DESCRIPTION OF THE OUJDA RESEARCH PROJECT 2011

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Abstract. The present research project is a joint enterprise of professors, doctors, and dissertants of the Faculté des Sciences d'Oujda (FSO) at the Université Mohammed Premier in Oujda, Morocco, and scientific collaborators in Fès, Taza, and Graz, Austria. It is devoted to the investigation of p-class field towers over algebraic number fields, the structure of their Galois groups, and the capitulation of ideal classes of order a power of p. The project is based on algebraic number theory, Kummer theory, Iwasawa theory, class field theory, and genus field theory and aims to close up to the research frontier of the theory of pro-p-groups and the coclass theory of finite p-groups, which are of central interest in current theoretical and computational research.

1. Introduction

1.1. **Coclass theory of finite and profinite p-groups.** The idea of using the coclass $\mathrm{cc}(G)$ of a finite p-group G as the primary invariant for classification is due to Mike F. Newman and Charles R. Leedham-Green [18]. Together with Judith Ascione [2, 3], George Havas [1], Susan McKay [17], Eamonn O'Brien [22], Bettina Eick, et al. [10, 9, 12] they are investigating periodicity phenomena of the coclass graphs $\mathcal{G}(p,r)$ of all finite p-groups of coclass r, which lead to parameterised presentations of infinite chains of p-groups. Calculating the transfers of a metabelian p-group G to its maximal subgroups with the aid of its presentation immediately yields the solution of the capitulation problem for the second p-class group $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ [50, 51] of an algebraic number field K with p-class group isomorphic to the abelianisation G/G'.

1.2. **Second p-class groups of algebraic number fields.** A central aim of the present research project is to determine which metabelian (resp. non-metabelian) p-groups can occur as the second p-class group $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ (resp. as the higher p-class groups $G = \mathrm{Gal}(\mathrm{F}_p^n(K)|K)$ with $n \geq 3$) of certain base fields K. Due to the connection between the transfers of the metabelian p-group $G = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ to its maximal subgroups and the capitulation of ideal classes of order p of the base field K in unramified cyclic extensions of K of degree p, this problem can be solved essentially by determining the capitulation type of K and comparing with the transfer types of metabelian p-groups G having an abelianisation G/G' isomorphic to the p-class group of K.

1.3. **Research of project participants.** In their previous research, the participants of the project have gained considerable experience in the capitulation problem of various base fields, such as quadratic fields (Raymond Couture and Aïssa Derhem [67, 42, 60, 50] for $p = 2$ and Mohamed Talbi [57] and Daniel C. Mayer [51, 52, 56] for $p = 3, 5$), cyclic cubic fields (Aïssa Derhem [43] for $p = 2$ and Mohammed Ayadi [25] for $p = 3$), cyclic or bicyclic biquadratic fields (Abdelmalek Azizi, Ali Mouhib, Mohammed Talbi, Mohammed Taous, and Abdelkader Zekhnini [27, 34, 38, 41] for $p = 2$), and non-abelian sextic fields (Moulay Chrif Ismaïli [45] and Rachid El Masaoudi [48] for $p = 3$). Other activities concern Voronoi's algorithms for computing lattice minima and differential principal factors in cubic orders (Ouafae Lahlou and Mohamed El Hassani Charkani [49] and Daniel C. Mayer [53, 54]) and p-integral bases (L Houssain El Fadil [44]).

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1.4. **Summary of project targets.** The Oujda Research Project 2011 will start with class field theoretical applications of p-groups G on the well-known coclass graphs $\mathcal{G}(p,1)$ for $p = 2,3,5$, having abelianisations G/G' of type (p,p). However, it will then continue by a break through to the coclass graphs $\mathcal{G}(p,2)$ for $p = 2,3,5$, which need a great deal of investigation for $p = 3,5$, and to $\mathcal{G}(p,r)$ for $p = 2,3$ and $r \geq 3$. The types of abelianisations addressed by the continuation are mainly (p,p), (p^2,p), and (p,p,p).

In the following sections we summarize known results and open problems concerning the coclass graphs $\mathcal{G}(2,1)$ in section 3, $\mathcal{G}(3,1)$ in section 4, $\mathcal{G}(p,1)$ in section 5, and $\mathcal{G}(p,2)$ for $p = 2,3,5$ in section 6. The group theoretical aspects of $\mathcal{G}(2,2)$ are essentially known [22, 12]. However, the number theoretical applications of $\mathcal{G}(2,2)$ to the above mentioned base fields with 2-class groups of type $(4,2)$ still need a lot of investigation. The same is true for $\mathcal{G}(p,2)$ with $p = 3,5$, where even the group theory is still a current object of research, and the application to interesting cases, like G/G' of type $(3,3)$, $(9,3)$, and $(5,5)$, will be focussed in the present research project.

