STEM AND BRANCH GROUPS OF ISOCLINISM FAMILIES

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Abstract. Presentations for the p-groups G of order $|G| = p^5$ with an odd prime p, coclass $\mathrm{cc}(G) = 2$, and abelianisation of type (p,p) are used for $p \in \{3,5\}$ to determine explicit expressions for the transfers V_i from these groups to their maximal normal subgroups M_i, $1 \le i \le p{+}1$, and to calculate the transfer kernels $\ker(\mathrm{V}_i)$ in G/G'. For $p = 3$ it is shown, that the groups can also be characterised by the structure of their transfer targets M_i/M_i'.

1. Introduction

The *stem groups* G of Philip Hall's isoclinism family Φ_6 [9, p.139] are p-groups of order $|G| = p^5$ with an odd prime p, nilpotency class $\mathrm{cl}(G) = 3$, and coclass $\mathrm{cc}(G) = 2$. They were discovered in 1898 by Giuseppe Bagnera [4, pp.182–183], and constructed as extensions of $C(p)^3$ by $C(p)^2$, for $p \ge 5$, in 1926 by Otto Schreier [21, pp.341–345]. Bagnera also pointed out that these groups do not have an analogon for $p = 2$. Presentations were given in 1980 by Rodney James [10, pp.620–621], for arbitrary odd primes p, and, for the special prime $p = 3$, in 1977 by Ascione, Havas, and Leedham-Green [1, p.265], and in 1989 by Nebelung [18, pp.1–3].

These groups are represented by top vertices of the coclass graph $\mathcal{G}(p,2)$ in the sense of Eick and Leedham-Green [7, 6, 8], and play an important role as starting groups for the p-group generation algorithm [1, 2, 3], since the coclass graphs $\mathcal{G}(3,2)$ and $\mathcal{G}(5,2)$ are central objects of current research with interesting number theoretical applications such as abelianisations G/G' of type $(3,3)$, $(9,3)$, and $(5,5)$, of second p-class groups G of number fields [14].

James [10, pp.623–624] has also given presentations for the *first branch groups* of isoclinism family Φ_6, which are of order $|G| = p^6$, nilpotency class $\mathrm{cl}(G) = 3$, and coclass $\mathrm{cc}(G) = 3$.

In section 2 we recall common features of stem groups, resp. first branch groups, in Φ_6, such as lower central series, centers, 2-step centralisers and maximal subgroups. In section 3 we use the presentations by James to derive explicit expressions for the transfers and to compute the transfer kernels. Finally, we characterise the groups by the structure of their transfer targets in section 5.

2. Common features of the groups in Φ_6

2.1. Stem groups.

Every stem group of isoclinism family Φ_6 is a 2-generator group $G = \langle x, y\rangle$ with main commutator $s_2 = [y,x]$ in $\gamma_2(G)$ and higher commutators $s_3 = [s_2,x]$, $t_3 = [s_2,y]$ in $\gamma_3(G)$, satisfying the power relations $s_2^p = s_3^p = t_3^p = 1$. The lower central series of G is given by

$$\gamma_2(G) = \langle s_2, s_3, t_3\rangle \text{ of type } (p,p,p), \quad \gamma_3(G) = \langle s_3, t_3\rangle \text{ of type } (p,p), \quad \gamma_4(G) = 1,$$

and the center by $\zeta_1(G) = \gamma_3(G)$. The central quotient $G/\zeta_1(G)$ is of type $\Phi_2(1^3)$ (the extra special p-group of order p^3 and exponent p) and the abelianisation $G/\gamma_2(G)$ is of type (p,p). Therefore, the lower central structure of these groups uniformly consists of two diamonds $G/\gamma_2(G)$ and $\gamma_3(G)/\gamma_4(G)$ separated by the cyclic factor $\gamma_2(G)/\gamma_3(G)$.

For any stem group G in Φ_6, there exists a very nice 1-to-1 correspondence between the two diamonds by taking the derived subgroups.

