5-CLASS TOWERS OF CYCLIC QUARTIC FIELDS ARISING FROM QUINTIC REFLECTION

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Abstract. Let ζ_5 be a primitive fifth root of unity and $d \neq 1$ be a quadratic fundamental discriminant not divisible by 5. For the 5-dual cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ of the quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$ in the sense of the quintic reflection theorem, the possibilities for the isomophism type of the Galois group $\mathrm{G}_5^{(2)}M = \mathrm{Gal}(M_5^{(2)}/M)$ of the second Hilbert 5-class field $M_5^{(2)}$ of M are investigated, when the 5-class group $\mathrm{Cl}_5(M)$ is elementary bicyclic of rank two. Usually, the maximal unramified pro-5 extension $M_5^{(\infty)}$ of M coincides with $M_5^{(2)}$ already. The precise length $\ell_5 M$ of the 5-class tower of M is determined, if $\mathrm{G}_5^{(2)}M$ is of order less than or equal to 5^5. Theoretical results are underpinned by the actual computation of all 83, respectively 109, cases in the range $0 < d < 10^4$, respectively $-2 \cdot 10^5 < d < 0$.

1. Introduction

The present article arose from the desire to generalize our results [1] for the second 3-class group $\mathrm{Gal}\left(k_3^{(2)}/k\right)$ of the bicyclic biquadratic field $k = \mathbb{Q}\left(\sqrt{-3}, \sqrt{d}\right)$, which is the compositum of 3-dual quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{-3d})$ in the cubic reflection theorem, to the situation of the quintic reflection theorem.

The precise statement of both reflection theorems requires the concept of *virtual units*. Let p be a prime number and K be a number field with multiplicative group $K^\times = K \setminus \{0\}$, maximal order $\mathcal{O}$, unit group U, fractional ideal group $\mathcal{I}$, and p-class rank ϱ_p. The quotient $V_p = I_p/(K^\times)^p$, where $I_p = \{\alpha \in K^\times \mid \alpha\mathcal{O} = \mathfrak{a}^p \text{ for some } \mathfrak{a} \in \mathcal{I}\}$, is an elementary abelian p-group of rank $\sigma_p = \varrho_p + \dim_{\mathbb{F}_p}(U/U^p)$ and is called the *p-Selmer group* of non-trivial *p-virtual units*, that is, generators of principal pth powers of ideals of K. We refer to σ_p as the *p-Selmer rank* of K.

1.1. Cubic reflection theorem.

It is well known that the 3-Selmer ranks $\sigma_3(k_1)$ and $\sigma_3(k_2)$ of 3-dual quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{-3d})$ ($d > 0$ square free) with respect to the quadratic cyclotomic mirror field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $\zeta_3 = \exp(2\pi i/3)$, satisfy the *cubic reflection theorem*

$$\sigma_3(k_2) = \sigma_3(k_1) - \delta, \tag{1.1}$$

which is a consequence of comparing the numbers of cyclic cubic extensions of k_1 and k_2 which are unramified outside of 3 from the viewpoint of both, class field theory and Kummer theory. The invariant $0 \le \delta \le 1$ depends on the 3-virtual units of k_1 and k_2. More precisely, we have

$$\delta = \begin{cases} 0, & \text{if } V_3(k_2) \text{ (imaginary)} \\ 1, & \text{if } V_3(k_1) \text{ (real)} \end{cases} \text{ contains a 3-virtual unit which is not 3-primary.} \tag{1.2}$$

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1.2. **Quintic reflection theorem.** If $d \neq 1$ denotes a square free integer prime to 5, then the 5-Selmer ranks $\sigma_5(k_1)$, $\sigma_5(k_2)$ of associated quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$, $k_2 = \mathbb{Q}(\sqrt{5d})$ and the 5-class rank $\varrho_5(M)$ of their 5-dual cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $\zeta_5 = \exp(2\pi i/5)$, with respect to the quartic cyclotomic mirror field $k_0 = \mathbb{Q}(\zeta_5)$ satisfy the *quintic reflection theorem*

$$\varrho_5(M) = \sigma_5(k_1) + \sigma_5(k_2) - \delta_1 - \delta_2, \tag{1.3}$$

where the invariants $0 \leq \delta_1, \delta_2 \leq 1$ depend on the 5-virtual units of k_1 and k_2 [9, p. 2]. The formula is derived by comparing the numbers of cyclic quintic extensions of k_1, k_2 and M which are unramified outside of 5. The maximal real subfield of $k_0 = \mathbb{Q}(\zeta_5)$ is the quadratic field $k_0^+ = \mathbb{Q}(\sqrt{5})$.

1.3. **Overview.** The layout of this article is as follows. In § 2, we prove that the action of the absolute Galois group $\mathrm{Gal}(M/\mathbb{Q})$ on the 5-class group $\mathrm{Cl}_5(M)$ considerably reduces the possibilities for the metabelianization $\mathrm{G}_5^{(2)}M$ of the 5-class tower group $\mathrm{G}_5^{(\infty)}M$ of M. In § 3, it is shown that the six unramified cyclic quintic relative extensions E_i/M, $1 \leq i \leq 6$, give rise to absolute extensions $E_i/\mathbb{Q}$ which are either Frobenius or non-Galois. Using class number relations for the dihedral subextensions E_i/k_0^+ of $E_i/\mathbb{Q}$, we determine further constraints for the second 5-class group $\mathrm{G}_5^{(2)}M$, the 5-class tower group $\mathrm{G}_5^{(\infty)}M$, and the length $\ell_5 M$ of the 5-class tower in § 4. The paper concludes with tables of concrete numerical realizations in § 5 which underpin all theoretical statements and additionally reveal the statistical distribution of possible cases.

2. p-Principalization enforced by Galois action

The generating automorphism σ of a cyclic number field $F/\mathbb{Q}$ of degree d with Galois group $\mathrm{Gal}(F/\mathbb{Q}) = \langle\sigma\rangle$ acts on the class group $\mathrm{Cl}(F)$ of F and thus also on the higher p-class groups $\mathrm{G}_p^{(n)}F$ with $n \in \mathbb{N} \cup \{\infty\}$, for a fixed prime number p. When d and p are coprime, a remarkable restriction of the possibilities for the metabelian second p-class group $\mathfrak{M} = \mathrm{G}_p^{(2)}F$ and consequently for the transfer kernel type $\varkappa(F)$ of F is due to the fact that the trace $T_\sigma = \sum_{i=0}^{d-1} \sigma^i$ of σ annihilates the commutator quotient of all the groups $\mathrm{G}_p^{(n)}F$.

Definition 2.1. Let p be a prime number and G be a pro-p group with finite abelianization G/G'. Suppose that $d \geq 2$ is a fixed integer. G is said to be a *σ-group of degree d*, if G possesses an automorphism σ of order d whose trace $T_\sigma = \sum_{j=0}^{d-1} \sigma^j \in \mathbb{Z}[\mathrm{Aut}(G)]$ annihilates G modulo G', that is, if there exists $\sigma \in \mathrm{Aut}(G)$ such that $\mathrm{ord}(\sigma) = d$ and $x^{T_\sigma} = \prod_{j=0}^{d-1} \sigma^j(x) \in G'$ for all $x \in G$.

We show that an epimorphism with characteristic kernel preserves the property of being a σ-group of degree d.

Theorem 2.1. *Let $\phi : G \to H$ be an epimorphism of groups, whose kernel $\ker(\phi)$ is characteristic in G. If G is a σ-group of degree d then H is also a σ-group of degree d.*

Proof. If G is a σ-group of degree d, then there exists an automorphism $\sigma \in \mathrm{Aut}(G)$ of order $\mathrm{ord}(\sigma) = d$ such that $x^{T_\sigma} = \prod_{i=0}^{d-1} \sigma^i(x) \in G'$, for all $x \in G$. According to [16, Thm. 6.2], there exists an induced automorphism $\hat{\sigma} \in \mathrm{Aut}(H)$ such that $\hat{\sigma} \circ \phi = \phi \circ \sigma$. By induction we obtain $\hat{\sigma}^n \circ \phi = \phi \circ \sigma^n$, for all $n \in \mathbb{Z}$: let $n \geq 2$ be an integer and assume that $\hat{\sigma}^{n-1} \circ \phi = \phi \circ \sigma^{n-1}$, then

$$\hat{\sigma}^n \circ \phi = \hat{\sigma}^{n-1} \circ \hat{\sigma} \circ \phi = \hat{\sigma}^{n-1} \circ \phi \circ \sigma = \phi \circ \sigma^{n-1} \circ \sigma = \phi \circ \sigma^n.$$

Furthermore, $(\sigma^{-1})^\wedge = \hat{\sigma}^{-1}$. Now let $y \in H$. Since ϕ is surjective, there exists $x \in G$ with $\phi(x) = y$, and we obtain, as required,

$$y^{T_{\hat{\sigma}}} = \prod_{i=0}^{d-1} \hat{\sigma}^i(y) = \prod_{i=0}^{d-1} \hat{\sigma}^i(\phi(x)) = \prod_{i=0}^{d-1} \phi(\sigma^i(x)) = \phi\left(\prod_{i=0}^{d-1} \sigma^i(x)\right) = \phi\left(x^{T_\sigma}\right) \in \phi(G') = H'.$$

□

Corollary 2.1. *In a descendant tree $\mathcal{T}$ of finite p-groups with edges $\pi : G \to \pi G$, the property of* ***not*** *being a σ-group of degree d is inherited from the parent πG by the immediate descendant G.*

Proof. The parent operator $\pi : G \to \pi G$ is the canonical projection from G onto the quotient $\pi G = G/\gamma_c G$ by the last non-trivial member $\gamma_c G$, $c = \mathrm{cl}(G)$, of the lower central series $(\gamma_i G)_{i\geq 1}$ of G, and thus π is an epimorphism with characteristic kernel $\ker(\pi) = \gamma_c G$, whence Theorem 2.1 justifies the claim. □

Remark 2.1. A σ-group G in the classical sense is a σ-group of degree 2 in the new sense, since $x\sigma(x) \in G'$ is equivalent with $\sigma(x)G' = x^{-1}G'$. Such a group is also referred to as a group with *generator inverting* automorphism or briefly GI-automorphism.

Theorem 2.2. *The p-class tower group $\mathrm{G}_p^{(\infty)}F$ and all higher p-class groups $\mathrm{G}_p^{(n)}F$ with $n \geq 2$ of a cyclic quartic number field F are σ-groups of degree 4.*

When the quadratic subfield $k < F$ has a trivial p-class group, the groups $\mathrm{G}_p^{(\infty)}F$ and $\mathrm{G}_p^{(n)}F$ with $n \geq 2$ are simultaneously σ-groups of degree 2.

Proof. The generating automorphism σ of $F/\mathbb{Q}$ annihilates the class group $\mathrm{Cl}(F)$ when it acts by its trace $T_\sigma = \sum_{i=0}^{3} \sigma^i \in \mathbb{Z}[\langle\sigma\rangle]$, since $x^{T_\sigma} = \prod_{i=0}^{3} \sigma^i(x) = \mathrm{N}_{F/\mathbb{Q}}(x) \in \mathrm{Cl}(\mathbb{Q}) = 1$, for all $x \in \mathrm{Cl}(F)$. Of course, the same is true for all p-class groups $\mathrm{Cl}_p(F)$ with primes p. Finally, we have isomorphisms $\mathrm{G}_p^n F/\left(\mathrm{G}_p^n F\right)' \simeq \mathrm{Cl}_p(F)$, for any $n \in \mathbb{N} \cup \{\infty\}$.

When the unique (real) quadratic subfield $k < F$ has trivial p-class group $\mathrm{Cl}_p(k) = 1$, then the relative automorphism $\tau = \sigma^2 \in \mathrm{Gal}(F/k)$ with order 2 acts by inversion on $\mathrm{Cl}_p(F)$, since $x^{T_\tau} = x^{1+\tau} = x \cdot \tau(x) = \mathrm{N}_{F/k}(x) \in \mathrm{Cl}_p(k) = 1$, and thus $x^\tau = x^{-1}$, for all $x \in \mathrm{Cl}_p(F)$. □

Remark 2.2. A pro-p group G with finite abelianization G/G' is called a *strong* σ-group if it possesses an automorphism σ of order 2 which acts as inversion on both cohomology groups $\mathrm{H}^1(G, \mathbb{F}_p)$ and $\mathrm{H}^2(G, \mathbb{F}_p)$. We emphasize the following two facts:

- An epimorphism does not necessarily preserve the property of being a strong σ-group.
- Whereas the group $\mathrm{G}_p^{(\infty)}F$ of a quadratic field F is a strong σ-group, according to Schoof [19, Lem. (4.1), p. 217], this is not necessarily the case for a cyclic quartic field F. See for instance the unusual cases in Theorem 4.5.

