PROPAGATION OF ARTIN TRANSFER PATTERNS BETWEEN ISOCLINIC 2-GROUPS

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Abstract. For isoclinism families Φ of finite non-abelian 2-groups, it is investigated by which general laws the abelian type invariants and the transfer kernels of the stem $\Phi(0)$ are connected with those of the branches $\Phi(n)$ with $n \geq 1$.

1. Introduction

As opposed to groups with odd prime power orders, finite 2-groups G with abelianizations G/G' of the types $(2,2)$, $(4,2)$ and $(2,2,2)$ possess sufficiently simple presentations to admit an easy calculation of invariants like central series, two-step centralizers, maximal subgroups, Artin transfers, and Artin patterns. Isoclinism between stem groups of type $(2,2)$ and branch groups of the types $(4,2)$ and $(2,2,2)$ ensures isomorphic central quotients and isomorphic lower central series of the former and latter [12], and is useful for comparing Artin patterns [19, Dfn. 4.3, p. 27].

2. Isoclinism of groups

For classifying finite p-groups, Hall introduced the concept of *isoclinism* [12] which admits to collect non-isomorphic groups with certain similarities in *isoclinism families*. Hall's idea was based on the following connection between commutators and central quotients (inner automorphisms).

Lemma 2.1. *Let G be a group, then the commutator mapping $[.,.] : G \times G \to G'$, $(g,h) \mapsto [g,h]$, factorizes through the central quotient by inducing a well-defined map*

$$c_G : (G/\zeta G) \times (G/\zeta G) \to G',\ (g \cdot \zeta G, h \cdot \zeta G) \mapsto [g,h]. \tag{2.1}$$

Proof. Let $\omega : G \to G/\zeta G$ be the natural projection onto the central quotient. The claim $[.,.] = c_G \circ (\omega \times \omega)$ follows immediately when we show that the map c_G is independent of the coset representatives, that is, $[gz, hw] = [g,h]$ for any $g,h \in G$ and $z,w \in \zeta G$. According to the right product rule for commutators, we have

$$[gz, h \cdot w] = [gz, w] \cdot [gz, h]^w = w^{-1}[gz, h]w = [gz, h],$$

since $w \in \zeta G$ and $\mathrm{cnj}(w) = \mathrm{id}$. According to the left product rule for commutators, we obtain

$$[g \cdot z, h] = [g,h]^z \cdot [z,h] = z^{-1}[g,h]z = [g,h],$$

since $z \in \zeta G$ and $\mathrm{cnj}(z) = \mathrm{id}$. □

Definition 2.1. Two groups G and H are *isoclinic*, if there exists

- an isomorphism $\psi_1 : G/\zeta G \to H/\zeta H$ of the central quotients
- and an isomorphism $\psi_2 : G' \to H'$ of the commutator subgroups
- such that $\psi_2 \circ c_G = c_H \circ (\psi_1 \times \psi_1)$, i.e. the diagram in Table 1 commutes.

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The pair $\Psi = (\psi_1, \psi_2)$ is called an *isoclinism* from G to H.

TABLE 1. Isoclinism $\Psi = (\psi_1, \psi_2)$ from G to H

$$\begin{array}{ccccc} & & \psi_1 \times \psi_1 & & \\ & (G/\zeta G) \times (G/\zeta G) & \longrightarrow & (H/\zeta H) \times (H/\zeta H) & \\ c_G & \downarrow & /// & \downarrow & c_H \\ & G' & \longrightarrow & H' & \\ & & \psi_2 & & \end{array}$$

Isoclinism is an equivalence relation. The equivalence classes are called *isoclinism families.* They are coarser than isomorphism classes, because isomorphic groups are also isoclinic (see [16, Lem. 2.1, p. 852]). An important example of isoclinic but non-isomorphic groups will be given in Theorem 2.1. First, however, we have to recall some properties of direct products.

2.1. **Direct products.** Let G, H be groups with centres $\zeta G, \zeta H$ and commutator subgroups G', H'. The *direct product* $G \times H = \{(g, h) \mid g \in G, h \in H\}$ of G and H is endowed with

$$(G \times H) \times (G \times H) \to G \times H,\ ((g_1, h_1), (g_2, h_2)) \mapsto (g_1 g_2, h_1 h_2).$$

as group operation. The neutral element is $1_{G\times H} = (1_G, 1_H)$ and the inverse of (g, h) is (g^{-1}, h^{-1}).