Theorem 2.1. *The maximal normal subgroups of G contain the commutator subgroup $G' = \gamma_2(G)$ and are given by*

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$$M_i = \langle g_i, G'\rangle \text{ with generators } g_1 = y \text{ and } g_i = xy^{i-2} \text{ for } 2 \le i \le p+1,$$

and their derived subgroups $M_i' = (G')^{g_i - 1}$ *are given by*

$$M_1' = \langle t_3\rangle \text{ and } M_i' = \langle s_3 t_3^{i-2}\rangle \text{ for } 2 \le i \le p+1.$$

Proof. According to Blackburn [5, p.52, Lem.2.1], the derived subgroups of the M_i can be represented in the form

$$M_i' = [M_i, M_i] = [G', M_i] = (G')^{g_i-1}.$$

Taking into account that $G' = \langle s_2, s_3, t_3\rangle$ and s_3, t_3 generate the center $\zeta_1(G)$ of G, we can reduce the set of generators of each derived subgroup to a single element only

$$M_i' = \langle s_2^{g_i-1}\rangle.$$

We immediately obtain $M_1' = \langle s_2^{y-1}\rangle = \langle [s_2, y]\rangle = \langle t_3\rangle$ and for the other maximal subgroups M_i, the right product rule and the power rule for commutators yield: $s_2^{g_i-1} = s_2^{xy^{i-2}-1} = [s_2, xy^{i-2}] = [s_2, y^{i-2}][s_2, x]^{y^{i-2}} = [s_2, y]^{\mathrm{tr}_{i-2}(y)} s_3^{y^{i-2}} = t_3^{\sum_{j=1}^{i-2} y^{j-1}} s_3 = s_3 t_3^{i-2}$, for $2 \le i \le p+1$. □

As a consequence of theorem 2.1, we only have trivial 2-step centralisers $G' = \chi_2(G) < \chi_3(G) = G$ and the invariant $s = 3$ [14, Def.2.1].

The individual relations for the isomorphism classes [10, pp.620–621] are given by table 1, where ν denotes the smallest positive quadratic non-residue modulo p and g denotes the smallest positive primitive root modulo p.

Table 1. Relations for the stem groups of Φ_6

stem group	x^p	y^p	parameters
$\Phi_6(221)_a$	s_3	t_3	
$\Phi_6(221)_{b_r}$	s_3	t_3^k	$1 \le r \le \frac{p-1}{2}$, $k = g^r$
$\Phi_6(221)_{c_r}$	$s_3^r t_3^r$	$s_3^{-\frac{r}{4}}$	$r \in \{1, \nu\}$
$\Phi_6(221)_{d_0}$	t_3^{ν}	s_3	
$\Phi_6(221)_{d_r}$	$s_3 t_3$	s_3^k	$1 \le r \le \frac{p-1}{2}$, $k = \frac{g^{2r+1}-1}{4}$
$\Phi_6(21^3)_a$	1	t_3	$p \ge 5$
$\Phi_6(21^3)_{b_r}$	t_3^r	1	$r \in \{1, \nu\}$, $p \ge 5$
$\Phi_6(1^5)$	1	1	

2.2. **First branch groups.** There are two possibilities for first branch groups G in the isoclinism family Φ_6. Either the abelianisation G/G' is of type (p^2, p) and G is a 2-generator group $G = \langle x, y\rangle$ with $x^{p^2} \in G'$ or the abelianisation G/G' is of type (p, p, p) and G is a 3-generator group $G = \langle x, y, z\rangle$ with $z^p \in G'$.
In both cases, we have the main commutator $s_2 = [y, x]$ in $\gamma_2(G)$ and the higher commutators $s_3 = [s_2, x]$, $t_3 = [s_2, y]$ in $\gamma_3(G)$, which satisfy the power relations $s_2^p = s_3^p = t_3^p = 1$, completely similar as for the stem groups.
We do not deal further with groups having abelianisation of type (p, p, p) here.

For the groups G with abelianisation of type (p^2, p) we declare a slightly different order of the maximal normal subgroups

$$M_i = \langle g_i, G'\rangle \text{ with } g_i = xy^{i-1} \text{ for } 1 \le i \le p, \text{ and } M_{p+1} = \langle x^p, y, G'\rangle,$$

because we want to indicate the unique bicyclic factor M_{p+1}/G' by the special subscript $i = p+1$, in contrast to the cyclic factors M_i/G' of order p^2 for $1 \le i \le p$.
Since $[s_2, x^p] = [s_2, x]^{\mathrm{tr}_p(x)} = s_3^p = 1$, we obtain the derived subgroups

$$M_{p+1}' = \langle t_3\rangle \text{ and } M_i' = \langle s_3 t_3^{i-1}\rangle \text{ for } 1 \le i \le p.$$

As for the stem groups, the lower central series of G is given by

$$\gamma_2(G) = \langle s_2, s_3, t_3\rangle \text{ of type } (p,p,p), \quad \gamma_3(G) = \langle s_3, t_3\rangle \text{ of type } (p,p), \quad \gamma_4(G) = 1.$$

However, since the central quotient $G/\zeta_1(G)$ of type $\Phi_2(1^3)$ is an invariant of the isoclinism family Φ_6, we must now have a bigger center than for the stem groups. Indeed, by means of commutator calculus we deduce that $[y, x^p] = s_2^p s_3^{\binom{p}{2}} = 1$ for $p \neq 2$, in addition to the trivial relations $[s_2, x^p] = s_3^p = 1$ and $[x, x^p] = 1$, whence the center is given by $\zeta_1(G) = \langle x^p, \gamma_3(G)\rangle$, of type either (p, p, p) or (p^2, p).