TABLE 1. The Artin pattern of the twelve 5-groups of order 5^5 in the stem of Φ_6

Identifier of the 5-Group		Flag	5-Principalization Type		
James	SmallGroup	f	$\varkappa$	Cycle Pattern	Property
$\Phi_6(2^21)_a$	$\langle 3125, 14\rangle*$	1	(123456)	(1)(2)(3)(4)(5)(6)	identity
$\Phi_6(2^21)_{b_1}$	$\langle 3125, 11\rangle*$	1	(125364)	(1)(2)(3564)	4-cycle
$\Phi_6(2^21)_{b_2}$	$\langle 3125, 7\rangle$	1	(126543)	(1)(2)(36)(45)	two 2-cycles
$\Phi_6(2^21)_{c_1}$	$\langle 3125, 8\rangle*$	0	(612435)	(16532)(4)	5-cycle
$\Phi_6(2^21)_{c_2}$	$\langle 3125, 13\rangle*$	0	(612435)	(16532)(4)	5-cycle
$\Phi_6(2^21)_{d_0}$	$\langle 3125, 10\rangle$	0	(214365)	(12)(34)(56)	three 2-cycles
$\Phi_6(2^21)_{d_1}$	$\langle 3125, 12\rangle*$	0	(512643)	(154632)	6-cycle
$\Phi_6(2^21)_{d_2}$	$\langle 3125, 9\rangle*$	0	(312564)	(132)(456)	two 3-cycles
$\Phi_6(21^3)_a$	$\langle 3125, 4\rangle$	1	(022222)		nrl. const. with fp.
$\Phi_6(21^3)_{b_1}$	$\langle 3125, 5\rangle$	1	(011111)		nearly constant
$\Phi_6(21^3)_{b_2}$	$\langle 3125, 6\rangle$	1	(011111)		nearly constant
$\Phi_6(1^5)$	$\langle 3125, 3\rangle$	1	(000000)		constant

In view of our special situation with $p = 5$, $F = M$, $\mathrm{Cl}_5(M) = (5, 5)$ and $k = k_0^+$, we tested finite metabelian 5-groups G with $G/G' \simeq (5, 5)$ of order $|G| = 3125 = 5^5$ and coclass $\mathrm{cc}(G) = 2$, for the property of simultaneously being a σ-group of degree 4 and degree 2. These groups are crucial contestants for second 5-class groups $\mathrm{G}_5^2 M$ and form the stem of Hall's isoclinism family Φ_6. (See [13, § 3.5, pp. 445–448] and [17, § 7, pp. 93–98].) In Table 1, the groups are characterized by their identifiers according to James [8] and the SmallGroups Library [2]. An asterisk $*$ marks a Schur σ-group, and a flag $f \in \{0,1\}$ indicates a σ-group of simultaneous degrees 4 and 2.

Theorem 2.3. *A finite* 5*-group* G *with* $G/G' \simeq (5,5)$ *which is a* σ*-group of degree* 4 *is either of coclass* $\mathrm{cc}(G) = 1$ *or isomorphic to one of the two Schur* σ*-groups* $\langle 3125, i\rangle$ *with* $i \in \{11, 14\}$ *or isomorphic to a descendant of one of the capable groups* $\langle 3125, i\rangle$ *with* $i \in \{3,4,5,6,7\}$.

Proof. Using permutation representations, we compiled a program script in Magma [12] for testing whether an assigned 5-group G with $G/G' \simeq (5,5)$ is a σ-group of degree 4. $\square$

3. Frobenius and non-Galois unramified 5-extensions

3.1. **On the cyclic quartic fields** M**.** Let ζ_5 be a primitive 5th root of unity, then the irreducible polynomial of ζ_5 is given by $\mathrm{Irr}_{\mathbb{Q}}(\zeta_5) = X^4 + X^3 + X^2 + X + 1$, and $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \langle \mu \rangle$ is a cyclic group of order 4 which admits one subgroup $\langle \mu^2 \rangle$ of order 2. By Galois correspondence, this subgroup corresponds to $\mathbb{Q}(\zeta_5)^+ = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$. (See Figure 1.)

Figure 1. Galois correspondence between $\mathrm{Gal}(k_0/\mathbb{Q})$ and k_0

$$\begin{tikzcd}
1 \arrow[d, dash, "2"'] & & k_0 = \mathbb{Q}(\zeta_5) \arrow[d, dash] \\
H = \langle \mu^2 \rangle \arrow[d, dash, "2"'] & \overset{\text{Galois}}{\longleftrightarrow} & k_0^+ = \mathrm{inv}(H) = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5}) \arrow[d, dash] \\
\langle \mu \rangle & & \mathbb{Q}
\end{tikzcd}$$

Let d be a square free integer prime to 5, then $L = \mathbb{Q}(\sqrt{d}, \zeta_5)$ is a normal extension over $\mathbb{Q}$ of degree 8, and the Galois group is

$$\mathrm{Gal}\,(L/\mathbb{Q}) = \langle \tau, \mu \rangle = \langle 1, \tau, \mu, \mu^2, \mu^3, \tau\mu, \tau\mu^2, \tau\mu^3 \rangle, \text{ where } \tau(\sqrt{d}) = -\sqrt{d}.$$

This is an abelian group of type $(2,4)$ and has six proper subgroups ordered as follows :

$$H_1 = \langle \tau \rangle,\ H_2 = \langle \mu \rangle,\ H_3 = \langle \mu^2 \rangle,\ H_4 = \langle \tau\mu \rangle,\ H_5 = \langle \tau\mu^2 \rangle \text{ and } H_6 = \langle \tau, \mu^2 \rangle.$$

Note that the subgroups $H_1,\ H_3,\ H_5$ are cyclic of order 2, the subgroups $H_2,\ H_4$ are cyclic of order 4, and the group H_6 is bicyclic of order 4. (See Figure 2.)

We consider the field M fixed by the subgroup $\langle \tau\mu^2 \rangle$. Then M is a cyclic quartic field and can be generated by adjunction $M = \mathbb{Q}(\alpha)$ of $\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{-\frac{5d}{2} - \frac{d}{2}\sqrt{5}}$ to $\mathbb{Q}$. With respect to the quartic cyclotomic mirror field $k_0 = \mathbb{Q}(\zeta_5)$, M satisfies the *quintic reflection theorem* (Equation (1.3)).

Lemma 3.1. *Let* K *be a number field and* F/K *be a cyclic quartic extension, then there exist* n, e *and* $f \neq 0$ *in* K *such that*

(1) n *is not a square in* K,
(2) $n(e^2 - f^2 n)$ *is a square in* K,
(3) $F = K(\alpha)$, *where* $\alpha = \sqrt{e + f\sqrt{n}}$,

and the minimal polynomial of α *over* K *is given by* $\mathrm{Irr}_K(\alpha) = X^4 - 2eX^2 + (e^2 - f^2 n)$.

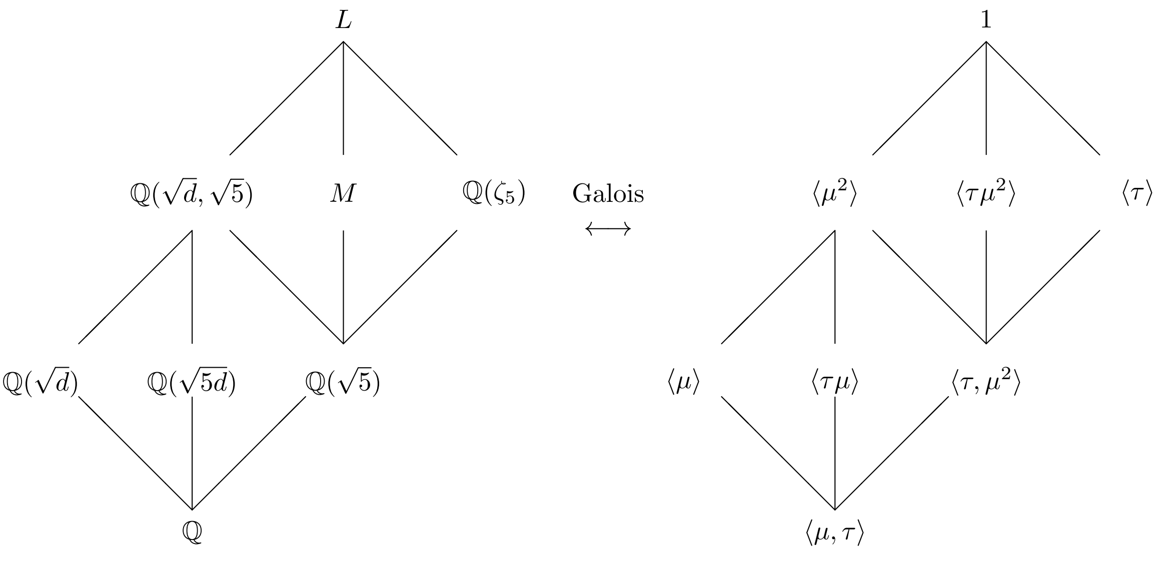
Conversely, if there exist numbers $n,\ e,$ *and* $f \neq 0$ *in* K *which satisfy the conditions* (1), (2) *and* (3)*, then* F/K *is a cyclic quartic extension and the polynomial* $P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$ *is irreducible over* K*. In fact,* F *is the splitting field of* $P(X)$*.*

Proof. It is known that the group $\mathbb{Z}/4\mathbb{Z}$ has a single subgroup of order 2. By Galois theory, there exists a corresponding intermediary field R of the cyclic quartic extension F/K. Thus we can find $n \in K$, no square in K, such that $R = K(\sqrt{n})$. Since F is a quadratic extension of R, there exists $\alpha \in F$ such that $\alpha^2 = e + f\sqrt{n} \in R$ with $e, f \in K$, $f \neq 0$, and $F = R(\alpha)$. Thus we have $K(\alpha) = F$, because $\alpha \notin R$. Furthermore, it is obvious that the minimal polynomial of α is $P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$ and the splitting field of $P(X)$ over $\mathbb{Q}$ is F.

The discriminant of P is given by $D = 16(e^2 - f^2 n)(2f\sqrt{n})^4 = 2^8 f^4 n^2 (e^2 - f^2 n)$. Therefore the Galois group $\mathrm{Gal}\,(F/K)$ can be seen as a subgroup of the permutation group of $P(X)$, which is isomorphic to S_4, and cannot be injected into $\mathcal{A}_4$, since the group $\mathcal{A}_4$ does not have a subgroup of order 4. We conclude that the discriminant is not a square in K, whence $K\left(\sqrt{e^2 - f^2 n}\right)/K$ is of degree 2 and is contained in F. It follows that $R = K\left(\sqrt{n}\right) = K\left(\sqrt{e^2 - f^2 n}\right)$, so $\frac{e^2 - f^2 n}{n}$ is a square in K. Consequently, we see that $n\left(e^2 - f^2 n\right) = n^2 \frac{e^2 - f^2 n}{n}$ is a square in K.

Conversely, let $P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$ with $n, e, f \in K$, $f \neq 0$, such that the conditions (1), (2) and (3) are satisfied. Since α is a root of $P(X)$, the degree $[F : K]$ must be a divisor of 4. Since $K\left(\sqrt{n}\right) \subseteq F$, we have either $F = K\left(\sqrt{n}\right)$ or $[F : K] = 4$. If we have $F = K\left(\sqrt{n}\right)$, there exist $u, v \in K$ such that $\sqrt{e + f\sqrt{n}} = u + v\sqrt{n}$. Thus $e^2 - f^2 n = \left(u^2 - v^2 n\right)^2$, and by (2) we conclude that n is a square in K, which is a contradiction. So $[F : K] = 4$, and this enforces that $P(X)$ is the minimal polynomial of α. From the fact $e - f\sqrt{n} = \left(\frac{1}{\sqrt{n}}\right)^2 \frac{n(e^2 - f^2 n)}{\alpha^2}$, we conclude that F is the splitting field of $P(X)$ over K. Moreover, F is normal and $\#\mathrm{Gal}\,(F/K) = 4$. Now we prove that $\mathrm{Gal}\,(F/K)$ is cyclic of order 4. If the Galois group $\mathrm{Gal}\,(F/K)$ were isomorphic to V_4, then the discriminant would be a square in K. This would imply that n were a square in K, which is a contradiction. $\square$

FIGURE 2. Galois correspondence between L and $\mathrm{Gal}(L/\mathbb{Q})$



Corollary 3.1. *Let d be a square free integer prime to* 5 *and ζ_5 be a primitive* 5*th root of unity. Then the mirror image M of $k_1 = \mathbb{Q}(\sqrt{d})$ can always be generated by adjoining the algebraic number $\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{\frac{-5d}{2} + \frac{-d}{2}\sqrt{5}}$ to the rational, whence M is complex for $d > 0$ and M is real for $d < 0$. For the construction of M one can therefore use the minimal polynomial of α over $\mathbb{Q}$, which is given by*

$$\mathrm{Irr}_{\mathbb{Q}}(\alpha) = X^4 + 5dX^2 + 5d^2. \tag{3.1}$$

Remark 3.1. The conductor $c(M)$ and the discriminant $d(M)$ of M are given by

(3.2)
$$c(M) = \begin{cases} 20d & \text{if } d \equiv 1 \pmod 4, \\ 5d & \text{if } d \equiv 2,3 \pmod 4, \end{cases} \qquad d(M) = c(M)^2 d(k_0^+) = \begin{cases} 2000d^2 & \text{if } d \equiv 1 \pmod 4, \\ 125d^2 & \text{if } d \equiv 2,3 \pmod 4, \end{cases}$$

where $d(k_0^+) = 5$ is the discriminant of the quadratic subfield $k_0^+ = \mathbb{Q}(\sqrt{5})$ of M. (See [5, 21].)