Lemma 2.2. *The centre of the direct product $G \times H$ is $\zeta G \times \zeta H$.*

Proof. If $(z, w) \in \zeta G \times \zeta H$, then $zg = gz$ for all $g \in G$ and $wh = hw$ for all $h \in H$. Consequently, $(z, w)(g, h) = (zg, wh) = (gz, hw) = (g, h)(z, w)$ for all $(g, h) \in G \times H$, and thus $(z, w) \in \zeta(G \times H)$. This proves the inclusion $\zeta G \times \zeta H \le \zeta(G \times H)$. The opposite inclusion follows from reading the proof in the opposite direction. □

Lemma 2.3. *The commutator subgroup of the direct product $G \times H$ is $G' \times H'$.*

Proof. An element of $(G \times H)'$ is a finite product of commutators

$$\prod_i [(g_i, h_i), (\tilde{g}_i, \tilde{h}_i)] = \prod_i ([g_i, \tilde{g}_i], [h_i, \tilde{h}_i]) = \left(\prod_i [g_i, \tilde{g}_i], \prod_i [h_i, \tilde{h}_i]\right) \in G' \times H'.$$

Conversely, if $(c, d) \in G' \times H'$, then $c = \prod_{j\in J} [g_j, \tilde{g}_j]$ and $d = \prod_{k\in K} [h_k, \tilde{h}_k]$ are finite products of commutators. Assuming $\#J \le \#K$ and multiplying c with $\#K - \#J$ trivial commutators $[1, 1] = 1$, we obtain

$$(c, d) = \left(\prod_{k\in K} [g_k, \tilde{g}_k], \prod_{k\in K} [h_k, \tilde{h}_k]\right) = \prod_{k\in K} [(g_k, \tilde{g}_k), (h_k, \tilde{h}_k)] \in (G \times H)'. \qquad \square$$

Theorem 2.1. *If G is a group and A is an abelian group, then the direct product $G \times A$ is isoclinic to G. (However, if G and A are finite and non-trivial, then $G \times A$ cannot be isomorphic to G.)*

Proof. Since A is abelian, we have $\zeta A = A$ and $A' = 1$. According to Lemma 2.2, $\zeta(G \times A) = \zeta G \times \zeta A = \zeta G \times A$, and according to Lemma 2.3, $(G \times A)' = G' \times A' = G' \times 1 \simeq G'$.

First of all, we prove the existence of an isomorphism $\psi_1 : (G \times A)/\zeta(G \times A) \to G/\zeta G$. It is induced by the (epimorphic) projection onto the first component, $\pi : G \times A \to G$, $(g, a) \mapsto g$, according to [20, Thm. 7.1, p. 98], since $\zeta(G \times A) = \zeta G \times A = \pi^{-1}(\zeta G)$. Explicitly, we have $\psi_1((g, a) \cdot \zeta(G \times A)) = g \cdot \zeta G$ for $(g, a) \in G \times A$.

The second isomorphism $\psi_2 : (G \times A)' \to G'$ is obvious: $\psi_2(g, 1) = g$ for $(g, 1) \in G' \times 1$.

Finally, we show that $\psi_2 \circ c_{G\times A} = c_G \circ (\psi_1 \times \psi_1)$. Let $(g, a), (h, b) \in G \times A$, then

$$c_G\big((\psi_1 \times \psi_1)((g, a) \cdot \zeta(G \times A), (h, b) \cdot \zeta(G \times A))\big) = c_G(g \cdot \zeta G, h \cdot \zeta G) = [g, h]$$

and, since $[a, b] = 1$,

$$\psi_2\big(c_{G\times A}((g, a) \cdot \zeta(G \times A), (h, b) \cdot \zeta(G \times A))\big) = \psi_2([(g, a), (h, b)]) = \psi_2([g, h], [a, b]) = [g, h]. \qquad \square$$

3. 2-groups of coclass 1

The simplest non-abelian 2-groups G are those of maximal nilpotency class, that is of coclass $\mathrm{cc}(G)=1$. Since their centre $\zeta(G)=\langle s_c\rangle$ is cyclic of order 2 and is contained in the commutator subgroup $G'=\langle s_2,\ldots,s_c\rangle$, they are *stem groups* $G\in\Phi(0)$ of isoclinism families Φ. A parametrized polycylic power-commutator presentation, with two bounded parameters $\alpha,\beta\in\{0,1\}$ and the nilpotency class $c:=\mathrm{cl}(G)\in\mathbb{N}$ as unbounded paramenter, is given for all these groups by

$$\begin{aligned}G_c(\alpha,\beta)=\langle\ x,y,s_2,\ldots,s_c\ \mid\ & s_2=[y,x],\ s_i=[s_{i-1},x]=[s_{i-1},y] \text{ for } 3\le i\le c,\\ & x^2=s_c^\alpha,\ y^2=s_c^\beta,\ s_i^2=s_{i+1}s_{i+2} \text{ for } 2\le i\le c-2,\ s_{c-1}^2=s_c\ \rangle\end{aligned}\tag{3.1}$$

These 2-groups form the vertices of the coclass tree $\mathcal{T}^1(R)$ with abelian root $R=\langle 4,2\rangle\simeq C_2\times C_2$, which is visualized in [18, Fig. 3.1, p. 419]. They can be classified into three infinite periodic sequences, the capable vertices of the *dihedral* mainline with parameters $(\alpha,\beta)=(0,0)$, the terminal *semidihedral* groups with $(\alpha,\beta)=(1,0)$, and the terminal *quaternion* groups with $(\alpha,\beta)=(1,1)$. In dependence on the logarithmic order $n:=\mathrm{cl}(G)+\mathrm{cc}(G)=c+1$, i.e. $\#(G)=2^n$, their identifiers $\langle 2^n,i\rangle$ in the SmallGroups database [1, 2] are given by Table 2.