2. Participants of the Oujda Research Project 2011

1. Professors	1.1.	Abdelmalek AZIZI
	1.2.	Mohammed AYADI
	1.3.	Mohamed El Hassani CHARKANI (Fès)
	1.4.	Mustapha CHELLALI
	1.5.	Moulay Chrif ISMAILI
	1.6.	Ali MOUHIB (Taza)
2. Doctors	2.1.	Aïssa DERHEM (Président, Al Dar Al Bayda)
	2.2.	L Houssain EL FADIL (Ouarzazate)
	2.3.	Rachid EL MESAOUDI (Melilla)
	2.4.	Ouafae LAHLOU (Abha, Saudi Arabia)
	2.5.	Mohamed TALBI
	2.6.	Mohammed TALBI
	2.7.	Mohammed TAOUS
3. Dissertants	3.1.	Abdelkader ZEKHNINI
	3.2.	Khalid MRABET
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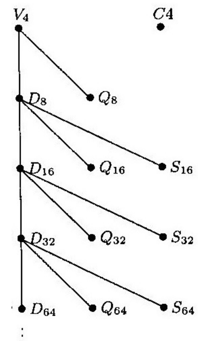
3. Second 2-class groups on coclass 1 graph $\mathcal{G}(2,1)$

To give a first impression of the impact exerted by coclass theory on algebraic number theory, we begin by a survey of all metabelian 2-groups G which are known to occur as the second 2-class group $G \simeq \mathrm{Gal}(\mathrm{F}_2^2(K)|K)$ of a complex quadratic number field $K = \mathbb{Q}(\sqrt{d_K})$ with 2-class group $\mathrm{Cl}_2(K)$ of type $(2,2)$, identified by its discriminant d_K [67, 50, 51]. These groups are necessarily of maximal class [6] and thus are vertices of the coclass graph $\mathcal{G}(2,1)$ of all 2-groups G of coclass $\mathrm{cc}(G) = 1$ which contains exactly one coclass tree $\mathcal{T}_{V_4}$ starting with Klein's four group V_4 [9, 12].



In table 1 the groups are arranged according to the branches $\mathcal{B}_i$ $(i \geq 0)$ of this coclass tree with periodicity $\mathcal{B}_{i+1} \simeq \mathcal{B}_i$ of length 1 starting with $i = 1$. An important recent observation is that all groups within one of the three *coclass families* (dihedral, quaternion, semidihedral) share the same transfer type $\varkappa$ [50]. This feature fits perfectly into the periodicity phenomena of coclass theory. The repeated appearances of a transfer type on higher branches are denoted by arrows and are called *excited states* of that type. Numerical examples for the ground states are due to H. Kisilevsky [67]. The excited states, however, have been discovered by D. C. Mayer [50] with the aid of [51, Th.8.1]. ℓ denotes the number of stages of the 2-class field tower of K, which cannot exceed 2 by [7, Th.4]. We don't claim the given discriminants are minimal in their absolute value.

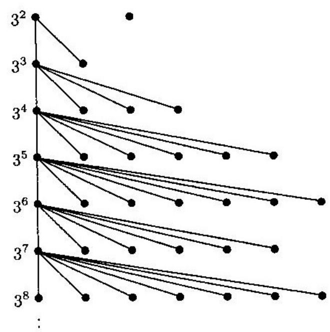
Table 1. Realisation of metabelian 2-groups of coclass 1

branch	depth	2-group	type	$\varkappa$	ℓ	D
$\mathcal{B}_0$	0	V_4	a.1	(000)	1	$-84 = -4 \cdot 3 \cdot 7$
	1	Q_8	Q.5	(123)	2	$-120 = -8 \cdot 3 \cdot 5$
$\mathcal{B}_1$	0	D_8	d.8	(210)	2	$-408 = -8 \cdot 3 \cdot 17$
	1	Q_{16}	Q.6	(132)	2	$-312 = -8 \cdot 3 \cdot 13$
	1	S_{16}	S.4	(211)	2	$-340 = -4 \cdot 5 \cdot 17$
$\mathcal{B}_2$	0	D_{16}	d.8 $\uparrow$	(210)	2	$-6168 = -8 \cdot 3 \cdot 257$
	1	Q_{32}	Q.6 $\uparrow$	(132)	2	$-888 = -8 \cdot 3 \cdot 37$
	1	S_{32}	S.4 $\uparrow$	(211)	2	$-2260 = -4 \cdot 5 \cdot 113$
$\mathcal{B}_3$	0	D_{32}	d.8 $\uparrow\uparrow$	(210)	2	$-29208 = -8 \cdot 3 \cdot 1217$
	1	Q_{64}	Q.6 $\uparrow\uparrow$	(132)	2	$-3768 = -8 \cdot 3 \cdot 157$
	1	S_{64}	S.4 $\uparrow\uparrow$	(211)	2	$-5140 = -4 \cdot 5 \cdot 257$

The numerical results of D. C. Mayer [51] suggest the conjecture that all groups G of $\mathcal{G}(2,1)$, with the exception of the sporadic isolated group C_4, occur as $G \simeq \mathrm{Gal}(\mathrm{F}_2^2(K)|K)$, even for the very special set of complex quadratic fields as base fields.