3.2. **Imaginary cyclic quartic fields M with $d > 0$.** In the following, the two Frobenius groups $F_{5,w}$ of order 20 with primitive root $w \in \{2, 3\}$ modulo 5 will be denoted by

$$\begin{aligned} F_{5,2} &= \langle\ \sigma, \iota \mid \sigma^5 = 1,\ \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^2\ \rangle, \\ F_{5,3} &= \langle\ \sigma, \iota \mid \sigma^5 = 1,\ \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^3\ \rangle, \end{aligned} \tag{3.3}$$

where $\iota|_M = \mu|_M$.

Table 2. All possible 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$ and invariants δ_1, δ_2 for the associated quadratic fields k_1, k_2 which are 5-dual to an imaginary cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$, $d > 0$, with 5-class rank $r := \varrho_5(M) = 2$

Case	r_1	δ_1	r_2	δ_2
(a)	1	0	0	1
(b)	0	1	1	0
(c)	1	1	1	1
(d)	0	0	0	0
(e)	1	1	0	0
(f)	0	0	1	1
(g)	2	1	0	1
(h)	0	1	2	1

Proposition 3.1. *Let $E_1, \ldots, E_6$ be the six unramified cyclic quintic extensions of the imaginary cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $d > 0$, with 5-class group $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$ or, more generally, of 5-class rank 2. The properties of these fields as absolute extensions $E_i/\mathbb{Q}$, in dependence on the eight cases in Table 2, are given as follows.*

- *In case (a) and (g), all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$, for $1 \le i \le 6$.*
- *In case (b) and (h), all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,3}$, for $1 \le i \le 6$.*
- *In all the other cases (c), (d), (e), (f), two extensions are normal with non-isomorphic automorphism groups, say $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$ and $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$, but the other four extensions are non-Galois and form two conjugate pairs $E_3 \simeq E_4$ and $E_5 \simeq E_6$.*

Proof. According to the quintic reflection theorem [9], the assumption $r = 2$ implies that one of the eight disjoint cases in Table 2 is satisfied.

- In case (a), the 5-Selmer group of k_1 is given by $V_5(k_1) = \langle \alpha_{11}, \varepsilon_1 \rangle$. See [9, p.2, l.3]. Let $E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \alpha_{11})$ and $E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_1)$. In virtue of $\delta_1 = 0$, E_1 and E_2 are unramified cyclic quintic extensions of M. According to [9, p. 17, l.9–23], $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$, for $1 \le i \le 2$. Let $L := E_1 \cdot E_2$ be the compositum. Then, by [9, Lem.2.5], all proper subextensions E of L/M have $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,2}$.
- In case (b), the 5-Selmer group of k_2 is given by $V_5(k_2) = \langle \alpha_{21}, \varepsilon_2 \rangle$. See [9, p. 2, l.3]. Let $E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \alpha_{21})$ and $E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_2)$. In virtue of $\delta_2 = 0$, E_1 and E_2 are unramified cyclic quintic extensions of M. According to [9, p. 17, l.9–23], $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,3}$, for $1 \le i \le 2$. Let $L := E_1 \cdot E_2$ be the compositum. Then, by [9, Lem.2.5], all proper subextensions E of L/M have $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,3}$.

- Exemplarily, we consider case (d). Then the 5-Selmer groups of k_1 and k_2 are given by $V_5(k_1) = \langle\varepsilon_1\rangle$, $V_5(k_2) = \langle\varepsilon_2\rangle$. Let $E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_1)$ and $E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_2)$. Then, in virtue of $\delta_1 = \delta_2 = 0$, E_1 and E_2 are unramified cyclic quintic extensions of M. According to [9, p. 17, l.9–23], $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$ and $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$. Let $L := E_1 \cdot E_2$ be the compositum. Then E/M is also an unramified cyclic quintic extension, for any proper subextension E of L/M distinct from E_1 and E_2. Assume that $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,2}$. Since $L = E_1 \cdot E$, all proper subextensions E' of L/M have $\mathrm{Gal}(E'/\mathbb{Q}) \simeq F_{5,2}$, by [9, Lem.2.5]. This is a contradiction to $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$. In the same manner, the assumption that $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,3}$ leads to a contradiction. Therefore $E/\mathbb{Q}$ must be a non-Galois extension.

□

3.3. **Infinite family of imaginary cyclic quartic fields M whose 5-rank is at least 2.** As before, let $d \neq 1$ be a square free integer prime to 5, and let $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$ be the associated quadratic fields. For $\gamma \in k = k_i$, $i \in \{1, 2\}$, Y. Kishi [9, p. 6] has defined the polynomial

$$f(X, \gamma) = X^5 - 5\mathrm{N}_k(\gamma)X^3 + 5\mathrm{N}_k(\gamma)^2 X - \mathrm{N}_k(\gamma)\mathrm{Tr}_k(\gamma),$$

where $\mathrm{N}_k, \mathrm{Tr}_k$ are the norm map and the trace map of $k/\mathbb{Q}$. The minimal splitting field of $f(X, \gamma)$ is noted by K_γ. Furthermore, M. Imaoka and Y. Kishi [7], have characterized all $F_{5,w}$-extensions with $w \in \{2, 3\}$ as K_γ for a suitable elements $\gamma \in k_i$ with $i \in \{1, 2\}$. If $\gamma \in k_1$, then $\mathrm{Gal}\,(K_\gamma/\mathbb{Q}) \simeq F_{5,2}$, and if $\gamma \in k_2$, then $\mathrm{Gal}\,(K_\gamma/\mathbb{Q}) \simeq F_{5,3}$. Now, we consider the real quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$, $d = (\alpha+\beta)^2 - 4$, given by Kishi in [10, Exm. 3.5, p. 489] for $p = 5$, where the pair of integers $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ such that $\alpha \geq 2$, $\beta \geq 2$ satisfies the simultaneous conditions

$$\begin{cases} \alpha^2 - 5^3\beta^2 = 4, \\ \alpha + \beta \equiv 0 \pmod{5^2}. \end{cases} \tag{3.4}$$

Remark 3.2. The Pellian equation $\alpha^2 - 5^3\beta^2 = 4$ has infinitely many solutions (α, β), which correspond to the powers $\eta^n = \frac{\alpha+\beta\sqrt{5^3}}{2}$ of the normpositive fundamental unit $\eta = \frac{123+11\sqrt{5^3}}{2} = \frac{123+11\cdot5\sqrt{5}}{2}$ of the suborder with conductor $f = 5$ of $\mathbb{Q}(\sqrt{5})$. The solution (α, β) satisfies the additional constraint $\alpha + \beta \equiv 0 \bmod 5^2$ in (3.4) if and only if $n = 7 + 5^2 k$ with an integer $k \geq 0$.

Proposition 3.2. *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha+\beta)^2 - 4}\right)$, where α, β satisfy the conditions* (3.4)*. Then the 5-rank of the class group of M is greater than or equal to 2.*

Proof. Let $\epsilon_1 = \frac{\alpha+\beta+\sqrt{d}}{2}$, resp. $\epsilon_2 = \frac{\alpha+5^3\beta+5\sqrt{5d}}{2}$, be an element of $k_1 = \mathbb{Q}\left(\sqrt{(\alpha+\beta)^2 - 4}\right)$, resp. $k_2 = \mathbb{Q}\left(\sqrt{5((\alpha+\beta)^2 - 4)}\right)$. According to [10, Exm. 3.5, p. 489], ϵ_1 and ϵ_2 are units of k_1 and k_2, respectively. They satisfy the conditions

$$\begin{cases} \mathrm{N}_{\mathbb{Q}(\sqrt{d})}\left(\epsilon_1^2\right) = \mathrm{N}_{\mathbb{Q}(\sqrt{5d})}\left(\epsilon_2\right) = 1, \\ \mathrm{Tr}_{\mathbb{Q}(\sqrt{d})}\left(\epsilon_1^2\right) \equiv \mathrm{Tr}_{\mathbb{Q}(\sqrt{5d})}\left(\epsilon_2\right) \equiv \pm 2 \pmod{5^3}. \end{cases} \tag{3.5}$$

By applying [10, Thm. 1.1, p. 482, Prop. 3.1, p. 487], we prove that $K_{\epsilon_1^2}$ and K_{ϵ_2} are two different absolute Galois F_5-extensions, unramified over M, it suffices to show that ϵ_1^2, resp. ϵ_2, cannot be the fifth power of an element of k_1, resp. k_2.

According to [18, Lem. 1, p. 16], we have the following general fact: Let p be a prime number and ξ be an element of $\mathbb{Q}(\sqrt{\delta})$ such that $\xi = \frac{u+v\sqrt{\delta}}{2}$. If $0 < |v| < \frac{\delta^{p-1/2}}{2^{p-1}}$, then $\xi \notin \mathbb{Q}(\sqrt{\delta})^p$.

Let us apply this result to ϵ_2 and ϵ_1^2. By the assumptions (3.4), $\alpha + \beta = 5^2 c$, for some $c \geq 1$. Hence $(\alpha+\beta)^2 = 5^4c^2$ and $5(\alpha+\beta)^2 = 5^5c^2$. Furthermore, $5^5c^2 \geq 5^5 > 36$ and thus $5(\alpha+\beta)^2 - 20 > 16$, whence $5d > 16$, $(5d)^2 > 16^2$, and $\frac{(5d)^2}{16} > 16$. Finally $5 < 16 < \frac{(5d)^2}{2^4}$, and if we put $v := 5$ and $\delta := 5d$, then $v < \frac{\delta^2}{2^4}$, whence ϵ_2 cannot be the fifth power of an element in k_2.

For ϵ_1^2, we express the square in the form $\epsilon_1^2 = \frac{\frac{(\alpha+\beta)^2+d}{2}+(\alpha+\beta)\sqrt{d}}{2}$. Moreover we have,

$$\alpha+\beta < \frac{d^2}{16} \Longleftrightarrow 16(\alpha+\beta) < (\alpha+\beta)^4 - 8(\alpha+\beta)^2 + 16.$$

Put $u := \alpha+\beta$, then $\alpha+\beta < \frac{d^2}{16} \Longleftrightarrow u^4 - 8u^2 - 16u + 16 > 0$. Since $\alpha \geq 2$ and $\beta \geq 2$, it follows that $u \geq 3$, whence $\phi(u) = u^4 - 8u^2 - 16u + 16$ is positive. Thus we get $\alpha+\beta < \frac{d^2}{2^4}$, and putting $v := \alpha+\beta$ and $\delta := d$ we conclude that ϵ_1^2 cannot be a fifth power in k_1 either.

□

Corollary 3.2. *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha+\beta)^2 - 4}\right)$, where the integers α, β satisfy the conditions* (3.4)*. Assume that the 5-class group $\mathrm{Cl}_5(M)$ of M is of type $(5,5)$. Then $M_5^{(1)}/M$ contains six unramified cyclic quintic extensions E_i/M, which give rise to absolute extensions of degree* 20 *over $\mathbb{Q}$, ordered as*

- *$E_1 = K_{\epsilon_1^2}$ of Type (I) with $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$, the splitting field of the polynomial $f(X, \epsilon_1^2) = X^5 - 5X^3 + 5X - (d+2)$, and*
- *$E_2 = K_{\epsilon_2}$ of Type (II) with $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$, the splitting field of the polynomial $f(X, \epsilon_2) = X^5 - 5X^3 + 5X - (\alpha + 5^3\beta)$, and*
- *the other four extensions E_3, $E_4 = E_3^{\varphi}$, E_5, $E_6 = E_5^{\varphi}$, which are non-Galois of Type (III) over $\mathbb{Q}$ and form two conjugate pairs.*

Proof. The claims are a consequence of Proposition 3.2, the Formulas (3.5) and the fact that $\mathrm{Tr}_{\mathbb{Q}(\sqrt{d})}\left(\epsilon_1^2\right) = d+2$ and $\mathrm{Tr}_{\mathbb{Q}(\sqrt{5d})}\left(\epsilon_2\right) = \alpha+5^3\beta$. By φ we denote the generator of $\mathrm{Gal}\left(\mathbb{Q}(\sqrt{5})/\mathbb{Q}\right)$. □

Remark 3.3. For $d > 0$, if the fundamental units of the real quadratic fields k_i, $i = 1, 2$, are 5-primary, then the field M has a non-Galois unramified cyclic quintic extension of Type (III) [9]. In this case, there are four pairwise conjugate extensions of Type (III), and among the remaining two Frobenius-extensions one is of Type (I) and one is of Type (II) [9]. Note that in the case $d > 0$ the cyclic quartic field M is imaginary. Also, if 5 divides the class number of k_i, $i = 1, 2$, there exists at most one 5-primary element of k_i, $i = 1, 2$, which gives rise to the Frobenius-extensions of Type (I) and Type (II).