Table 2. SmallGroup Identifiers of 2-groups G with $G/G'\simeq(2,2)$

Sequence	$n=$	3	4	5	6	7	8	9
dihedral	$i=$	3	7	18	52	161	539	2042
semidihedral			8	19	53	162	540	2043
quaternion		4	9	20	54	163	541	2044

Since our parametrized presentation in Formula (3.1), which is also used by the computational algebra system MAGMA [17], is slightly different from Blackburn's parametrized presentation [3], which is also given in [18, § 3.1.2, pp. 412–413], we show that exactly two of the elements x,y,xy produce the sequence $(s_i)_{i\ge 3}$ of polycyclic generators, whereas the remaining one commutes with all s_i, $i\ge 2$, and gives rise to the *polarization* by the two-step centralizers $\chi_j(G)$, $2\le j\le c-1$.

Lemma 3.1. *Let G be a finite 2-group with two main generators, $G=\langle x,y\rangle$, and nilpotency class $c:=\mathrm{cl}(G)$. Assume that generators of a polycyclic series of subgroups are defined by $s_2=[y,x]$ and $s_i=[s_{i-1},x]$ for $3\le i\le c$ such that $s_i^2=s_{i+1}s_{i+2}$ for $2\le i\le c-2$ and $s_{c-1}^2=s_c$.*

1. *If $[s_i,y]=1$ for $2\le i\le c$, then $[s_{i-1},xy]=s_i$ for $3\le i\le c$.*
2. *If $[s_{i-1},y]=s_i$ for $3\le i\le c$, then $[s_i,xy]=1$ for $2\le i\le c$.*

For a group G having the presentation in Formula (3.1), the centre is $\zeta(G)=\langle s_c\rangle$ and all two-step centralizers $\chi_jG=\langle xy,G'\rangle$ coincide for $2\le j\le c-1$, whereas $\chi_cG=G$.

Proof. Since x,y are the main generators of the metabelian group G, the centre is given by $\zeta(G)=\{z\in G\mid [z,x]=[z,y]=1\}$. The jth *two-step centralizer* is defined by $\chi_j(G)=\{c\in G\mid [c,s]\in\gamma_{j+2}(G) \text{ for } s\in\gamma_j(G)\}$, where $j\ge 2$ and $(\gamma_i(G))_{i\ge 1}$ denotes the lower central series of G. According to the right product rule for commutators, we have

1. $[s_{i-1},xy]=[s_{i-1},y]\cdot[s_{i-1},x]^y=1\cdot s_i^y=s_i\cdot s_i^{-1+y}=s_i\cdot[s_i,y]=s_i$,
2. $[s_{i-1},xy]=[s_{i-1},y]\cdot[s_{i-1},x]^y=s_i\cdot s_i^y=s_i^2\cdot s_i^{-1+y}=s_i^2s_{i+1}=s_{i+1}^2s_{i+2}=\ldots=s_c^2=1$.

Aside from the neutral element 1, the generator s_c of the last non-trivial lower central $\gamma_c(G)$ is the unique element which satisfies $[s_c,x]=[s_c,y]=s_{c+1}=1$. According to Kaloujnine, we have $[\gamma_2(G),\gamma_j(G)]\le\gamma_{j+2}(G)$, and thus $G'=\gamma_2(G)\le\chi_j(G)$. If $2\le j\le c-1$, then $[s_j,xy]=1\in\gamma_{j+2}(G)$, whereas $[s_j,x]=[s_j,y]=s_{j+1}\in\gamma_{j+1}(G)\setminus\gamma_{j+2}(G)$. Therefore, $\chi_jG=\langle xy,G'\rangle$. □

Theorem 3.1. *The invariants of 2-groups G of maximal class with parametrized presentation in Formula (3.1) are given by Table 3 in dependence on the parameters (α,β). The centre $\zeta(G)$ is denoted by ζ_1, the commutator subgroup G' by γ_2, the Artin pattern by $(\tau,\varkappa)$, the number of abelian maximal subgroups by n_a, the relation rank by μ, the nuclear rank by ν, the number of all immediate descendants (children) by N_1, and the number of capable children by C_1. They reveal a uniform regular behaviour for class $c\ge 3$, whereas exceptions arise for the irregular class $c=2$.*

