5 - \zeta_5^{-1}\right)\sqrt{d}\right)$  with  $-100000 < d < 0$ , resp.  $-175000 < d < -100000$ , resp.  $-200000 < d < -175000$ .

The *complete statistics* of the 109 *real* cyclic quartic fields  $M$  with  $-2 \cdot 10^5 < d < 0$  is as follows. There are 66 (61%) cases with  $G_5^{(2)}M \simeq \langle 5^3, 3 \rangle$ , 26 (24%) cases with  $G_5^{(2)}M \simeq \langle 5^4, 8 \rangle$ , and 11 (10%) cases with  $G_5^{(2)}M \simeq \langle 5^4, 7 \rangle$ . For only 5 cases we have  $G_5^{(2)}M \simeq \langle 5^2, 2 \rangle$ , and for a unique case  $G_5^{(2)}M \simeq \langle 5^5, 11 \rangle$ .

**5.3. The Galois action confirmed.** All numerical results in the Tables 4 to 8 are in perfect accordance with Theorems 2.2, 2.3 and Corollary 2.1. A rigorous check with the computational algebra system MAGMA [11, 2] proves that only the two terminal Schur  $\sigma$ -groups  $\langle 5^5, 11 \rangle$ ,  $\langle 5^5, 14 \rangle$  and five other capable top vertices  $\langle 5^5, 3 \rangle$ ,  $\langle 5^5, 4 \rangle$ ,  $\langle 5^5, 5 \rangle$ ,  $\langle 5^5, 6 \rangle$ ,  $\langle 5^5, 7 \rangle$  in the stem of Hall's isoclinism family  $\Phi_6$ , and the abelian root  $\langle 5^2, 2 \rangle$ , together with their descendants [14], are admissible for  $G_5^{(2)}M$  of any cyclic quartic field  $M$ , as drawn in Figure 4.

TABLE 4. The group  $G_5^{(2)}M$  of  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $0 < d < 5000$ 

No.	Discriminant		Principalization		$G_5^{(2)}M$	Invariants				
	$d$	Factors	$\varkappa$	Remark		$r_1$	$\delta_1$	$r_2$	$\delta_2$	Case
1	257	prime	(660666)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
2	457	prime	(234156)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
3	501	3, 167	(521346)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
4	508	4, 127	(653421)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
5	509	prime	(216453)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
6	581	7, 83	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
7	629	17, 37	(154326)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
8	753	3, 251	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
9	764	4, 191	(666066)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
10	881	prime	(463152)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
11	1113	3, 7, 53	(653421)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
12	1137	3, 379	(444444)	constant	$\langle 5^7, 891 894 897 \rangle$	0	0	0	0	(d)
13	1192	8, 149	(463152)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
14	1704	8, 3, 71	(653421)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
15	1708	4, 7, 61	(404444)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
16	1829	31, 59	(216453)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
17	1853	17, 109	(550555)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
18	1996	4, 499	(613254)	4-cycle	$\langle 5^5, 11 \rangle$	1	1	0	0	(e)
19	2008	8, 251	(550555)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
20	2189	11, 199	(505555)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
21	2296	8, 7, 41	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
22	2573	31, 83	(613254)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
23	2829	3, 23, 41	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
24	3121	prime	(532416)	two 2-cycles	$\langle 5^5, 7 \rangle$	1	1	0	0	(e)
25	3129	3, 7, 149	(333303)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	0	0	(e)
26	3169	prime	(444444)	constant	$\langle 5^7, 891 894 897 \rangle$	0	0	0	0	(d)
27	3253	prime	(243651)	4-cycle	$\langle 5^5, 11 \rangle$	1	1	0	0	(e)
28	4189	59, 71	(243651)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
29	4357	prime	(000000)	abelian	$\langle 5^2, 2 \rangle$	1	0	0	1	(a)
30	4444	4, 11, 101	(000000)	abelian	$\langle 5^2, 2 \rangle$	1	0	0	1	(a)
31	4461	3, 1487	(653421)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
32	4504	8, 563	(444404)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	1	1	(c)
33	4553	29, 157	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
34	4697	7, 11, 61	(000000)	tot., non-ab.	$\langle 5^5, 3 \rangle \downarrow$	0	0	0	0	(d)
35	4709	17, 277	(444444)	constant	$\langle 5^7, 885 \rangle$	0	0	0	0	(d)
36	4861	prime	(333303)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
37	4957	prime	(135246)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)

TABLE 5. The group  $G_5^{(2)}M$  of  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $5000 < d < 10000$ 