3.4. **Real cyclic quartic fields M with $d < 0$.** As before, the two Frobenius groups $F_{5,w}$ of order 20 with primitive root $w \in \{2, 3\}$ modulo 5 will be denoted as in formula (3.3).

TABLE 3. Some possible 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$ and invariants δ_1, δ_2 for the associated quadratic fields k_1, k_2 which are 5-dual to a real cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$, $d < 0$, with 5-class rank $r := \varrho_5(M) = 2$

Case	r_1	δ_1	r_2	δ_2
(a)	2	0	0	0
(b)	0	0	2	0
(c)	1	0	1	0
(d)	2	1	1	0
(e)	1	0	2	1

Proposition 3.3. *Let $E_1, \ldots, E_6$ be the six unramified cyclic quintic extensions of the real cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $d < 0$, with 5-class group $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$ or, more generally, of* 5*-class rank* 2*. The properties of these fields as absolute extensions $E_i/\mathbb{Q}$, in dependence on the five cases in Table 3, are given as follows.*

- *In case* (a)*, all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$, for $1 \leq i \leq 6$.*

- *In case* (b)*, all six fields* $E_1,\ldots,E_6$ *are normal and share isomorphic automorphism groups* $\mathrm{Gal}(E_i/\mathbb{Q})\simeq F_{5,3}$*, for* $1\le i\le 6$.
- *In all the other cases* (c), (d), (e)*, two extensions are normal with non-isomorphic automorphism groups, say* $\mathrm{Gal}(E_1/\mathbb{Q})\simeq F_{5,2}$ *and* $\mathrm{Gal}(E_2/\mathbb{Q})\simeq F_{5,3}$*, but the other four extensions are non-Galois and form two conjugate pairs* $E_3\simeq E_4$ *and* $E_5\simeq E_6$.

Proof. Similar to the proof of Proposition 3.1. □

4. The second 5-class group $\mathrm{G}_5^{(2)}M$ of M

Based on class number formula [11] for dihedral relative extensions E of degree 10 over a base field F with class number coprime to 5, we are now in the position to determine the isomorphism type of the Galois group $G_5^{(2)}M=\mathrm{Gal}(M_5^{(2)}/M)$ of the second Hilbert 5-class field $M_5^{(2)}$ of a cyclic quartic field $M=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{d}\right)$ with 5-class group of type $(5,5)$, because its unramified cyclic quintic extensions E_i, $1\le i\le 6$, turn out to be relatively dihedral over the quadratic subfield $k_0^+=\mathbb{Q}(\sqrt{5})$ of M, which has class number 1.

Theorem 4.1. *The relation between the* 5*-class numbers* $\mathrm{h}_5(E_i)$ *of the six unramified cyclic quintic extensions* E_i *of* M *and the* 5*-class numbers* $\mathrm{h}_5(L_i)$ *of their non-Galois subfields* L_i*, which are of relative degree* 5 *over the field* $k_0^+=\mathbb{Q}(\sqrt{5})$*, is given by*

$$\mathrm{h}_5(E_i)=\begin{cases}\mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M})=25,\ (U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i}))=1,\\ 5\cdot\mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M})=25,\ (U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i}))=5,\\ 5\cdot\mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M})=5,\ (U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i}))=1,\\ 25\cdot\mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M})=5,\ (U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i}))=5,\end{cases} \tag{4.1}$$

where U_F *is the unit group of the field* F.

Proof. According to Lemmermeyer [11, eq. (5.2), p. 685], we have the class number relation

$$\mathrm{h}_5(E_i)=\frac{(U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i}))}{\#\ker(j_{E_i/M})}\cdot\mathrm{h}_5(M)\cdot\mathrm{h}_5(L_i)^2,$$

where $\mathrm{h}_5(M)=25$, due to our general assumption on M. Distinction between total principalization, $\#\ker(j_{E_i/M})=25$, and partial principalization, $\#\ker(j_{E_i/M})=5$, immediately yields the four claimed cases, in dependence on the unit norm indices $u_i:=(U_{k_0^+}:\mathrm{N}_{E_i/k_0^+}(U_{L_i}))$. □

Remark 4.1. To prove Theorem 4.1 in a different manner we can use the class number formula, due to Lemmermeyer [11, Thm. 2.4, p. 681], and the following Lemma 4.1.

Lemma 4.1. *Let* p *be an odd prime and let* F *be a number field with class number coprime to* p*. Let* k *be a quadratic extension of* F*. Assume that* L *is an unramified cyclic extension of* k *of degree* p*, then the extension* L/F *is Galois, dihedral of degree* $2p$*, and we have the formula*

$$a:=(U_L:U_KU_{K'}U_k)=\frac{(U_k:U_k^p)\left(U_F:\mathrm{N}_{K/F}(U_K)\right)}{(U_F:U_F^p)\left(U_k:\mathrm{N}_{L/k}(U_L)\right)}$$

for the subfield unit index a*, where* $K\ne K'$ *denote two conjugate non-Galois subfields of* L.

Proof. Since $p\ge 3$ is an odd prime and the existence of an unramified cyclic extension L/k of degree p excludes the irregular case $p=3$, $F=\mathbb{Q}$, $k=\mathbb{Q}(\sqrt{-3})$ with $h_k=1$, either both fields k and F contain the pth roots of unity or both not. Therefore, $\frac{(U_k:U_k^p)}{(U_F:U_F^p)}=p^{r(k)-r(F)}$ with the torsionfree Dirichlet unit ranks $r(k)$ of k and $r(F)$ of F. For an unramified extension L/k, the Theorem on the Herbrand quotient of U_L is equivalent with $\#\ker(j_{L/k})=p\cdot b$ with $b:=\left(U_k:\mathrm{N}_{L/k}(U_L)\right)$. Using Lemma 4.1, which can be found in [11, p. 686], we can express the factor on the right hand side of the class number relation [11, Thm. 2.4, p. 681],

$$\mathrm{h}_p(L)=\frac{a}{p^{1+r(k)-r(F)}}\cdot\mathrm{h}_p(k)\cdot\mathrm{h}_p(K)^2,$$

in the form $\frac{a}{p^{1+r(k)-r(F)}} = \frac{a\cdot\left(U_F:U_F^p\right)}{p\cdot\left(U_k:U_k^p\right)} = \frac{\left(U_F:\mathrm{N}_{K/F}(U_K)\right)}{\#\ker(j_{L/k})}$, which we have used for $p=5$ in the proof of Theorem 4.1. □

4.1. Imaginary cyclic quartic fields M with $d>0$.

Theorem 4.2. *The 5-class field tower of M has length $\ell_5 M = 1$ if and only if the second 5-class group $\mathrm{G}_5^2 M$ of M is the abelian 5-group $\langle 25,2\rangle$ of type $(5,5)$. In this case,*

1. *the 5-class groups $\mathrm{Cl}_5(E_i)$ are cyclic of order 5, for $1\le i\le 6$,*
2. *the 5-class groups $\mathrm{Cl}_5(L_i)$ are trivial, for $1\le i\le 6$,*
3. *the 5-principalization of M is of type a.1, $\varkappa(M)=(000000)$.*

Proof. For $\mathrm{G}_5^2 M \simeq \langle 25,2\rangle$, we have cyclic 5-class groups $\mathrm{Cl}_5(E_i)\simeq C_5$ and six total principalizations $\#\ker(j_{E_i/M})=25$. According to Theorem 4.1, we obtain $\mathrm{h}_5(E_i)=5=u_i\cdot \mathrm{h}_5(L_i)^2$, which enforces $\mathrm{h}_5(L_i)=1$ and $u_i=5$, for all $1\le i\le 6$. □

Example 4.1. The values $d=4357$ and $d=4444$ give rise to fields $M=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{d}\right)$ with 5-class group of type $(5,5)$ having a single-stage 5-class tower. Fields of this type are extremely rare, since they from a fraction of $\frac{2}{83}$ among the fields with $0<d<10000$. Therefore, only 2% of the cases possess a single-stage tower.

Proposition 4.1. *Let $M=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{d}\right)$ with $d>0$ be an imaginary cyclic quartic field with 5-class group of type $(5,5)$. Let E_i, $1\le i\le 6$, be the six unramified cyclic quintic extensions of M and L_i their non-Galois subfields of relative degree 5 over the field $k_0^+=\mathbb{Q}(\sqrt{5})$. Then,*

1. *the subfield unit indices $a_i := \left(U_{E_i} : U_{L_i}U_{L_i'}U_M\right)$ are equal to 1,*
2. *the unit norm indices u_i satisfy the equivalence $u_i=1 \iff \#\ker(j_{E_i/M})=5$, and*
3. *the relations between the 5-class numbers $\mathrm{h}_5(E_i)$ and $\mathrm{h}_5(L_i)$ are given by*

$$\mathrm{h}_5(E_i)=5\cdot \mathrm{h}_5(L_i)^2.$$

Proof. According to Lemma 4.1, we can deduce that

$$a_ib_i=\frac{\left(U_M:U_M^5\right)\left(U_{k_0^+}:\mathrm{N}_{L_i/k_0^+}(U_{L_i})\right)}{\left(U_{k_0^+}:U_{k_0^+}^5\right)},$$

where b_i denotes the unit norm index $\left(U_M:\mathrm{N}_{E_i/M}(U_{E_i})\right)$. Since $d>0$, the field M is imaginary and it is a CM-field with maximal real subfield $M^+=k_0^+$. Hence, the torsionfree Dirichlet unit rank of M is $r(M)=1$, and $U_M=\langle -1,\epsilon_5\rangle$, where ϵ_5 denotes the fundamental unit of the quadratic field $k_0^+=\mathbb{Q}(\sqrt{5})$. This implies that

$$\left(U_M:U_M^5\right)=\left(U_{k_0^+}:U_{k_0^+}^5\right) \quad\text{and}\quad a_ib_i=u_i.$$

(1) To prove the assertion (1) it suffices to show the following equivalence

$$u_i=1 \text{ if and only if } \quad b_i=1.$$

So it suffices to show that the fundamental unit ϵ_5 of k_0^+, which is also the fundamental unit of M, is norm of a unit of E_i if and only if it is norm of a unit of L_i.
If $u_i=1$, for $i\in\{1,\cdots,6\}$, then ϵ_5 is norm of a unit of L_i (a non-Galois subfield of E_i), hence it is also norm of the same unit in E_i, and $b_i=1$.
Now suppose that $b_i=1$, for $1\le i\le 6$, then there exists a unit $\xi\in U_{E_i}$ such that $\epsilon_5=\mathrm{N}_{E_i/M}(\xi)$, and we obtain the following chain of implications:
$\mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}(\epsilon_5)=\mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/M}(\xi)\right)$
$\Rightarrow \epsilon_5^2=\mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/M}(\xi)\right)=\mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi)\right)$
$\Rightarrow \epsilon_5^6=\mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi^3)\right)$
$\Rightarrow \epsilon_5\cdot \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}(\epsilon_5)=\mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi^3)\right.,$
whence

$$\epsilon_5=\mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\epsilon_5^{-1}\cdot \mathrm{N}_{E_i/L_i}(\xi^3)\right).$$

Since the element $\epsilon_5^{-1} \cdot \mathrm{N}_{E_i/L_i}(\xi^3)$ is a unit of L_i, we obtain the index $u_i = 1$. On the other hand, the possible values of b_i and u_i are $\{1, 5\}$, and we can deduce that $u_i = b_i$. Finally, it follows from the equation $a_i b_i = u_i$ that $a_i = 1$.

(2) The result follows immediately from the fact that $\#\ker(j_{E_i/M}) = 5 \cdot b_i$.

(3) According to Theorem 4.1, we have two possible cases

$$u_i = 1 \text{ and } \#\ker j_{E_i/M} = 5$$

and

$$u_i = 5 \text{ and } \#\ker j_{E_i/M} = 25.$$

In both cases, the class number formula is given by $\mathrm{h}_5(E_i) = 5 \cdot \mathrm{h}_5(L_i)^2$.