Table 3. Invariants of 2-groups G with $G/G' \simeq (2,2)$

Class	ζ_1	γ_2	τ	$\varkappa$	n_a	μ	ν	N_1	C_1	α	β	Sequence	Symbol
$c=2$	1	1	$(1^2)^2,\mathbf{2}$	$21\mathbf{0}$	3	3	1	3	1	0	0	dihedral	$D(8)$
	1	1	$(2)^2,\mathbf{2}$	$12\mathbf{3}$	3	2	0	0	0	1	1	quaternion	$Q(8)$
$c\ge 3$	1	$c-1$	$(1^2)^2,\mathbf{c}$	$21\mathbf{0}$	1	3	1	3	1	0	0	dihedral	$D(2^{c+1})$
	1	$c-1$	$(1^2)^2,\mathbf{c}$	$21\mathbf{1}$	1	2	0	0	0	1	0	semidihedral	$S(2^{c+1})$
	1	$c-1$	$(1^2)^2,\mathbf{c}$	$21\mathbf{3}$	1	2	0	0	0	1	1	quaternion	$Q(2^{c+1})$

Proof. By Formula (3.1), the maximal subgroups M_i of G and their derived subgroups M_i' are
$M_1 = \langle y, G'\rangle, \quad M_1' = (G')^{y-1} = \langle [s_2, y]\rangle = \langle s_3\rangle,$
$M_2 = \langle x, G'\rangle, \quad M_2' = (G')^{x-1} = \langle [s_2, x]\rangle = \langle s_3\rangle,$ and, by Lemma 3.1,
$M_3 = \langle xy, G'\rangle, \quad M_3' = (G')^{xy-1} = \langle [s_2, xy]\rangle = 1.$ In particular, M_3 is abelian.
The Artin transfers from G to the M_i are given by $T_i : G/G' \to M_i/M_i'$ with outer transfers $T_i(gG') = g^2 M_i'$ for $g \in G \setminus M_i$ and inner transfers $T_i(gG') = g^{1+h} M_i'$ for $g \in M_i$ and $h \in G \setminus M_i$. Explicitly, the images of the generators x, y under the homomorphisms T_i are given by
$T_1(xG') = x^2\langle s_3\rangle = s_c^{\alpha}\langle s_3\rangle, \quad T_1(yG') = y^{1+x}\langle s_3\rangle = y^2 \cdot y^{-1}x^{-1}yx\langle s_3\rangle = s_c^{\beta} s_2\langle s_3\rangle,$
$T_2(xG') = x^{1+y}\langle s_3\rangle = x^2 \cdot x^{-1}y^{-1}xy\langle s_3\rangle = s_c^{\alpha} s_2^{-1}\langle s_3\rangle, \quad T_2(yG') = y^2\langle s_3\rangle = s_c^{\beta}\langle s_3\rangle,$ and
$T_3(xG') = x^2 \cdot 1 = s_c^{\alpha}, \quad T_3(yG') = y^2 \cdot 1 = s_c^{\beta}.$
Now we use the transfer images for determining the transfer kernel type by solving $T_i(gG') = M_i'$. Independently of (α, β), we obtain a 2-cycle (transposition) in the TKT $\varkappa$ for $c \ge 3$:
$T_1(xG') = \langle s_3\rangle, \quad T_1(yG') = s_2\langle s_3\rangle, \quad \varkappa(1) \hat{=} \ker(T_1) = M_2/G' \hat{=} 2,$
$T_2(xG') = s_2^{-1}\langle s_3\rangle, \quad T_2(yG') = \langle s_3\rangle, \quad \varkappa(2) \hat{=} \ker(T_2) = M_1/G' \hat{=} 1.$
The smallest value $c = 2$, however, is responsible for the irregularity $\langle s_3\rangle = 1$ and consequently
$T_1(xG') = s_2^{\alpha}, \quad T_1(yG') = s_2^{\beta+1}, \quad T_2(xG') = s_2^{\alpha-1}, \quad T_2(yG') = s_2^{\beta},$
which implies $\varkappa(1) \hat{=} \ker(T_1) = M_2/G' \hat{=} 2, \quad \varkappa(2) \hat{=} \ker(T_2) = M_1/G' \hat{=} 1,$ for $(\alpha, \beta) = (0, 0)$,
but $\varkappa(1) \hat{=} \ker(T_1) = M_1/G' \hat{=} 1, \quad \varkappa(2) \hat{=} \ker(T_2) = M_2/G' \hat{=} 2,$ two f.p. for $(\alpha, \beta) = (1, 1)$.
Finally, we have to find the kernel of the transfer to the abelian subgroup M_3, for any $c \ge 2$:
$\varkappa(3) \hat{=} \ker(T_3) = G/G' \hat{=} 0,$ characteristic for the dihedral mainline with $(\alpha, \beta) = (0, 0)$,
$\varkappa(3) \hat{=} \ker(T_3) = M_1/G' \hat{=} 1,$ for semidihedral vertices with $(\alpha, \beta) = (1, 0)$, and
$\varkappa(3) \hat{=} \ker(T_3) = M_3/G' \hat{=} 3,$ a fixed point for quaternion groups with $(\alpha, \beta) = (1, 1)$,
because $T_3(xy) = T_3(x) \cdot T_3(y) = s_c \cdot s_c = s_c^2 = 1.$ □