No.	Discriminant		Principalization		$G_5^{(2)}M$	Invariants				
	$d$	Factors	$\varkappa$	Remark		$r_1$	$\delta_1$	$r_2$	$\delta_2$	Case
38	5116	4, 1279	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	1	1	(f)
39	5129	23, 223	(526431)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
40	5233	prime	(142536)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
41	5241	3, 1747	(660666)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	0	0	(e)
42	5269	11, 479	(222220)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	0	0	(e)
43	5308	4, 1327	(513462)	4-cycle	$\langle 5^5, 11 \rangle$	1	1	0	0	(e)
44	5361	3, 1787	(625413)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
45	5393	prime	(440444)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
46	5464	8, 683	(440444)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
47	5557	prime	(111111)	constant	$\langle 5^7, 885 \rangle$	0	0	0	0	(d)
48	5736	8, 3, 239	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
49	5989	53, 113	(440444)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
50	6072	8, 3, 11, 23	(613254)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
51	6073	prime	(000000)	tot., non-ab.	$\langle 5^5, 3 \rangle \downarrow$	0	0	0	0	(d)
52	6113	prime	(421653)	4-cycle	$\langle 5^5, 11 \rangle$	1	1	0	0	(e)
53	6524	4, 7, 233	(513462)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
54	6761	prime	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
55	6949	prime	(666066)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	1	1	(c)
56	6952	8, 11, 79	(220222)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
57	7032	8, 3, 293	(213546)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
58	7041	3, 2347	(666666)	constant	$\langle 5^7, 885 \rangle$	0	0	0	0	(d)
59	7221	3, 29, 83	(444404)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	1	1	(c)
60	7229	prime	(444444)	constant	$\langle 5^7, 885 \rangle$	1	1	1	1	(c)
61	7336	8, 7, 131	(606666)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
62	7361	17, 433	(653421)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
63	7489	prime	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	0	0	(d)
64	7628	4, 1907	(164253)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
65	7656	8, 3, 11, 29	(444404)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
66	7752	8, 3, 17, 19	(623145)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
67	7833	3, 7, 373	(326154)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
68	7996	4, 1999	(022222)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
69	8008	8, 7, 11, 13	(625413)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
70	8012	4, 2003	(165432)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
71	8309	7, 1187	(111110)	nearly const.	$\langle 5^5, 4 \rangle$	1	1	0	0	(e)
72	8689	prime	(002001)	coclass 4	$\langle 5^7, 115 \rangle \downarrow$	1	1	0	0	(e)
73	8789	11, 17, 47	(362451)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
74	8877	3, 11, 269	(463152)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
75	8972	4, 2243	(362451)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	1	1	(f)
76	9013	prime	(123456)	identity	$\langle 5^5, 14 \rangle$	0	0	1	1	(f)
77	9052	4, 31, 73	(333303)	nearly const.	$\langle 5^5, 4 \rangle$	0	0	0	0	(d)
78	9544	8, 1193	(125364)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
79	9564	4, 3, 797	(425136)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
80	9573	3, 3191	(216453)	two 2-cycles	$\langle 5^5, 7 \rangle$	0	0	0	0	(d)
81	9669	3, 11, 293	(362451)	4-cycle	$\langle 5^5, 11 \rangle$	1	1	1	1	(c)
82	9752	8, 23, 53	(513462)	4-cycle	$\langle 5^5, 11 \rangle$	0	0	0	0	(d)
83	9829	prime	(123456)	identity	$\langle 5^5, 14 \rangle$	1	1	0	0	(e)

TABLE 6. The group  $G_5^{(2)}M$  of  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $-100000 < d < 0$ 

No.	Discriminant		Principalization		$G_5^{(2)}M$	Invariants				
	$d$	Factors	$\varkappa$	Type		$r_1$	$\delta_1$	$r_2$	$\delta_2$	Case
1	-12883	13, 991	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
2	-13147	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
3	-14339	13, 1103	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
4	-15419	17, 907	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
5	-16724	4, 37, 113	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
6	-23336	8, 2917	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
7	-23732	4, 17, 349	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
8	-26743	47, 569	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
9	-27528	8, 3, 31, 37	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
10	-27939	3, 67, 139	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
11	-28696	8, 17, 211	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
12	-31103	19, 1637	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
13	-35067	3, 11689	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
14	-35839	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
15	-38984	8, 11, 443	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
16	-39947	43, 929	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	2	1	1	0	(d)
17	-40823	prime	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
18	-42899	prime	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
19	-47172	4, 3, 3931	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
20	-52276	4, 7, 1867	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
21	-54347	prime	(100000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
22	-55667	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
23	-56167	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
24	-58424	8, 67, 109	(000000)	abelian	$\langle 5^2, 2 \rangle$	2	0	0	0	(a)
25	-64103	13, 4931	(000000)	a.1	$\langle 5^3, 3 \rangle$	2	1	1	0	(d)
26	-64415	5, 13, 991	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
27	-64724	4, 11, 1471	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
28	-65735	5, 13147	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
29	-67128	8, 3, 2797	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	2	1	1	0	(d)
30	-69619	11, 6329	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
31	-70763	7, 11, 919	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
32	-71695	5, 13, 1103	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
33	-74019	3, 11, 2243	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
34	-75103	7, 10729	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
35	-75892	4, 18973	(100000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
36	-77095	5, 17, 907	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
37	-78747	3, 26249	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
38	-83620	4, 5, 37, 113	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
39	-83636	4, 7, 29, 103	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
40	-86404	4, 21601	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
41	-91127	prime	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
42	-92219	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
43	-99428	4, 7, 53, 67	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)