□

Theorem 4.3. *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $d > 0$ be an imaginary cyclic quartic field with 5-class group $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$. If the second 5-class group $G := \mathrm{G}_5^2(M)$ of M is non-abelian, then the coclass $\mathrm{cc}(G)$ of G is greater than or equal to 2, $\mathrm{cc}(G) \geq 2$.*

Proof. Assume that G is non-abelian of coclass $cc(G) = 1$, then the possible capitulation types of M in the six intermediary cyclic quintic extensions of $M_5^{(2)}/M$, noted by $E_6 = E_5^{\varphi}$, E_5, $E_4 = E_3^{\varphi}$, E_3, E_2, E_1, are given by $\varkappa(G) = (111111)$ or $\varkappa(G) = (\ell 00000)$, $\ell \in \{0, 1, 2\}$.
First we consider the type $\varkappa(G) = (111111)$. In this case the group G is the extra special 5-group of order 5^3 and exponent 5^2, whose maximal normal subgroups are of order 5^2. This implies that the 5-class number of all E_i is equal to 5^2. Using Proposition 4.1, we conclude that the valuation $v_5(h_5(E_i))$ of the 5-class number of E_i must be odd, which is a contradiction. Thus the type $\varkappa(G) = (111111)$ can not occur.

For the three other types, we have total capitulation in the four extensions $E_6 = E_5^{\varphi}$, E_5, $E_4 = E_3^{\varphi}$, E_3, E_2, so the value of the index b_i, $2 \leq i \leq 6$, is $b_i = 5$, whence $u_i = 5$. On the other hand, for $2 \leq i \leq 6$ we have $\mathrm{h}_5(E_i) = 5^2$, and, by Proposition 4.1, we have $u_i = 1$, which is a contradiction. □

Proposition 4.2 (Number of fields)**.** *In the range $0 < d < 10000$ of fundamental discriminants d of real quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ with $\gcd(5, d) = 1$ there exist precisely* **83** *cases such that the 5-dual field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ of k_1 has a 5-class group $\mathrm{Cl}_5(M)$ of type $(5, 5)$.*

Proof. See Tables 4 and 5. □

Theorem 4.4 (Two-stage towers of 5-class fields with Schur σ-groups)**.**

(1) *If the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (125643)$ with two fixed points and a 4-cycle, then the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^3)^2, (21)^4]$, and the 5-class tower group is the Schur σ-group $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^5, \mathbf{11} \rangle$.*
(Examples are **23** *(***28***%) real quadratic fields k_1 starting with the following discriminants $d \in \{457,\ 501,\ 1996,\ 2573,\ 3253,\ 4189,\ 4957,\ 5129,\ 5233,\ 5308,\ 5361,\ \ldots\}$.)*

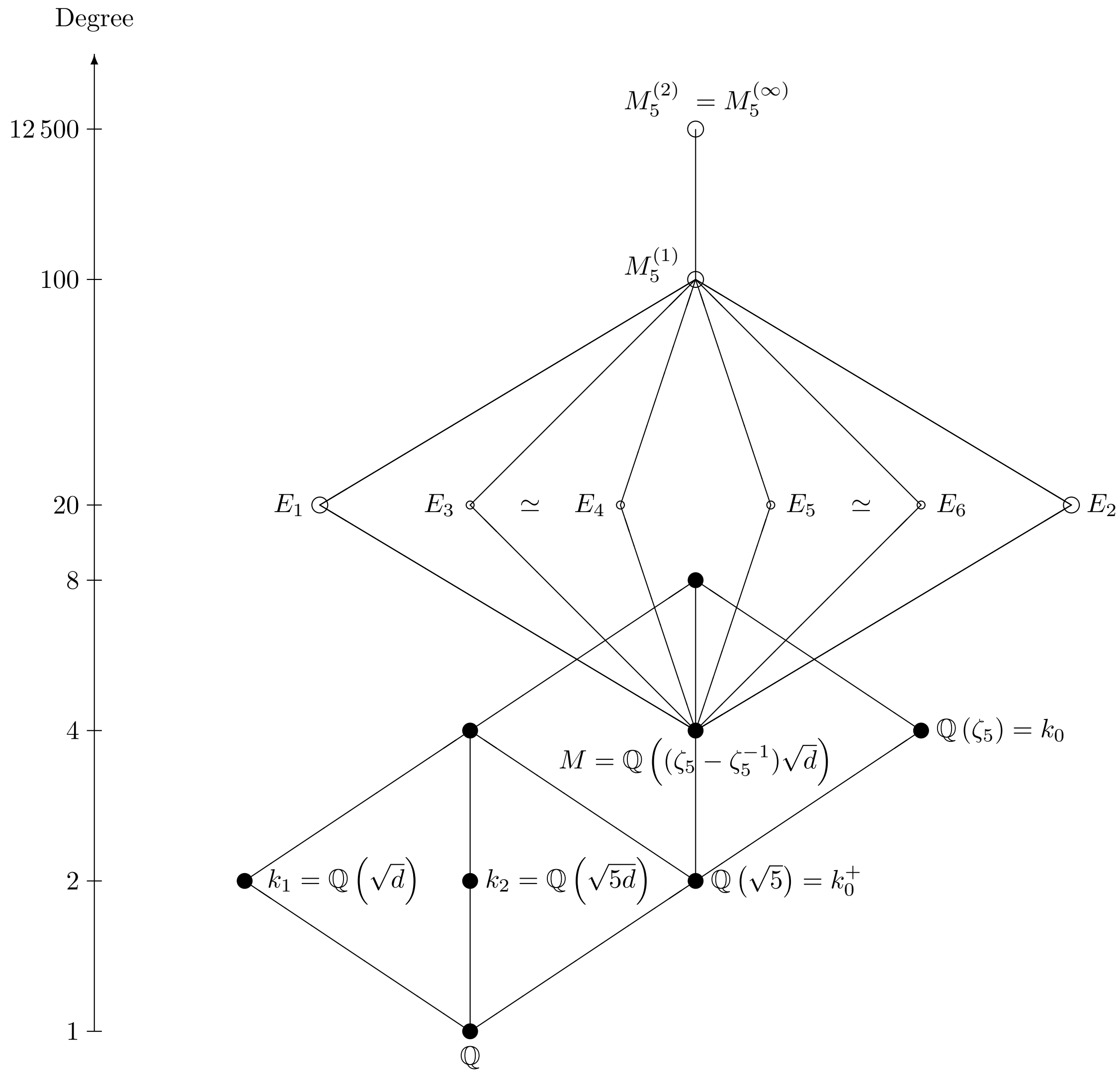
(2) *If the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (123456)$, the identity permutation, then the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^3)^6]$, and the 5-class tower group is the Schur σ-group $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^5, \mathbf{14} \rangle$.*
(Examples are **11** *(***13***%) real quadratic fields k_1 with the following discriminants $d \in \{581,\ 753,\ 2296,\ 2829,\ 4553,\ 5116,\ 5736,\ 6761,\ 7489,\ 9013,\ 9829\}$, verifying a conjecture by O. Taussky in* [22]*, and announced in* [13, § 3.5.2, p. 448]*, except* 2829.*)*

Proof. In each case, the length of the 5-class tower of M is given by $\ell_5 M = 2$, since $G := \mathrm{G}_5^{(2)}M$ is a Schur σ-group with balanced presentation, i.e., relation rank $d_2(G) = d_1(G) = \varrho_5(M) = 2$. □

Remark 4.2. The pairs of conjugate non-Galois extensions $E_3 \simeq E_4$ and $E_5 \simeq E_6$ of M are not adjacent in the factor (3546) of the cycle pattern (1)(2)(3546) of the 4-cycle $\varkappa(M) = (125643)$, and the Frobenius extensions E_1, E_2 correspond to the fixed points (1), (2). The identity $\varkappa(M) = (123456)$, which does not have two distinguished fixed points a priori, is endowed with a random arithmetical bipolarization by the two Frobenius extensions E_1, E_2.

Figure 3 visualizes the situation of a two-stage 5-class tower in the Theorems 4.4, 4.5, 4.9.

FIGURE 3. 5-class tower $M_5^{(\infty)}$ of the field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ if $\#\mathrm{G}_5^{(2)}M = 5^5$



Theorem 4.5 (Two-stage towers of 5-class fields with unusual capable weak σ-groups)**.**

(1) *If the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (022222)$, nearly constant with a single total capitulation and a single fixed point, then the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^3)^2, (21)^4]$, and the 5-tower group is $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^5, \mathbf{4}\rangle$. (Examples are* **22** *(***27***%) real quadratic fields k_1 starting with the following discriminants $d \in \{257,\ 764,\ 1708,\ 1853,\ 2008,\ 2189,\ 3129,\ 4504,\ 4861,\ 5241,\ 5269,\ \ldots\}$.)*

(2) *If the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (124365)$ with two fixed points and two disjoint 2-cycles, then the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^3)^2, (21)^4]$, and the 5-class tower group is $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^5, \mathbf{7}\rangle$. (Examples are* **16** *(***19***%) real quadratic fields k_1 starting with the following discriminants $d \in \{508,\ 509,\ 629,\ 881,\ 1113,\ 1192,\ 1704,\ 1829,\ 3121,\ 4461,\ 7032,\ \ldots\}$.)*

Proof. In each case, the length of the 5-class tower of M is given by $\ell_5 M = 2$, since $G := \mathrm{G}_5^{(2)}M$ is a metabelian σ-group with trivial cover [15, Dfn. 5.1, p. 30], according to Heider and Schmithals [6, p. 20]. The presentation of G is not balanced, since the relation rank $d_2(G) = 3$ is too big.

However, the Shafarevich Theorem [20] in its corrected version [15, Thm. 5.1, p. 28] ensures that $d_2(G) \le d_1(G) + r = 3$ just reaches the admissible upper bound, since the generator rank of G and the torsionfree Dirichlet unit rank of M with signature $(0,2)$ are given by $d_1(G) = \varrho_5(M) = 2$ and $r = 0 + 2 - 1 = 1$. The 5-tower groups $\langle 5^5, \mathbf{4}\rangle$ and $\langle 5^5, \mathbf{7}\rangle$ are unusual, because they are not strong σ-groups and thus are forbidden for (imaginary and real) quadratic base fields [19]. □

Remark 4.3. The pairs of conjugate non-Galois extensions $E_3 \simeq E_4$ and $E_5 \simeq E_6$ of M correspond to the factors (34) and (56) of the cycle pattern (1)(2)(34)(56) of the two disjoint 2-cycles $\varkappa(M) = (124365)$, and the Frobenius extensions E_1, E_2 correspond to the fixed points (1), (2). For the nearly constant type $\varkappa(M) = (022222)$, the first, resp. second, Frobenius extension E_1, resp. E_2, corresponds to the single total capitulation, resp. the single fixed point.

Theorem 4.6 (Single-stage towers of 5-class fields with abelian group)**.**
For the **2** *(***2***%) real quadratic fields k_1 with discriminants $d \in \{4357,\ 4444\}$, the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (000000)$, a constant with six total capitulations, the abelian type invariants of $E_1, \dots, E_6$ are $\tau(M) = [(1)^6]$, and thus the 5-class tower is abelian with group $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(1)}M \simeq \langle 5^2, \mathbf{2}\rangle$ and length $\ell_5 M = 1$.*

Proof. Here, the 5-class tower is abelian with length $\ell_5 M = 1$, according to Theorem 4.2. □

Remark 4.4. Outside of the range $0 < d < 10^4$ of our systematic investigations, we have discovered three occurrences of case (g) in Table 2. For the real quadratic fields k_1 with discriminants $d \in \{244641,\ 1277996,\ 1915448\}$ the 5-dual field M of k_1 has 5-principalization type $\varkappa(M) = (000000)$, a constant with six total capitulations, abelian type invariants $\tau(M) = [(1)^6]$, and abelian 5-class tower with group $G_5^{(\infty)}M = G_5^{(1)}M \simeq \langle 5^2, \mathbf{2}\rangle$ and length $\ell_5 M = 1$. The invariants are given by $(r_1, r_2, \delta_1, \delta_2) = (2, 0, 1, 1)$.