4. 2-groups of coclass 2 with abelianization (4, 2)

These groups G in the first branch $\Phi(1)$ of isoclinism families Φ containing groups of maximal class in their stem $\Phi(0)$ are characterized by an *exponent increment* of their abelianization. Their centre $\zeta_1 G = \langle \tau, s_c\rangle$ is enlarged by the power $\tau = x^2$ of the non-elementary generator x. A parametrized polycylic power-commutator presentation, with two bounded parameters $\alpha, \beta \in \{0, 1\}$ and the nilpotency class $c := \mathrm{cl}(G) \in \mathbb{N}$ as unbounded paramenter, is given by

$$\begin{aligned} G_c(\alpha,\beta) = \langle\ & x, y, \tau, s_2, \ldots, s_c \mid s_2 = [y, x],\ s_i = [s_{i-1}, x] \text{ for } 3 \le i \le c, \\ & \tau = x^2,\ \tau^2 = s_c^{\alpha},\ y^2 = s_2 s_3 s_c^{\beta},\ s_i^2 = s_{i+1}s_{i+2} \text{ for } 2 \le i \le c-2,\ s_{c-1}^2 = s_c\ \rangle \end{aligned} \tag{4.1}$$

5. 2-groups of coclass 2 with abelianization (2, 2, 2)

These groups G in the first branch $\Phi(1)$ of isoclinism families Φ containing groups of maximal class in their stem $\Phi(0)$ are characterized by a *rank increment* of their abelianization. Their centre $\zeta_1 G = \langle z, s_c\rangle$ is enlarged by the independent generator z. A parametrized polycylic power-commutator presentation, with four bounded parameters $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and the nilpotency

Table 4. Invariants of 2-groups G with $G/G' \simeq (4,2)$

ζ_1	γ_2	τ	$\varkappa$	n_a	μ	ν	N_1	C_1	α	β
1^2	2	$(21)^2,\mathbf{31}$	$33\mathbf{P}$	1	3	1	3	2	0	0
		$(3)^2,21$	O^3	3						
1^2	2	$(21)^2,\mathbf{31}$	$33\mathbf{3}$	1	3	1	1	0	0	1
		$(3)^2,21$	O^3	3						
2	2	$(21)^2,\mathbf{31}$	$33\mathbf{2}$	1	2	0	0	0	1	1
		$(3)^2,21$	P^3	3						
1^2	$c-1$	$(21)^2,(\mathbf{c},\mathbf{1})$	$33\mathbf{P}$	1	3	1	3	2	0	0
		$(c)^2,(c-1,1)$	O^3	3						
1^2	$c-1$	$(21)^2,(\mathbf{c},\mathbf{1})$	$33\mathbf{3}$	1	3	1	1	0	0	1
		$(c)^2,(c-1,1)$	O^3	3						
2	$c-1$	$(21)^2,(\mathbf{c},\mathbf{1})$	$33\mathbf{1}$	1	2	0	0	0	1	0
		$(c)^2,(c-1,1)$	P^3	3						
1	$c-1$	$(21)^2,(\mathbf{c}-\mathbf{1},\mathbf{1})$	$33\mathbf{3}$	0	2	0	0	0		
		$(c)^2,(c-1,1)$	Q^2R	3						

class $c := \mathrm{cl}(G) \in \mathbb{N}$ as unbounded paramenter, is given by

$$\begin{aligned} G_c(\alpha,\beta,\gamma,\delta) =\langle\ & x,y,z,t_2,s_2,\ldots,s_c \ \mid\ s_2=[y,x],\ s_i=[s_{i-1},x]=[s_{i-1},y] \text{ for } 3\le i\le c, \\ & x^2=s_c^\alpha,\ y^2=s_c^\beta,\ z^2=s_c^\gamma,\ s_i^2=s_{i+1}s_{i+2} \text{ for } 2\le i\le c-2,\ s_{c-1}^2=s_c, \\ & t_2=[z,x],\ t_2=s_c^\delta\ \rangle \end{aligned} \tag{5.1}$$