4. Second 3-class groups on coclass 1 graph $\mathcal{G}(3,1)$

We continue with all metabelian 3-groups G of maximal class [6] which can be realised as the second 3-class group $G \simeq \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ either of a cyclic cubic field K with conductor $f = pq$ or $f = 9q$ and discriminant $d_K = f^2$ divisible by two primes with certain cubic residuacity conditions by Ayadi [25] or of a real quadratic field $K = \mathbb{Q}(\sqrt{d_K})$ with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ [51, 52]. Similarly as $\mathcal{G}(2,1)$, the coclass graph $\mathcal{G}(3,1)$ of these 3-groups G of coclass $\mathrm{cc}(G) = 1$, which are necessarily metabelian by [7, Th.6], contains exactly one coclass tree $\mathcal{T}_{C_3\times C_3}$ starting with the elementary abelian bicyclic 3-group $C_3 \times C_3$ [20].



In table 2 the groups are arranged according to the branches $\mathcal{B}_i$ $(i \geq 0)$ of this coclass tree with periodicity $\mathcal{B}_{i+2} \simeq \mathcal{B}_i$ of length 2 starting with $i = 2$. The stem groups of the isoclinism families Φ_s for $s \in \{2, 3, 35, 36\}$ in the sense of P. Hall [15] are taken from the listing of R. James [16]

Table 2. Realisation of metabelian 3-groups of coclass 1

branch	depth	3-group	type	$\varkappa$	ℓ	D	#	#
$\mathcal{B}_0$	0	$C_3 \times C_3 = \Phi_1(11)$	a.1	(0000)	1	657^2		
	1	$G_0^{(3)}(0,1) = \Phi_2(21)$	A.1	(1111)	2	$2\,439^2$		
$\mathcal{B}_1$	1	$G_0^{(4)}(0,1) = \Phi_3(211)_a$	a.2	(1000)	2	72 329	34	1 386
	1	$G_0^{(4)}(-1,0) = \Phi_3(211)_{b_2}$	a.3	(2000)	2	32 009	52	
	1	$G_0^{(4)}(1,0) = \Phi_3(211)_{b_1} = \mathrm{Syl}_3\mathrm{A}_9$	a.3*	(2000)	2	142 097	34	697
$\mathcal{B}_3$	1	$G_0^{(6)}(0,1) = \Phi_{35}(21111)_a$	a.2 $\uparrow$	(1000)	2	790 085	1	72
	1	$G_0^{(6)}(1,0) = \Phi_{35}(21111)_{b_0}$	a.3 $\uparrow$	(2000)	2	494 236	1	
	1	$G_0^{(6)}(-1,0) = \Phi_{35}(21111)_{b_1}$						
	1	$G_1^{(6)}(0,0) = \Phi_{36}(1^6)$	a.1	(0000)	2	62 501	11	147
	1	$G_1^{(6)}(0,1) = \Phi_{36}(21111)_{a_0}$						
	1	$G_1^{(6)}(0,-1) = \Phi_{36}(21111)_{a_1}$						
$\mathcal{B}_5$	1	$G_1^{(8)}(0,0)$	a.1 $\uparrow$	(0000)	2	2 905 160	0	1
	1	$G_1^{(8)}(0,1)$						
	1	$G_1^{(8)}(0,-1)$						

According to Ayadi [25], only the two sporadic 3-groups of branch $\mathcal{B}_0$ occur for cyclic cubic fields with conductor divisible by two primes. For real quadratic fields, the first count # concerns the range $0 < d_K < 10^6$ of discriminants, the second count # the range $0 < d_K < 10^7$, and class number relations enforce a restriction of the occurring 3-groups to branches $\mathcal{B}_i$ with odd subscripts i [51].

Again we have several *coclass families*, seven for odd i, sharing the same transfer type $\varkappa$ [50], which emphasises the periodicity phenomena of coclass theory. Similarly as in table 1, excited states of transfer type are denoted by arrows, and ℓ denotes the number of stages of the 3-class field tower of K, which is bounded by 2 [7, Th.4].

Here, we definitely know that the given discriminants are minimal, as the computations of D. C. Mayer [50, 51] show.