Theorem 4.7 (Frobenius and non-Galois extensions)**.**
The properties of the absolute extensions $E_i/\mathbb{Q}$ and the values of the invariants in the Quintic Reflection Theorem, Table 2, and Proposition 3.1, for the **83** *cases in Proposition 4.2 are:*

1. *For the* **2** *cases with $\ell_5 M = 1$ in Theorem 4.6, we have*
$$(r_1, r_2, \delta_1, \delta_2) = (1, 0, 0, 1), \ \textit{and}\ \mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}\ \textit{for}\ 1 \le i \le 6\ (\textit{Case (a)}).$$
2. *For the other* **81** *cases, including the* **34** *cases of $\ell_5 M = 2$ in Theorem 4.4 and the* **38** *cases of $\ell_5 M = 2$ in Theorem 4.5, we have pairwise conjugate non-Galois extensions $E_3 \simeq E_4$, $E_5 \simeq E_6$,*
$$\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2},\ \mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3},\ \textit{and}$$
$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (1, 0, 1, 0), & \textit{for}\ d \in \{1996,\ 3121,\ 3129,\ 3253,\ 5241,\ 5269,\ 5308, \\ & \qquad 6113,\ 8309,\ 8689,\ 9829\}\ (\textit{Case (e)}), \\ (r_1, r_2, \delta_1, \delta_2) = (0, 1, 0, 1), & \textit{for}\ d \in \{5116,\ 8972,\ 9013\}\ (\textit{Case (f)}), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 1, 1), & \textit{for}\ d \in \{4504,\ 6949,\ 7221,\ 7229,\ 9669\}\ (\textit{Case (c)}), \\ (r_1, r_2, \delta_1, \delta_2) = (0, 0, 0, 0), & \textit{otherwise}\ (\textit{Case (d)}). \end{cases}$$

Proof. See Tables 4 and 5. □

4.2. Real cyclic quartic fields M with $d < 0$.

Proposition 4.3. *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $d < 0$ be a real cyclic quartic field with 5-class group of type $(5,5)$. Denote by E_i, $1 \le i \le 6$, the six unramified cyclic quintic extensions of M and by L_i their non-Galois subfields, which are of relative degree 5 over the field $k_0^+ = \mathbb{Q}(\sqrt{5})$.*

1. *If $\#ker(j_{E_i/M}) = 5$, then the unit norm index is $u_i = (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 1$, and in this case the subfield unit index $a_i = \left(U_{E_i} : U_{L_i} U_{L_i'} U_M\right)$ is equal to 25.*
2. *The relation between the 5-class numbers $\mathrm{h}_5(E_i)$ and $\mathrm{h}_5(L_i)$ is given by*
$$\mathrm{h}_5(E_i) = \begin{cases} 5 \cdot \mathrm{h}_5(L_i)^2 & \textit{if}\ b_i = u_i, \\ \mathrm{h}_5(L_i)^2 & \textit{if}\ b_i \ne u_i. \end{cases}$$

Proof. Since $d < 0$, the field M is totally real, and the Dirichlet rank of its torsionfree unit group is given by $r(M) = 3$.

(1) Denote by $U_{M/\mathbb{Q}(\sqrt{5})}$ the group of relative units, that is $\{\epsilon \in U_M \mid \mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}(\epsilon) = 1\}$.
   For a cyclic quartic field $K/\mathbb{Q}$ with real quadratic subfield k, Hasse showed that the group $U_k U_{K/k}$ has index at most 2 in the full group of units U_K.
   In our case, $U_{\mathbb{Q}(\sqrt{5})} U_{M/\mathbb{Q}(\sqrt{5})}$ has index at most 2 in U_M, and $U_M = \langle -1, \epsilon_5, \eta, \eta^{\mu\tau} \rangle$, where η satisfies $\eta^{1+\mu\tau} = \pm 1$.
   If $\#\ker(j_{E_i/M}) = 5$, which means that the unit norm index $b_i = (U_M : \mathrm{N}_{E_i/M}(U_{E_i}))$ is equal to 1, then all units of M are norm of a unit of E_i, in particular ϵ_5. In the same manner as in the proof of claim 1 of Proposition 4.1, we deduce that ϵ_5 is also norm of a unit of L_i, whence $u_i = 1$. On the other hand, by applying Lemma 4.1, we deduce that $a_i \cdot b_i = 25 \cdot u_i$, and consequently $a_i = 25$.
(2) According to Theorem 4.1 or the Lemmermeyer class number formula [11, Thm. 4.1, p. 456], we conclude that $\mathrm{h}_5(E_i) = 5 \cdot \mathrm{h}_5(L_i)^2$ if $b_i = 1$ or ($b_i = 5$ and $u_i = 1$). But if $b_i = 5$ and $u_i = 5$, we have $\mathrm{h}_5(E_i) = \mathrm{h}_5(L_i)^2$, which completes the proof.

□

Remark 4.5. For totally real or imaginary cyclic quartic fields M, the last case $\mathrm{h}_5(E_i) = 25 \cdot \mathrm{h}_5(L_i)^2$, is impossible for any $1 \le i \le 6$.

Proposition 4.4. *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $d < 0$ be a real cyclic quartic field with 5-class group of type $(5,5)$. Let E_i, $1 \le i \le 6$, be the six unramified cyclic quintic extensions of M. Denote by $G := \mathrm{G}_5^{(2)}(M)$ the second 5-class group of M and assume that the order of G is equal to $|G| = 5^3$, then the transfer kernel type of G is $\varkappa(G) = (000000)$ (capitulation type of M in the six unramified extensions E_i) and the transfer target type of G is $\tau(G) = \left[(1^2)^6\right]$.*

Proof. In this case, the group G is extra special of maximal class. Thus, the possible types of capitulation are (111111) and (000000). First, we know that the type (111111) is not possible, because in this case $\mathrm{h}_5(E_i) = 5^2$ and $b_i = 1$, which contradicts claim (2) of Proposition 4.3. Thus, the transfer kernel type of G is $\varkappa(G) = (000000)$.

On the other hand, for all $1 \le i \le 6$, the unit norm index is $b_i = 5$, and u_i must be equal to 1. Otherwise, the Lemmermeyer class number formula [11, Thm. 4.1, p. 456] implies $|G| \ge 5^4$. Thus, for all $1 \le i \le 6$, we have $\mathrm{h}_5(E_i) = \mathrm{h}_5(L_i)^2$ and $\mathrm{h}_5(L_i) = 5$. Since the six extensions E_i are of type A in the sense of Taussky and $\mathrm{h}_5(E_i) = 5^2$, we deduce that $\mathrm{Cl}_5(E_i)$ is of type $(5,5)$. Thus $\tau(G) = \left[(1^2)^6\right]$. □

Remark 4.6. Assume that the group G is not abelian and $\varkappa(G) = (000000)$, then the prime 5 must divide the class number of the fields L_i. Because, in this case $b_i = 5$ and $u_i = 1$ or 5. The case $u_i = 1$ is obvious. Now suppose that $u_i = 5$. If 5 does not divide $\mathrm{h}(L_i)$, then $\mathrm{h}_5(E_i) = 5$, Hence E_i is an unramified extension of M and $\mathrm{h}_5(E_i) = \frac{\mathrm{h}_5(M)}{5}$. Then $M_5^{(2)} = M_5^{(1)}$ and the group G is abelian, which is a contradiction.

Proposition 4.5 (Number of fields)**.** *In the range $-200000 < d < 0$ of fundamental discriminants d of imaginary quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ with $\gcd(5,d) = 1$ there exist precisely* **109** *cases such that the 5-dual field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ of k_1 has a 5-class group $\mathrm{Cl}_5(M)$ of type $(5,5)$.*

Proof. See Tables 6, 7 and 8. □

Theorem 4.8 (Two-stage towers of 5-class fields with groups of low order)**.**

(1) *If the 5-dual field M of k_1 has 5-principalization type* a.1, *$\varkappa(M) = (000000)$, a constant with six total capitulations, and the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^2)^6]$, then the 5-tower group is the extra special group $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^3, \mathbf{3} \rangle$.*
(Examples are **66** *(***61***%) imaginary quadratic fields k_1 starting with the discriminants $d \in \{-12883,\ -13147,\ -14339,\ -23336,\ -23732,\ -26743,\ -28696,\ -35067,\ -35839,\ -38984,\ -47172,\ \ldots\}$.)*

(2) *If the* 5*-dual field* M *of* k_1 *has* 5*-principalization type* a.2*,* $\varkappa(M) = (100000)$ *with a fixed point and five total capitulations, and the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [1^3, (1^2)^5]$*, then the* 5*-tower group is the Schur+*1 σ*-group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^4, \mathbf{8}\rangle$*.*
(Examples are **26** *(***24**%*) imaginary quadratic fields* k_1 *starting with the discriminants* $d \in \{-27528,\ -27939,\ -39947,\ -40823,\ -54347,\ -75892,\ -91127,\ -99428,\ -101784,\ -105431,\ -114679,\ \ldots\}$*.)*

(3) *If the* 5*-dual field* M *of* k_1 *has* 5*-principalization type* a.1*,* $\varkappa(M) = (000000)$*, a constant with six total capitulations, and the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [1^3, (1^2)^5]$*, then the* 5*-tower group is the mainline group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^4, \mathbf{7}\rangle$*.*
(Examples are **11** *(***10**%*) imaginary quadratic fields* k_1 *with the following discriminants* $d \in \{-15419,\ -16724,\ -31103,\ -42899,\ -67128,\ -70763,\ -77095,\ -83620,\ -105784,\ -115515,\ -194487\}$*.)*

Proof. In each case, the length of the 5-class tower of M is given by $\ell_5 M = 2$, according to Blackburn [3], since $G := \mathrm{G}_5^{(2)}M$ is a σ-group with at most two-generated commutator subgroup $G' \in \{1, 1^2\}$. The presentation of G is not balanced, since the relation rank $d_2(G) \in \{3, 4\}$ is too big. However, the Shafarevich Theorem [20] in its corrected version [15, Thm. 5.1, p. 28] ensures that $d_2(G) \le d_1(G) + r = 5$ does not exceed the admissible upper bound, since the generator rank of G and the torsionfree Dirichlet unit rank of M with signature $(4, 0)$ are given by $d_1(G) = \varrho_5(M) = 2$ and $r = 4 + 0 - 1 = 3$. □

Theorem 4.9 (Two-stage tower of 5-class fields with Schur σ-group)**.**
If the 5*-dual field* M *of* k_1 *has* 5*-principalization type* $\varkappa(M) = (124563)$*, a* 4*-cycle and two fixed points, then the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [(1^3)^2, (21)^4]$ *and the* 5*-class tower group is* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^5, \mathbf{11}\rangle$*. In this case, the length of the* 5*-class tower of* M *is given by* $\ell_5 M = 2$*, and* $G := \mathrm{G}_5^{(2)}M$ *is a Schur* σ*-group with balanced presentation, that is, relation rank* $d_2(G) = d_1(G) = \varrho_5(M) = 2$*.*
(The ***unique*** *example is the imaginary quadratic field* k_1 *with discriminant* $d = -114303$*.)*

Proof. Similar to the proof of Theorem 4.4. □

Theorem 4.10 (Single-stage towers of 5-class fields with abelian group)**.**
For the **5** *(***5**%*) imaginary quadratic fields* k_1 *with discriminants* $d \in \{-58424,\ -115912,\ -148507,\ -151879,\ -154408\}$ *the* 5*-dual field* M *of* k_1 *has* 5*-principalization type* $\varkappa(M) = (000000)$*, a constant with six total capitulations, the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [(1)^6]$*, and the* 5*-class tower is abelian with group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(1)}M \simeq \langle 5^2, \mathbf{2}\rangle$ *and length* $\ell_5 M = 1$*.*

Proof. Similar to the proof of Theorem 4.6. □

Theorem 4.11 (Frobenius and non-Galois extensions)**.**
The properties of the absolute extensions $E_i/\mathbb{Q}$ *and the values of the invariants in the Quintic Reflection Theorem, Table 3, and Proposition 3.3, for the* **109** *cases in Proposition 4.5 are:*

(1) *For the* **5** *cases with* $\ell_5 M = 1$ *in Theorem 4.10, we have*

$$(r_1, r_2, \delta_1, \delta_2) = (2, 0, 0, 0),\ \text{and } \mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2} \text{ for } 1 \le i \le 6\ (\text{Case (a)}).$$