Table 5. Invariants of 2-groups G with $G/G' \simeq (2,2,2)$

ζ_1	γ_2	τ	$\varkappa$	n_a	μ	ν	N_1	C_1	α	β	γ	δ
1^2	1	$(1^3)^2,(1^2)^3,\mathbf{21},1^2$	$P_2P_1O^3\mathbf{O}O$	3	6	1	6	1	0	0	0	0
		$(1^2)^5,(2)^2$	O^7	7								
1^2	1	$(21)^2,(1^2)^3,\mathbf{21},1^2$	$P_1P_2O^3\mathbf{P_6}O$	3	5	0	0	0	1	1	0	0
		$1^2,(2)^6$	O^7	7								
2	1	$(21)^2,(1^2)^3,\mathbf{21},1^2$	$P_4P_5O^3\mathbf{P_3}O$	3	5	0	0	0	0	0	1	0
		$(1^2)^3,(2)^4$	O^7	7								
1^2	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{O}O$	1	6	1	6	1	0	0	0	0
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								
1^2	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{P_1}O$	1	5	0	0	0	1	0	0	0
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								
1^2	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{P_6}O$	1	5	0	0	0	1	1	0	0
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								
2	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{P_3}O$	1	5	0	0	0	0	0	1	0
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								
1	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c}-\mathbf{1},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{O}O$	0	5	0	0	0	0	0	0	1
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								
1	$c-1$	$(1^3)^2,(1^2)^3,(\mathbf{c}-\mathbf{1},\mathbf{1}),1^2$	$P_2P_1O^3\mathbf{O}O$	0	5	0	0	0	0	1	0	1
		$(1^2)^4,(c-1,1),(c)^2$	O^7	3								

5.1. **Propagation of Artin patterns.** In Theorem 2.1, we have seen that the direct product $G\times A$ of a group G with an abelian group A is isoclinic to G. In the case of finite non-trivial groups G and A, however, $G\times A$ cannot be isomorphic to G, for cardinality reasons $\#(G\times A) = \#G\cdot\#A > \#G$. An important instance of this general theorem is realized by those 2-groups of coclass 2 with abelianization $(2,2,2)$ which have the special presentations with parameters $\gamma=\delta=0$ in Formula (5.1), that is, with $(\alpha,\beta)\in\{(0,0),(1,0),(1,1)\}$. These presentations show that the branch group $G_c(\alpha,\beta,0,0)$ is the direct product of the stem group $G_c(\alpha,\beta)$ in Formula (3.1) with the cyclic group C_2 of order 2. Thus $G_c(\alpha,\beta,0,0)$ is isoclinic to $G_c(\alpha,\beta)$. At the first sight, it seems impossible to compare the Artin transfer patterns of these isoclinic groups since, due to the rank increment from two to three, $G_c(\alpha,\beta,0,0)$ has 7 maximal subgroups, whereas $G_c(\alpha,\beta)$ has only 3. Nevertheless, the following theorem shows that the Artin pattern of $G_c(\alpha,\beta)$ can be viewed as imbedded into the Artin pattern of $G_c(\alpha,\beta,0,0)$ in a certain sense, and we speak about the *propagation* of the Artin transfer pattern from a stem group to a branch group with rank increment. The $7-3=4$ additional components are simply padded with stable invariants having low contents of information.

Theorem 5.1. *The Artin pattern*

$$(\tau,\varkappa)=\big([1^2,1^2,\mathbf{c}],(21*)\big)$$

of $G_c(\alpha,\beta)$ with polarization at the third component propagates to the Artin pattern

$$(\tau,\varkappa)=\big([1^3,1^3,1^2,1^2,1^2,(\mathbf{c},\mathbf{1}),1^2],(P_2,P_1,O,O,O,*,O)\big)$$

of $G_c(\alpha,\beta,0,0)$ with stable imbedding at the first and second component, polarized imbedding at the sixth component and trivial padding at the third, fourth, fifth and seventh component.

Proof. We always observe that $\zeta_1 G=\langle z,s_c\rangle$ is the centre of G, if $\gamma=\delta=0$.