TABLE 7. The group  $G_5^{(2)}M$  of  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $-175000 < d < -100000$ 

No.	Discriminant		Principalization		$G_5^{(2)}M$	Invariants				
	$d$	Factors	$\varkappa$	Type		$r_1$	$\delta_1$	$r_2$	$\delta_2$	Case
44	-100708	4, 17, 1481	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
45	-101011	83, 1217	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
46	-101784	8, 3, 4241	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
47	-104503	7, 14929	(000000)	a.1	$\langle 5^3, 3 \rangle$	2	1	1	0	(d)
48	-105431	19, 31, 179	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
49	-105784	8, 7, 1889	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
50	-107791	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
51	-110479	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	2	1	(e)
52	-114303	3, 7, 5443	(263415)	4-cycle	$\langle 5^5, 11 \rangle$	1	0	1	0	(c)
53	-114679	prime	(000006)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
54	-115912	8, 14489	(000000)	abelian	$\langle 5^2, 2 \rangle$	2	0	0	0	(a)
55	-116680	8, 5, 2917	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
56	-118660	4, 5, 17, 349	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
57	-119191	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	2	1	1	0	(d)
58	-123028	4, 30757	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
59	-124099	193, 643	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
60	-125547	3, 41849	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
61	-127259	11, 23, 503	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
62	-127519	7, 18217	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
63	-133188	4, 3, 11, 1009	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
64	-133715	5, 47, 569	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
65	-134392	8, 107, 157	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
66	-136311	3, 7, 6491	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
67	-137640	8, 3, 5, 31, 37	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
68	-139695	3, 5, 67, 139	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
69	-139703	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
70	-140232	8, 3, 5843	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
71	-142904	8, 17863	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
72	-143480	8, 5, 17, 211	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
73	-145007	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
74	-145668	4, 3, 61, 199	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
75	-148004	4, 163, 227	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
76	-148507	97, 1531	(000000)	abelian	$\langle 5^2, 2 \rangle$	2	0	0	0	(a)
77	-151879	7, 13, 1669	(000000)	abelian	$\langle 5^2, 2 \rangle$	2	0	0	0	(a)
78	-154408	8, 19301	(000000)	abelian	$\langle 5^2, 2 \rangle$	2	0	0	0	(a)
79	-155515	5, 19, 1637	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
80	-155603	7, 22229	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
81	-157028	4, 37, 1061	(003000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
82	-157031	7, 22433	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
83	-159679	13, 71, 173	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
84	-160571	211, 761	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
85	-163427	11, 83, 179	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
86	-164116	4, 89, 461	(000006)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
87	-165364	4, 41341	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
88	-169752	8, 3, 11, 643	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)

TABLE 8. The group  $G_5^{(2)}M$  of  $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$  with  $-200000 < d < -175000$ 

No.	Discriminant		Principalization		$G_5^{(2)}M$	Invariants				
	$d$	Factors	$\varkappa$	Type		$r_1$	$\delta_1$	$r_2$	$\delta_2$	Case
89	-175076	4, 11, 23, 173	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
90	-175335	3, 5, 11689	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
91	-176459	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
92	-177428	4, 44357	(100000)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
93	-179195	5, 35839	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
94	-180583	13, 29, 479	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
95	-181847	43, 4229	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
96	-182968	8, 22871	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
97	-185883	3, 61961	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
98	-186187	prime	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
99	-186271	prime	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
100	-190387	prime	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
101	-193483	191, 1013	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
102	-193571	7, 27653	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
103	-194487	3, 241, 269	(000000)	a.1 $\uparrow$	$\langle 5^4, 7 \rangle$	1	0	1	0	(c)
104	-194920	8, 5, 11, 443	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
105	-196648	8, 47, 523	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
106	-196707	3, 7, 17, 19, 29	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)
107	-197752	8, 19, 1301	(000000)	a.1	$\langle 5^3, 3 \rangle$	1	0	1	0	(c)
108	-199735	5, 43, 929	(000050)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	2	1	(e)
109	-199947	3, 11, 73, 83	(000400)	a.2, fixed point	$\langle 5^4, 8 \rangle$	1	0	1	0	(c)

## 6. ACKNOWLEDGEMENTS

The third author gratefully acknowledges that his research is supported financially by the Austrian Science Fund (FWF): P 26008-N25.

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