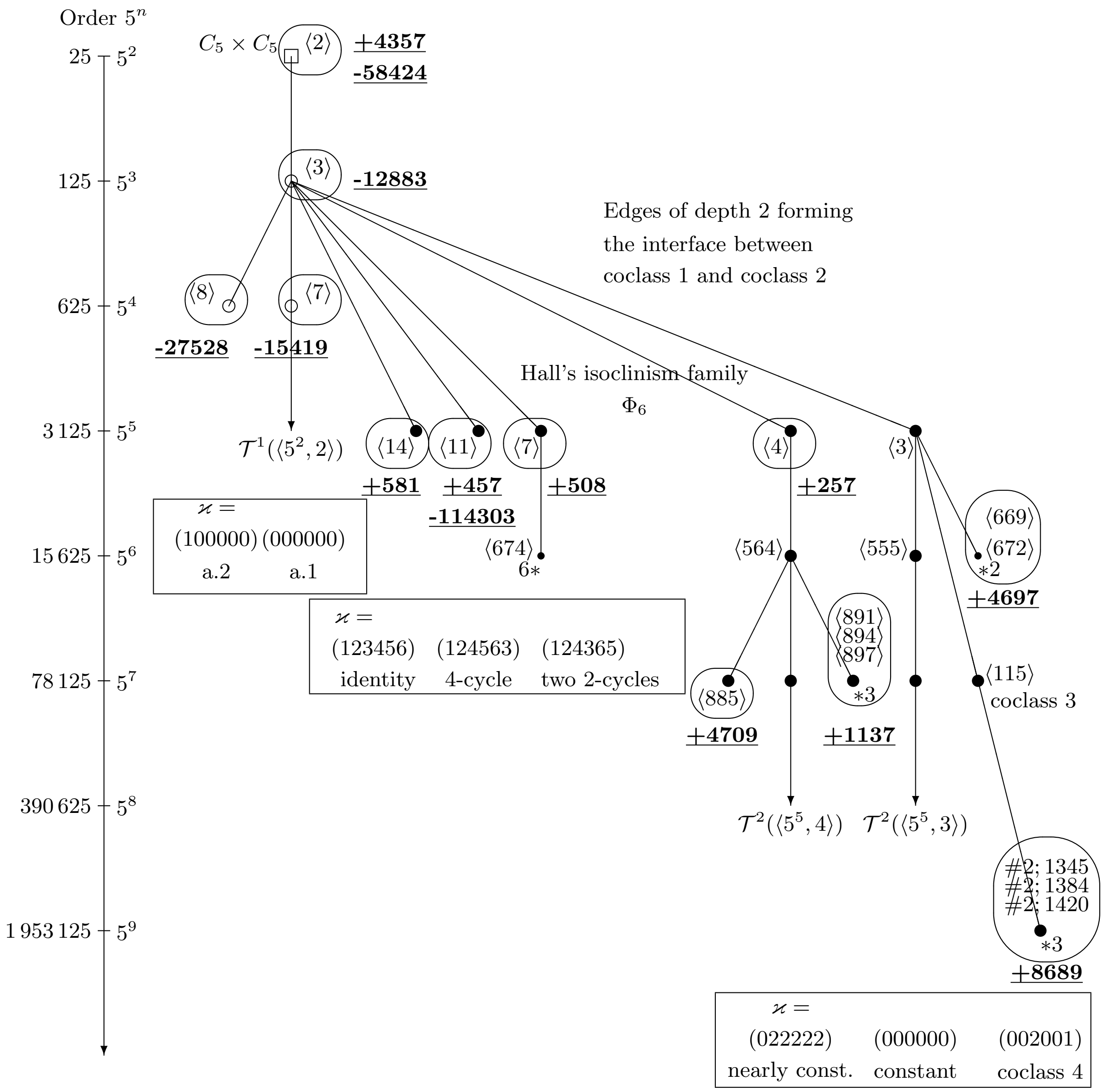
(2) *For the other* **104** *cases, including the* **103** *cases of* $\ell_5 M = 2$ *in Theorem 4.8, and the* ***unique*** *case of* $\ell_5 M = 2$ *in Theorem 4.9, we have pairwise conjugate non-Galois extensions* $E_3 \simeq E_4$*,* $E_5 \simeq E_6$*,* $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$*,* $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$*, and*

$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (2, 1, 1, 0), & \text{for } d \in \{-39947,\ -64103,\ -67128,\ -104503,\ -119191\}\ (\text{Case (d)}), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 2, 0, 1), & \text{for } d \in \{-110479,\ -199735\}\ (\text{Case (e)}), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 0, 0), & \text{otherwise}\ (\text{Case (c)}). \end{cases}$$

Proof. See Tables 6, 7 and 8. □

Figure 4 visualizes the relevant part of the descendant tree of finite 5-groups, beginning at the abelian root $C_5 \times C_5 = \langle 5^2, 2\rangle$, on which the second 5-class groups $\mathrm{G}_5^{(2)}M$ of the fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ are located as vertices. The figure is a modification of the diagram [13, Fig. 3.8, p. 448]. The minimal positive, resp. maximal negative, discriminants d are indicated by underlined boldface integers adjacent to the oval surrounding the vertex realized by $\mathrm{G}_5^{(2)}M$. The identifiers are due to the packages [2, 4] which are implemented in [12]. (For trees, see [14].)

FIGURE 4. Tree position of second 5-class groups $\mathrm{G}_5^{(2)}M$ of the fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$



5. Tables of second 5-class groups $\mathrm{G}_5^{(2)}M$

5.1. **Imaginary cyclic quartic fields M with $d > 0$.** Table 4, resp. Table 5, shows the factorized fundamental discriminant d of the dual quadratic field k_1, the 5-principalization type $\varkappa = \varkappa(M)$, the second 5-class group $\mathrm{G}_5^{(2)}M$, the length $\ell_5 M$ of the 5-class tower, the 5-class ranks

$r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$, the invariants δ_1, δ_2, and the case in Proposition 3.1 for the 37, resp. 46, cyclic quartic fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $0 < d < 5000$, resp. $5000 < d < 10000$.

For the fields with constant 5-principalization type, consisting of partial kernels, we have a *polarization* of the target type whose abelian invariants can be either homogeneous (1^5) or inhomogeneous (21^3). In the inhomogeneous case, there are three possibilities for the second 5-class group, namely $\langle 5^7, 891\rangle$, $\langle 5^7, 894\rangle$ and $\langle 5^7, 897\rangle$. In the homogeneous case, the second 5-class group $\langle 5^7, 885\rangle$ is unique. According to the Shafarevich Theorem [20, Thm. 6, Eqn. (18′)], whose misprint we have corrected in [15, Thm. 5.1, p. 28], these four groups, which possess relation rank $d_2 = 4$, are forbidden as 5-class tower groups for imaginary cyclic quartic fields with unit rank 1. Therefore, the length of the 5-class tower must be $\ell_5 M \geq 3$ at least, and we conjecture a precise three-stage tower $\ell_5 M = 3$.

The *complete statistics* of the 83 *imaginary* cyclic quartic fields M with $0 < d < 10^4$ is as follows. There are 23 (28%) cases with $\mathrm{G}_5^{(2)} M \simeq \langle 5^5, 11\rangle$ the Schur σ-group with transfer kernel type a 4-cycle, 22 (27%) cases with $\mathrm{G}_5^{(2)} M \simeq \langle 5^5, 4\rangle$, 16 (19%) cases with $\mathrm{G}_5^{(2)} M \simeq \langle 5^5, 7\rangle$, and 11 (13%) cases with $\mathrm{G}_5^{(2)} M \simeq \langle 5^5, 14\rangle$ the Schur σ-group with transfer kernel type the identity permutation. For only 4 cases we have $\mathrm{G}_5^{(2)} M \simeq \langle 5^7, 885\rangle$, for 2 cases $\mathrm{G}_5^{(2)} M \simeq \langle 5^7, 891|894|897\rangle$, for 2 cases $\mathrm{G}_5^{(2)} M \simeq \langle 5^2, 2\rangle$ the elementary bicyclic 5-group of rank 2, for 2 cases $\mathrm{G}_5^{(2)} M$ is a descendant of $\langle 5^5, 3\rangle$ indicated by the symbol $\downarrow$, and for a unique case $\mathrm{G}_5^{(2)} M \simeq \langle 5^7, 115\rangle - \#2; 1345|1384|1420$. The last three groups have the biggest order 5^9 and coclass 4. They posses relation rank $d_2 = 5$, which clearly enforces $\ell_5 M \geq 3$ by the Shafarevich Theorem. Again, we conjecture equality $\ell_5 M = 3$. Furthermore, we point out that the groups $\langle 5^5, 4\rangle$ and $\langle 5^5, 7\rangle$ with relation rank $d_2 = 3$ are not strong σ-groups in the sense of Schoof [19]. They are forbidden as 5-class tower groups for any quadratic field, imaginary or real. However, they are admissible for our imaginary cyclic quartic fields M with unit rank 1, since the subfield $k_0^+ = \mathbb{Q}(\sqrt{5})$ also possesses unit rank 1, and so a strong σ-group is not required.

5.2. **Real cyclic quartic fields M with $d < 0$.** Table 6, resp. Table 7, resp. Table 8, shows the factorized fundamental discriminant d of the dual quadratic field k_1, the 5-principalization type $\varkappa = \varkappa(M)$, the 5-class tower group $\mathrm{G}_5^{(\infty)} M$, the length $\ell_5 M$ of the 5-class tower, the 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$, the invariants δ_1, δ_2, and the case in Proposition 3.3 for the 43, resp. 45, resp. 21, cyc. quart. fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-100000 < d < 0$, resp. $-175000 < d < -100000$, resp. $-200000 < d < -175000$.

The *complete statistics* of the 109 *real* cyclic quartic fields M with $-2 \cdot 10^5 < d < 0$ is as follows. There are 66 (61%) cases with $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^3, 3\rangle$ the extra special 5-group of exponent 5, 26 (24%) cases with $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^4, 8\rangle$ having a transfer kernel type with fixed point, and 11 (10%) cases with $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^4, 7\rangle$ having total transfer kernels exclusively. For only 5 cases we have $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^2, 2\rangle$ the elementary bicyclic 5-group of rank 2, and for a unique case $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^5, 11\rangle$ the Schur σ-group with transfer kernel type a 4-cycle. The 5-class tower of M possesses length $\ell_5 M = 1$ for the abelian $\mathrm{G}_5^{(\infty)} M \simeq \langle 5^2, 2\rangle$, and $\ell_5 M = 2$ in all other cases. According to the Shafarevich Theorem [20, Thm. 6, Eqn. (18′)], whose misprint we have corrected in [15, Thm. 5.1, p. 28], the mainline groups $\langle 5^3, 3\rangle$ and $\langle 5^4, 7\rangle$ with relation rank $d_2 = 4$ are forbidden as 5-class tower groups for real quadratic fields with unit rank 1, but they are admissible for real cyclic quartic fields, which have bigger unit rank 3.

5.3. **The Galois action confirmed.** All numerical results in the Tables 4 to 8 are in perfect accordance with Theorems 2.2, 2.3 and Corollary 2.1. A rigorous check with the computational algebra system MAGMA [12, 2] proves that only the two terminal Schur σ-groups $\langle 5^5, 11\rangle$, $\langle 5^5, 14\rangle$ and five other capable top vertices $\langle 5^5, 3\rangle$, $\langle 5^5, 4\rangle$, $\langle 5^5, 5\rangle$, $\langle 5^5, 6\rangle$, $\langle 5^5, 7\rangle$ in the stem of Hall's isoclinism family Φ_6, and the abelian root $\langle 5^2, 2\rangle$, together with their descendants [16], are admissible for $\mathrm{G}_5^{(2)} M$ of any cyclic quartic field M, as drawn in Figure 4.