- The seven Artin transfers $T_i: G/G'\to M_i/M_i'$ for $1\le i\le 7$ act as outer transfers $T_i(gG')=g^2M_i'$, if $g\in G\setminus M_i$, and as inner transfers $T_i(gG')=g^{1+h}M_i'$ with $h\in G\setminus M_i$, if $g\in M_i$.
- The derived subgroups of the seven maximal subgroups $M_i=\langle t_i,u_i,G'\rangle$ with quotients $M_i/G'\simeq P_i=\langle t_i,u_i\rangle$ for $1\le i\le 7$, which possess the generators t_i and u_i in Table 7 and the relations in Formula (5.1), are given by
  $M_1'=\langle [s_i,y]\mid i\ge 2\rangle=\langle s_3\rangle$, since $G=G_c(\alpha,\beta,0,0)$ is isoclinic to $G_c(\alpha,\beta)$,
  similarly $M_2'=\langle [s_i,x]\mid i\ge 2\rangle=\langle s_3\rangle$,
  $M_3'=\langle [y,x],[s_i,x]=[s_i,y]\mid i\ge 2\rangle=\langle s_2\rangle=G'$,
  similarly $M_4'=G'$, since $[g,yz]=[g,z][g,y]^z=[g,y]$ for all $g\in G$,
  similarly $M_5'=G'$, since $[g,zx]=[g,x][g,z]^x=[g,x]$ for all $g\in G$,
  $M_6'=1$, i.e. M_6 is abelian, since $[s_i,xy]=1$ for $i\ge 2$ and $[G',G']=1$,
  $M_7'=G'$, since $[y,xy]=[y,y][y,x]^y=s_2^y=s_2\cdot s_2^{-1+y}=s_2s_3$ and $[s_i,yz]=[s_i,y]=s_{i+1}$ for $i\ge 2$.
- To determine the kernels $\ker(T_i)$ for $1\le i\le 7$, it suffices to calculate the images of the generators x,y,z and to decide which images are trivial.
  $T_3(xG')=x^{1+z}G'=x^2G'=G'$, $T_3(yG')=y^{1+z}G'=y^2G'=G'$, $T_3(zG')=z^2G'=G'$,
  $T_4(xG')=x^{1+z}G'=x^2G'=G'$, $T_4(yG')=y^2G'=G'$, $T_4(zG')=z^2G'=G'$,
  $T_5(xG')=x^2G'=G'$, $T_5(yG')=y^{1+z}G'=y^2G'=G'$, $T_5(zG')=z^2G'=G'$,
  $T_7(xG')=x^2G'=G'$, $T_7(yG')=y^2G'=G'$, $T_7(zG')=z^2G'=G'$,
  and thus all the kernels $\ker(T_3)=\ker(T_4)=\ker(T_5)=\ker(T_7)=G/G'\simeq O$ are total.
  $T_1(xG')=x^2\langle s_3\rangle=s_c^\alpha\langle s_3\rangle$, $T_1(yG')=y^{1+x}\langle s_3\rangle=y^2y^{-1+x}\langle s_3\rangle=s_c^\beta s_2\langle s_3\rangle$,
  $T_1(zG')=z^{1+x}\langle s_3\rangle=z^2z^{-1+x}\langle s_3\rangle=s_c^\gamma s_c^\delta\langle s_3\rangle$,
  $T_2(xG')=x^{1+y}\langle s_3\rangle=x^2x^{-1+y}\langle s_3\rangle=s_c^\alpha s_2^{-1}\langle s_3\rangle$, $T_2(yG')=y^2\langle s_3\rangle=s_c^\beta\langle s_3\rangle$,
  $T_2(zG')=z^{1+y}\langle s_3\rangle=z^2z^{-1+y}\langle s_3\rangle=s_c^\gamma\langle s_3\rangle$,
  $T_6(xG')=x^2=s_c^\alpha$, $T_6(yG')=y^2=s_c^\beta$, $T_6(zG')=z^{1+x}=z^2z^{-1+x}=s_c^\gamma s_c^\delta$.
  We continue to consider the case $(\gamma,\delta)=(0,0)$: For class $c\ge 3$ and any (α,β), we have $T_1(xG')=\langle s_3\rangle$, $T_1(yG')=s_2\langle s_3\rangle$, $T_1(zG')=\langle s_3\rangle$, and $\ker(T_1)=M_2/G'\simeq P_2$,
  $T_2(xG')=s_2^{-1}\langle s_3\rangle$, $T_2(yG')=\langle s_3\rangle$, $T_2(zG')=\langle s_3\rangle$, and $\ker(T_2)=M_1/G'\simeq P_1$.
  Class $c=2$ is irregular, since $\langle s_3\rangle=1$, and thus
  $T_1(xG')=s_2^\alpha$, $T_1(yG')=s_2^{\beta+1}$, $T_1(zG')=1$, $T_2(xG')=s_2^{\alpha-1}$, $T_2(yG')=s_2^\beta$, $T_2(zG')=1$,
  $\ker(T_1)=M_2/G'\simeq P_2$ for $(\alpha,\beta)=(0,0)$, $\ker(T_1)=M_1/G'\simeq P_1$ for $(\alpha,\beta)=(1,1)$,
  $\ker(T_2)=M_1/G'\simeq P_1$ for $(\alpha,\beta)=(0,0)$, $\ker(T_2)=M_2/G'\simeq P_2$ for $(\alpha,\beta)=(1,1)$.
  Finally, $\ker(T_6)=G/G'\simeq O$, confirming the mainline principle, for $(\alpha,\beta)=(0,0)$, but