This fact has a considerable impact on the 2-step centralisers
$G' < \chi_1(G) = \chi_2(G) = \tilde{M}_{p+1} = \langle x^p, G'\rangle < \chi_3(G) = G$ and the invariant $s = 1$ [14, Def.2.1]. $\tilde{M}_{p+1} = \Phi(G)$ coincides with the Frattini subgroup of G.

The individual relations for the isomorphism classes [10, pp.623–624] are given by table 2.

Table 2. Relations for the first branch groups of Φ_6

branch group	x^{p^2}	y^p	parameters
$\Phi_6(321)_{a_r}$	s_3	t_3	$1 \leq r \leq p-1$
$\Phi_6(321)_{b_{r,s}}$	t_3^s	$s_3^{r-\binom{p}{3}}$	$r, s \in \{1, \nu\}$
$\Phi_6(31^3)_a$	s_3^{-1}	1	
$\Phi_6(31^3)_{b_r}$	t_3^r	$s_3^{-\binom{p}{3}}$	$r \in \{1, \nu\}$
$\Phi_6(2^21^2)_g$	1	t_3^{-1}	
$\Phi_6(2^21^2)_{h_r}$	1	s_3^{-r}	$r \in \{1, \nu\}$
$\Phi_6(21^4)_d$	1	1	

3. Transfer kernel types $\varkappa$ of the groups in Φ_6

3.1. **Stem groups.** The presentations [10, pp.620–621] for the 7 isomorphism classes of 3-groups, resp. the 12 isomorphism classes of 5-groups, among the stem groups of isoclinism family Φ_6 are used in this section to calculate the kernels of the transfers V_i from these metabelian p-groups G to their maximal normal subgroups M_i. For $p = 3$, the groups B_0, Q_0, U_0 give rise to three coclass trees in the graph $\mathcal{G}(3, 2)$, whose mainlines share a common *transfer kernel type* with their roots. In table 3, the transfer kernel types $\varkappa$ of the 3-groups in the notation of [13, sec.3.3] are due to Nebelung [17], but for the 5-groups they have been calculated by ourselves and are given in table 3 for the first time, except for $\Phi_6(221)_a$, which was used in the well-known paper by Taussky [22, pp.435–436] to show that the condition $(AAAAAA)$ can occur for $p = 5$, and similarly for $p \geq 7$.

Table 3. Transfer kernel types of corresponding p-groups in Φ_6 for $p = 3$ and $p = 5$

$p=3$			$p=5$		
type	$\varkappa$	3-group	$\varkappa$	property	5-group
D.5	(4334)	$G_0^{(4,5)}(1,1,-1,1)$	(123456)	identity	$\Phi_6(221)_a$
H.4	(4443)	$G_0^{(4,5)}(1,1,1,1)$	(125364)	4-cycle	$\Phi_6(221)_{b_1}$
	no analogon		(126543)	two transpos.	$\Phi_6(221)_{b_2}$
c.21	(1024)	$U_0 = G_0^{(4,5)}(0,0,0,1)$	(612435)	5-cycle	$\Phi_6(221)_{c_1}$
D.10	(4124)	$G_0^{(4,5)}(0,0,-1,1)$	(612435)	5-cycle	$\Phi_6(221)_{c_2}$
G.19	(2143)	$G_0^{(4,5)}(0,-1,-1,0)$	(214365)	three transpos.	$\Phi_6(221)_{d_0}$
c.18	(3023)	$Q_0 = G_0^{(4,5)}(0,-1,0,1)$	(512643)	6-cycle	$\Phi_6(221)_{d_1}$
	no analogon		(312564)	two 3-cycles	$\Phi_6(221)_{d_2}$
	no analogon		(022222)	alm.const.with fp.	$\Phi_6(21^3)_a$
	no analogon		(011111)	almost constant	$\Phi_6(21^3)_{b_1}$
	no analogon		(011111)	almost constant	$\Phi_6(21^3)_{b_2}$
b.10	(2100)	$B_0 = G_0^{(4,5)}(0,0,0,0)$	(000000)	constant	$\Phi_6(1^5)$

3.2. **First branch groups.** The presentations [10, pp.623–624] for the 13 isomorphism classes of 3-groups among the first branch groups of isoclinism family Φ_6 will be used in this section to calculate the kernels of the transfers V_i from these metabelian 3-groups G to their maximal normal subgroups M_i. It is unknown up to now, which of these groups give rise to coclass trees in the graph $\mathcal{G}(3,3)$, whose mainlines will probably share a common transfer kernel type with their roots. In table 4, the punctured transfer kernel types $\varkappa$ of these 3-groups are given, using the notation of [16]. The puncture always concerns the distinguished subscript 4 which is separated by a semicolon.