TABLE 4. The group $\mathrm{G}_5^{(2)}M$ of $M=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{d}\right)$ with $0<d<5000$

No.	Discriminant		Principalization		$\mathrm{G}_5^{(2)}M$	$\ell_5 M$	Invariants				
	d	Factors	$\varkappa$	Remark			r_1	δ_1	r_2	δ_2	Case
1	257	prime	(660666)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
2	457	prime	(234156)	4-cycle	$\langle 5^5,11\rangle$	2	0	0	0	0	(d)
3	501	$3,167$	(521346)	4-cycle	$\langle 5^5,11\rangle$	2	0	0	0	0	(d)
4	508	$4,127$	(653421)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
5	509	prime	(216453)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
6	581	$7,83$	(123456)	identity	$\langle 5^5,14\rangle$	2	0	0	0	0	(d)
7	629	$17,37$	(154326)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
8	753	$3,251$	(123456)	identity	$\langle 5^5,14\rangle$	2	0	0	0	0	(d)
9	764	$4,191$	(666066)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
10	881	prime	(463152)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
11	1113	$3,7,53$	(653421)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
12	1137	$3,379$	(444444)	constant	$\langle 5^7,891\vert 894\vert 897\rangle$	≥ 3	0	0	0	0	(d)
13	1192	$8,149$	(463152)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
14	1704	$8,3,71$	(653421)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
15	1708	$4,7,61$	(404444)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
16	1829	$31,59$	(216453)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
17	1853	$17,109$	(550555)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
18	1996	$4,499$	(613254)	4-cycle	$\langle 5^5,11\rangle$	2	1	1	0	0	(e)
19	2008	$8,251$	(550555)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
20	2189	$11,199$	(505555)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
21	2296	$8,7,41$	(123456)	identity	$\langle 5^5,14\rangle$	2	0	0	0	0	(d)
22	2573	$31,83$	(613254)	4-cycle	$\langle 5^5,11\rangle$	2	0	0	0	0	(d)
23	2829	$3,23,41$	(123456)	identity	$\langle 5^5,14\rangle$	2	0	0	0	0	(d)
24	3121	prime	(532416)	two 2-cycles	$\langle 5^5,7\rangle$	2	1	1	0	0	(e)
25	3129	$3,7,149$	(333303)	nearly const.	$\langle 5^5,4\rangle$	2	1	1	0	0	(e)
26	3169	prime	(444444)	constant	$\langle 5^7,891\vert 894\vert 897\rangle$	≥ 3	0	0	0	0	(d)
27	3253	prime	(243651)	4-cycle	$\langle 5^5,11\rangle$	2	1	1	0	0	(e)
28	4189	$59,71$	(243651)	4-cycle	$\langle 5^5,11\rangle$	2	0	0	0	0	(d)
29	4357	prime	(000000)	abelian	$\langle 5^2,2\rangle$	1	1	0	0	1	(a)
30	4444	$4,11,101$	(000000)	abelian	$\langle 5^2,2\rangle$	1	1	0	0	1	(a)
31	4461	$3,1487$	(653421)	two 2-cycles	$\langle 5^5,7\rangle$	2	0	0	0	0	(d)
32	4504	$8,563$	(444404)	nearly const.	$\langle 5^5,4\rangle$	2	1	1	1	1	(c)
33	4553	$29,157$	(123456)	identity	$\langle 5^5,14\rangle$	2	0	0	0	0	(d)
34	4697	$7,11,61$	(000000)	tot., non-ab.	$\langle 5^5,3\rangle\downarrow$	≥ 3	0	0	0	0	(d)
35	4709	$17,277$	(444444)	constant	$\langle 5^7,885\rangle$	≥ 3	0	0	0	0	(d)
36	4861	prime	(333303)	nearly const.	$\langle 5^5,4\rangle$	2	0	0	0	0	(d)
37	4957	prime	(135246)	4-cycle	$\langle 5^5,11\rangle$	2	0	0	0	0	(d)

TABLE 5. The group $\mathrm{G}_5^{(2)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $5000 < d < 10000$

No.	Discriminant		Principalization		$\mathrm{G}_5^{(2)}M$	$\ell_5 M$	Invariants				
	d	Factors	$\varkappa$	Remark			r_1	δ_1	r_2	δ_2	Case
38	5116	$4, 1279$	(123456)	identity	$\langle 5^5, 14\rangle$	2	0	0	1	1	(f)
39	5129	$23, 223$	(526431)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
40	5233	prime	(142536)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
41	5241	$3, 1747$	(660666)	nearly const.	$\langle 5^5, 4\rangle$	2	1	1	0	0	(e)
42	5269	$11, 479$	(222220)	nearly const.	$\langle 5^5, 4\rangle$	2	1	1	0	0	(e)
43	5308	$4, 1327$	(513462)	4-cycle	$\langle 5^5, 11\rangle$	2	1	1	0	0	(e)
44	5361	$3, 1787$	(625413)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
45	5393	prime	(440444)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
46	5464	$8, 683$	(440444)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
47	5557	prime	(111111)	constant	$\langle 5^7, 885\rangle$	≥ 3	0	0	0	0	(d)
48	5736	$8, 3, 239$	(123456)	identity	$\langle 5^5, 14\rangle$	2	0	0	0	0	(d)
49	5989	$53, 113$	(440444)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
50	6072	$8, 3, 11, 23$	(613254)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
51	6073	prime	(000000)	tot., non-ab.	$\langle 5^5, 3\rangle \downarrow$	≥ 3	0	0	0	0	(d)
52	6113	prime	(421653)	4-cycle	$\langle 5^5, 11\rangle$	2	1	1	0	0	(e)
53	6524	$4, 7, 233$	(513462)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
54	6761	prime	(123456)	identity	$\langle 5^5, 14\rangle$	2	0	0	0	0	(d)
55	6949	prime	(666066)	nearly const.	$\langle 5^5, 4\rangle$	2	1	1	1	1	(c)
56	6952	$8, 11, 79$	(220222)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
57	7032	$8, 3, 293$	(213546)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
58	7041	$3, 2347$	(666666)	constant	$\langle 5^7, 885\rangle$	≥ 3	0	0	0	0	(d)
59	7221	$3, 29, 83$	(444404)	nearly const.	$\langle 5^5, 4\rangle$	2	1	1	1	1	(c)
60	7229	prime	(444444)	constant	$\langle 5^7, 885\rangle$	≥ 3	1	1	1	1	(c)
61	7336	$8, 7, 131$	(606666)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
62	7361	$17, 433$	(653421)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
63	7489	prime	(123456)	identity	$\langle 5^5, 14\rangle$	2	0	0	0	0	(d)
64	7628	$4, 1907$	(164253)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
65	7656	$8, 3, 11, 29$	(444404)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
66	7752	$8, 3, 17, 19$	(623145)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
67	7833	$3, 7, 373$	(326154)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
68	7996	$4, 1999$	(022222)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
69	8008	$8, 7, 11, 13$	(625413)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
70	8012	$4, 2003$	(165432)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
71	8309	$7, 1187$	(111110)	nearly const.	$\langle 5^5, 4\rangle$	2	1	1	0	0	(e)
72	8689	prime	(002001)	coclass 4	$\langle 5^7, 115\rangle \downarrow$	≥ 3	1	1	0	0	(e)
73	8789	$11, 17, 47$	(362451)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
74	8877	$3, 11, 269$	(463152)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
75	8972	$4, 2243$	(362451)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	1	1	(f)
76	9013	prime	(123456)	identity	$\langle 5^5, 14\rangle$	2	0	0	1	1	(f)
77	9052	$4, 31, 73$	(333303)	nearly const.	$\langle 5^5, 4\rangle$	2	0	0	0	0	(d)
78	9544	$8, 1193$	(125364)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
79	9564	$4, 3, 797$	(425136)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
80	9573	$3, 3191$	(216453)	two 2-cycles	$\langle 5^5, 7\rangle$	2	0	0	0	0	(d)
81	9669	$3, 11, 293$	(362451)	4-cycle	$\langle 5^5, 11\rangle$	2	1	1	1	1	(c)
82	9752	$8, 23, 53$	(513462)	4-cycle	$\langle 5^5, 11\rangle$	2	0	0	0	0	(d)
83	9829	prime	(123456)	identity	$\langle 5^5, 14\rangle$	2	1	1	0	0	(e)

TABLE 6. The group $\mathrm{G}_5^{(\infty)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-100000 < d < 0$

No.	Discriminant		Principalization		$\mathrm{G}_5^{(\infty)}M$	$\ell_5 M$	Invariants				
	d	Factors	$\varkappa$	Type			r_1	δ_1	r_2	δ_2	Case
1	-12883	$13, 991$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
2	-13147	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
3	-14339	$13, 1103$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
4	-15419	$17, 907$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
5	-16724	$4, 37, 113$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
6	-23336	$8, 2917$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
7	-23732	$4, 17, 349$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
8	-26743	$47, 569$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
9	-27528	$8, 3, 31, 37$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
10	-27939	$3, 67, 139$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
11	-28696	$8, 17, 211$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
12	-31103	$19, 1637$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
13	-35067	$3, 11689$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
14	-35839	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
15	-38984	$8, 11, 443$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
16	-39947	$43, 929$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	2	1	1	0	(d)
17	-40823	prime	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
18	-42899	prime	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
19	-47172	$4, 3, 3931$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
20	-52276	$4, 7, 1867$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
21	-54347	prime	(100000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
22	-55667	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
23	-56167	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
24	-58424	$8, 67, 109$	(000000)	abelian	$\langle 5^2, 2\rangle$	1	2	0	0	0	(a)
25	-64103	$13, 4931$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	2	1	1	0	(d)
26	-64415	$5, 13, 991$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
27	-64724	$4, 11, 1471$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
28	-65735	$5, 13147$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
29	-67128	$8, 3, 2797$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	2	1	1	0	(d)
30	-69619	$11, 6329$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
31	-70763	$7, 11, 919$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
32	-71695	$5, 13, 1103$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
33	-74019	$3, 11, 2243$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
34	-75103	$7, 10729$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
35	-75892	$4, 18973$	(100000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
36	-77095	$5, 17, 907$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
37	-78747	$3, 26249$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
38	-83620	$4, 5, 37, 113$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
39	-83636	$4, 7, 29, 103$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
40	-86404	$4, 21601$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
41	-91127	prime	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
42	-92219	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
43	-99428	$4, 7, 53, 67$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)

TABLE 7. The group $\mathrm{G}_5^{(\infty)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-175000 < d < -100000$

No.	Discriminant		Principalization		$\mathrm{G}_5^{(\infty)}M$	$\ell_5 M$	Invariants				
	d	Factors	$\varkappa$	Type			r_1	δ_1	r_2	δ_2	Case
44	-100708	$4, 17, 1481$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
45	-101011	$83, 1217$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
46	-101784	$8, 3, 4241$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
47	-104503	$7, 14929$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	2	1	1	0	(d)
48	-105431	$19, 31, 179$	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
49	-105784	$8, 7, 1889$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
50	-107791	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
51	-110479	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	2	1	(e)
52	-114303	$3, 7, 5443$	(263415)	4-cycle	$\langle 5^5, 11\rangle$	2	1	0	1	0	(c)
53	-114679	prime	(000006)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
54	-115912	$8, 14489$	(000000)	abelian	$\langle 5^2, 2\rangle$	1	2	0	0	0	(a)
55	-116680	$8, 5, 2917$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
56	-118660	$4, 5, 17, 349$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
57	-119191	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	2	1	1	0	(d)
58	-123028	$4, 30757$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
59	-124099	$193, 643$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
60	-125547	$3, 41849$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
61	-127259	$11, 23, 503$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
62	-127519	$7, 18217$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
63	-133188	$4, 3, 11, 1009$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
64	-133715	$5, 47, 569$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
65	-134392	$8, 107, 157$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
66	-136311	$3, 7, 6491$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
67	-137640	$8, 3, 5, 31, 37$	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
68	-139695	$3, 5, 67, 139$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
69	-139703	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
70	-140232	$8, 3, 5843$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
71	-142904	$8, 17863$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
72	-143480	$8, 5, 17, 211$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
73	-145007	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
74	-145668	$4, 3, 61, 199$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
75	-148004	$4, 163, 227$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
76	-148507	$97, 1531$	(000000)	abelian	$\langle 5^2, 2\rangle$	1	2	0	0	0	(a)
77	-151879	$7, 13, 1669$	(000000)	abelian	$\langle 5^2, 2\rangle$	1	2	0	0	0	(a)
78	-154408	$8, 19301$	(000000)	abelian	$\langle 5^2, 2\rangle$	1	2	0	0	0	(a)
79	-155515	$5, 19, 1637$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
80	-155603	$7, 22229$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
81	-157028	$4, 37, 1061$	(003000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
82	-157031	$7, 22433$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
83	-159679	$13, 71, 173$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
84	-160571	$211, 761$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
85	-163427	$11, 83, 179$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
86	-164116	$4, 89, 461$	(000006)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
87	-165364	$4, 41341$	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
88	-169752	$8, 3, 11, 643$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)

TABLE 8. The group $\mathrm{G}_5^{(\infty)}M$ of $M=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{d}\right)$ with $-200000<d<-175000$

No.	Discriminant		Principalization		$\mathrm{G}_5^{(\infty)}M$	$\ell_5 M$	Invariants				
	d	Factors	$\varkappa$	Type			r_1	δ_1	r_2	δ_2	Case
89	-175076	$4, 11, 23, 173$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
90	-175335	$3, 5, 11689$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
91	-176459	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
92	-177428	$4, 44357$	(100000)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
93	-179195	$5, 35839$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
94	-180583	$13, 29, 479$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
95	-181847	$43, 4229$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
96	-182968	$8, 22871$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
97	-185883	$3, 61961$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
98	-186187	prime	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
99	-186271	prime	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
100	-190387	prime	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
101	-193483	$191, 1013$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
102	-193571	$7, 27653$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
103	-194487	$3, 241, 269$	(000000)	a.1 $\uparrow$	$\langle 5^4, 7\rangle$	2	1	0	1	0	(c)
104	-194920	$8, 5, 11, 443$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
105	-196648	$8, 47, 523$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
106	-196707	$3, 7, 17, 19, 29$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)
107	-197752	$8, 19, 1301$	(000000)	a.1	$\langle 5^3, 3\rangle$	2	1	0	1	0	(c)
108	-199735	$5, 43, 929$	(000050)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	2	1	(e)
109	-199947	$3, 11, 73, 83$	(000400)	a.2, fixed point	$\langle 5^4, 8\rangle$	2	1	0	1	0	(c)

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