$\ker(T_6) = M_1/G' \simeq P_1$, for $(\alpha, \beta) = (1, 0)$, and $\ker(T_6) = M_6/G' \simeq P_6$, for $(\alpha, \beta) = (1, 1)$, since $T_6(xy) = T_6(x) \cdot T_6(y) = s_c^{\alpha+\beta} = s_c^2 = 1$, for any $c \geq 2$. □

6. The elementary abelian 2-group of type (2, 2, 2)

We are going to analyze the capitulation of number fields F with 2-class group $\mathrm{Cl}_2 F$ of elementary abelian type $(2, 2, 2)$ in their $\frac{2^3-1}{2-1} = 7$ unramified quadratic extensions E_i/F with $1 \leq i \leq 7$. Such a group can be viewed as a *vector space* O with dimension $\dim_{\mathbb{F}_2}(O) = 3$ over the finite field $\mathbb{F}_2$ with two elements. The vector space O possesses $2^2 + 2 + 1 = 7$ *lines*, that is, subgroups L_i of index $(O : L_i) = 2^2$, and 7 *planes*, that is, subgroups P_i of index $(O : P_i) = 2$, where $1 \leq i \leq 7$. Let x, y, z be fixed generators of $O = \langle x, y, z\rangle$, then we shall arrange the generators of the lines $L_i = \langle g_i\rangle$ in the way shown in Table 6.

Table 6. Generators of the 7 lines L_i in O

i	1	2	3	4	5	6	7
g_i	x	y	z	xy	yz	zx	xyz

For the sake of brevity, we simply denote g_i by its subscript i, and we introduce identifiers for the planes $P_i = \langle t_i, u_i\rangle$ as shown in Table 7. The elements t_i, u_i can be viewed as generators of a transversal of $L_i = \langle g_i\rangle$, i.e. a system of coset representatives for L_i in O. Each set T_i contains the subscripts of generators g_j contained in P_i.

Table 7. Identifiers and generators of the 7 planes P_i in O

i	1	2	3	4	5	6	7
t_i	y	z	x	x	y	z	xy
u_i	z	x	y	yz	zx	xy	yz
T_i	2, 3, 5	1, 3, 6	1, 2, 4	1, 5, 7	2, 6, 7	3, 4, 7	4, 5, 6

It is also useful to list the *bundles* B_i of planes containing an assigned line L_i in Table 8.

Table 8. The 7 bundles B_i of planes in O

i	1	2	3	4	5	6	7
B_i	P_2, P_3, P_4	P_1, P_3, P_5	P_1, P_2, P_6	P_3, P_6, P_7	P_1, P_4, P_7	P_2, P_5, P_7	P_4, P_5, P_6

If $G := \mathrm{Gal}(F_2^{(\infty)}/F)$ denotes the pro-2 Galois group of the Hilbert 2-class field tower $F_2^{(\infty)}$ of the number field F, then we have the isomorphism $O := \mathrm{Cl}_2 F \simeq G/G'$.

Finally, we recall the connection between the size of the capitulation kernel $\ker(T_{E/F})$, that is the kernel of the transfer $T_{E/F} : \mathrm{Cl}_2 F \to \mathrm{Cl}_2 E$ of 2-classes from F to E, and the unit norm index $(U_F : \mathrm{Norm}_{E/F} U_E)$ of an unramified quadratic extension E/F of a totally complex quartic number field F.

Theorem 6.1. *The order of the capitulation kernel of E/F is given by*

$$\# \ker(T_{E/F}) = \begin{cases} 2, \\ 4, \\ 8, \end{cases} \quad \textit{if and only if} \quad (U_F : \mathrm{Norm}_{E/F} U_E) = \begin{cases} 1, \\ 2, \\ 4. \end{cases} \tag{6.1}$$

Proof. According to the Herbrand Theorem on the unit group U_E as a Galois module with respect to $\mathrm{Gal}(E/F)$ [13], we have the relation $\# \ker(T_{E/F}) = [E : F] \cdot (U_F : \mathrm{Norm}_{E/F} U_E)$, since E/F is unramified of prime degree $[E : F] = 2$. If the base field F has signature $(r_1, r_2) = (0, 2)$ and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 1$, then there are three possibilities for the unit norm index $(U_F : \mathrm{Norm}_{E/F} U_E) \in \{1, 2, 4\}$, since U_F contains the 2-torsion unit -1. □

Remark 6.1. When F is a number field with 2-class group $O := \mathrm{Cl}_2 F$ of elementary abelian type $(2,2,2)$, then $\#\ker(T_{E/F}) = 2$ if and only if $(\exists\, 1 \le i \le 7)\ \ker(T_{E/F}) = L_i$, $\#\ker(T_{E/F}) = 4$ if and only if $(\exists\, 1 \le i \le 7)\ \ker(T_{E/F}) = P_i$, and $\#\ker(T_{E/F}) = 8$ if and only if $\ker(T_{E/F}) = O$.

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