Table 4. Punctured transfer kernel types of the first branch groups in Φ_6

branch group	type	$\varkappa$
$\Phi_6(321)_{a_1}$	D.11	$(423;2)$
$\Phi_6(321)_{a_2}$	D.11	$(432;2)$
$\Phi_6(321)_{b_{1,1}}$	B.7	$(111;4)$
$\Phi_6(321)_{b_{1,2}}$	B.7	$(111;4)$
$\Phi_6(321)_{b_{2,1}}$	E.12	$(123;4)$
$\Phi_6(321)_{b_{2,2}}$	E.12	$(132;4)$
$\Phi_6(31^3)_a$	d.10	$(011;2)$
$\Phi_6(31^3)_{b_1}$	e.14	$(132;0)$
$\Phi_6(31^3)_{b_2}$	e.14	$(123;0)$
$\Phi_6(2^21^2)_g$	A.20	$(444;4)$
$\Phi_6(2^21^2)_{h_1}$	c.27	$(044;0)$
$\Phi_6(2^21^2)_{h_2}$	b.31	$(044;4)$
$\Phi_6(21^4)_d$	b.15	$(000;4)$

4. Transfer kernel types $\varkappa$ of the groups in Φ_{23}

4.1. **Stem groups.** James [10, p.633] has also given presentations for the *stem groups* of isoclinism family Φ_{23}, which are of order $|G| = p^6$, nilpotency class $\mathrm{cl}(G) = 4$, and coclass $\mathrm{cc}(G) = 2$. These groups are immediate descendants of three stem groups of isoclinism family Φ_6. B, D, G, J belong to the tree with root B_0, O, P, Q, R are contained in the coclass tree with root Q_0, and S, T, U, V are vertices of the coclass tree with root U_0. For $p = 3$, the presentations by Nebelung [17] and James [10] for these 12 isomorphism classes of 3-groups among the stem groups of isoclinism family Φ_{23} are now used to calculate the kernels of the transfers V_i from these metabelian 3-groups G to their maximal normal subgroups M_i. In table 5, the 12 different resulting transfer kernel types $\varkappa$, in the notation of [13, sec.3.3], reveal the correspondence between these 3-groups in the different presentations by James [10], Ascione [2], and Nebelung [17].

Table 5. Transfer kernel types of stem groups in Φ_{23} for $p = 3$

type	$\varkappa$	Ascione / Nebelung	$\varkappa$	James
E.9	(2231)	$V = G_0^{(5,6)}(0,0,1,1)$	(3134)	$\Phi_{23}(2211)_{b_1}$
d.19	(4043)	$J = G_0^{(5,6)}(1,0,1,0)$	(3043)	$\Phi_{23}(2211)_{b_2}$
G.16	(4231)	$S = G_0^{(5,6)}(1,0,0,1)$	(2134)	$\Phi_{23}(2211)_{c_{1,0}}$
d.25	(2043)	$D = G_0^{(5,6)}(0,0,1,0)$	(2043)	$\Phi_{23}(2211)_{c_{2,0}}$
E.6	(1313)	$P = G_0^{(5,6)}(1,-1,1,1)$	(1122)	$\Phi_{23}(21^4)_a$
c.21	(0231)	$U = G_0^{(5,6)}(0,0,0,1)$	(0134)	$\Phi_{23}(21^4)_{b_{1,0}}$
b.10	(0043)	$B = G_0^{(5,6)}(0,0,0,0)$	(0043)	$\Phi_{23}(21^4)_{b_{2,0}}$
E.8	(1231)	$T = G_0^{(5,6)}(1,0,-1,1)$	(1134)	$\Phi_{23}(21^4)_{b_{0,1}}$
d.23	(1043)	$G = G_0^{(5,6)}(1,0,0,0)$	(1043)	$\Phi_{23}(21^4)_{b_{1,1}}$
H.4	(3313)	$O = G_0^{(5,6)}(1,-1,-1,1)$	(2122)	$\Phi_{23}(21^4)_{c_0}$
E.14	(2313)	$R = G_0^{(5,6)}(0,-1,1,1)$	(3122)	$\Phi_{23}(21^4)_e$
c.18	(0313)	$Q = G_0^{(5,6)}(0,-1,0,1)$	(0122)	$\Phi_{23}(1^6)$

5. Transfer target types τ of the groups in Φ_6

5.1. **Stem groups.** In table 6 the structure of the commutator subgroup G' and of the abelianisations M_i/M_i' of the four maximal normal subgroups M_i, $1 \le i \le 4$, which are uniformly of order 3^3 [14, Thm.3.3], is given for each stem group G in Φ_6 for $p = 3$ [15, Thm.2.4]. This information is called the *transfer target type* τ of G. The isomorphism invariant ε of G is the number of abelianisations M_i/M_i' of type $(3,3,3)$. Each of the 7 isomorphism classes, and thus each of the 7 transfer kernel types $\varkappa$, is characterized uniquely by ε together with the isomorphism invariant ν of G which counts the total transfers $\varkappa(i) = 0$.

Table 6. Bi- and tricyclic abelianisations for $|G| = 3^5$, $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 2$

stem group	type	$\varkappa$	ν	G'	M_1/M_1'	M_2/M_2'	M_3/M_3'	M_4/M_4'	ε
$\Phi_6(221)_{d_0}$	G.19	(2143)	0	$(3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
$\Phi_6(221)_{c_2}$	D.10	(2241)	0	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	1
$\Phi_6(221)_{a}$	D.5	(4224)	0	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	2
$\Phi_6(221)_{b_1}$	H.4	(4443)	0	$(3,3,3)$	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	3
$\Phi_6(221)_{c_1}$	c.21	(0231)	1	$(3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
$\Phi_6(221)_{d_1}$	c.18	(0313)	1	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	1
$\Phi_6(1^5)$	b.10	(0043)	2	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2

5.2. **First branch groups.** In table 7 the structure of the commutator subgroup G' and of the abelianisations M_i/M_i' of the four maximal normal subgroups M_i, $1 \le i \le 4$, which are uniformly of order $|M_i/M_i'| = |M_i|/|M_i'| = 3^5/3 = 3^4$, is given for each first branch group G in Φ_6 for $p = 3$. Not all of the 13 isomorphism classes, but all 9 punctured transfer kernel types $\varkappa$ in the notation of [16] are characterized uniquely by the isomorphism invariant ε counting the abelianisations M_i/M_i' of 3-rank at least 3 together with the isomorphism invariant ν of G which counts the total transfers $\varkappa(i) = 0$.

Table 7. Abelianisations of 3-ranks $2, 3, 4$ for $|G| = 3^6$, $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 3$

branch group	type	$\varkappa$	ν	G'	M_1/M_1'	M_2/M_2'	M_3/M_3'	M_4/M_4'	ε
$\Phi_6(321)_{b_{1,1}}$	B.7	$(111;4)$	0	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1
$\Phi_6(321)_{b_{1,2}}$	B.7	$(111;4)$	0	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1
$\Phi_6(321)_{b_{2,1}}$	E.12	$(123;4)$	0	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,9,3)$	1
$\Phi_6(321)_{b_{2,2}}$	E.12	$(132;4)$	0	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,9,3)$	1
$\Phi_6(321)_{a_1}$	D.11	$(423;2)$	0	$(3,3,3)$	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2
$\Phi_6(321)_{a_2}$	D.11	$(432;2)$	0	$(3,3,3)$	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2
$\Phi_6(2^21^2)_g$	A.20	$(444;4)$	0	$(3,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4
$\Phi_6(31^3)_{b_1}$	e.14	$(132;0)$	1	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,9,3)$	1
$\Phi_6(31^3)_{b_2}$	e.14	$(123;0)$	1	$(3,3,3)$	$(27,3)$	$(27,3)$	$(27,3)$	$(3,9,3)$	1
$\Phi_6(31^3)_a$	d.10	$(011;2)$	1	$(3,3,3)$	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2
$\Phi_6(2^21^2)_{h_2}$	b.31	$(044;4)$	1	$(3,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,9,3)$	4
$\Phi_6(2^21^2)_{h_1}$	c.27	$(044;0)$	2	$(3,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,9,3)$	4
$\Phi_6(21^4)_d$	b.15	$(000;4)$	3	$(3,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4

Remark 5.1. Although this paper is exclusively concerned with group theory, we would like to emphasize that its statements about transfer kernels and targets are of considerable importance for class field theory and algebraic number theory. Whereas it is rather time consuming to determine the transfer kernels of the second p-class groups [14] of a series of number fields, we have developed an algorithm for determining the transfer kernel types via the structure of the transfer targets in [15], at least for $p = 3$ and quadratic number fields.

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