The principalisation types a.3 and a.3* differ by the structure of the 3-class group $\mathrm{Cl}_3(N_1)$ of the first dihedral field $N_1|K$, which is almost homogeneous of type $(9,3)$ in the first case and elementary abelian of type $(3,3,3)$ in the last case. Type a.3* occurs only on branch $\mathcal{B}_1$ for the sporadic group $G_0^{(4)}(1,0) \simeq \mathrm{Syl}_3\mathrm{A}_9$.

It is unknown why the groups $G_0^{(m)}(0,0)$ with odd parameters $m \geq 3$ on the main line of the coclass tree $\mathcal{T}_{C_3 \times C_3}$ do not occur for real quadratic base fields, though there would be no contradiction to class number relations.

5. Transfer types of metabelian p-groups on $\mathcal{G}(p,1)$

In table 3 we compare the multiplets $\varkappa$ of transfer types [50] of metabelian p-groups G of maximal class [6] and order $|G| = p^m$ for $p = 2$ and $p \geq 3$. They all have abelianisations G/G' of type (p,p). Here, κ denotes the multiplet of coarse transfer types expressed with the aid of condition (B) by O. Taussky [73, p.435], which is given by Kisilevsky [67, p.273, Th.2] and by E. Benjamin and C. Snyder [60, p.163, §2].

Similarly as in section 4, we denote by $G_a^{(m)}(z,w)$ the representative of an isomorphism class of metabelian p-groups G of maximal class and of order $|G| = p^m$, which satisfies the Blackburn relations for pth powers of the generators x and y of G,

$$x^p = s_{m-1}^w \quad \text{and} \quad y^p \prod_{\ell=2}^{p} s_\ell^{\binom{p}{\ell}} = s_{m-1}^z \quad \text{with exponents} \quad 0 \leq w, z \leq p-1\,, \tag{1}$$

and the main commutator relation of Miech [19, p.332, Th.2],

$$[y, s_2] = \prod_{r=1}^{k} s_{m-r}^{a(m-r)} \in [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G)\,, \tag{2}$$

with a fixed system of exponents $a = (a(m-1), \ldots, a(m-k))$ where $0 \leq a(m-r) \leq p-1$ for $1 \leq r \leq k$ and $a(m-k) > 0$.

Here, $s_2 = [y,x]$, $s_j = [s_{j-1}, x]$ for $j \geq 3$, the $\gamma_j(G)$ for $j \geq 2$ denote the members of the lower central series of G, $\chi_2(G)$ is the centraliser of the two-step factor group $\gamma_2(G)/\gamma_4(G)$, that is, the biggest subgroup of G such that $[\chi_2(G), \gamma_2(G)] \leq \gamma_4(G)$, and the isomorphism invariant k of G is defined by $[\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G)$. For $k = 0$, the symbol $a = 0$ means the empty family $(a(m-r))_{1 \leq r \leq k}$.

Table 3. Transfer types of corresponding p-groups of coclass 1 for $p = 2$ and $p \geq 3$

$p=2$				$p \geq 3$	
κ	$\varkappa$	2-group	m	$\varkappa$	p-group
(000)	(000)	$C_2 \times C_2 = V_4$	2	$(\overbrace{0,\ldots,0}^{p+1 \text{ times}})$	$C_p \times C_p$
(123)	(123)	$G_0^{(3)}(0,1) = Q_8$	3	$(\overbrace{1,\ldots,1}^{p+1 \text{ times}})$	$G_0^{(3)}(0,1)$
$(0BB)$	(032)	$G_0^{(m)}(0,0) = D_{2^m}$	≥ 3	$(\overbrace{0,\ldots,0}^{p+1 \text{ times}})$	$G_0^{(m)}(0,0)$
$(1BB)$	(132)	$G_0^{(m)}(0,1) = Q_{2^m}$	≥ 4	$(1,\overbrace{0,\ldots,0}^{p \text{ times}})$	$G_0^{(m)}(0,1)$
(BBB)	(232)	$G_0^{(m)}(1,0) = S_{2^m}$	≥ 4	$(2,\overbrace{0,\ldots,0}^{p \text{ times}})$	$G_0^{(m)}(z,0)$
no analogon			≥ 5	$(\overbrace{0,\ldots,0}^{p+1 \text{ times}})$	$G_a^{(m)}(z,w),\ a \neq 0$

6. Transfer types of the stem groups of isoclinism family Φ_6

Now we turn to the coclass graphs $\mathcal{G}(p, 2)$ with $p = 3, 5$, which are a central object of current research with interesting number theoretical applications such as abelianisations G/G' of type $(3, 3)$, $(9, 3)$, and $(5, 5)$.

Presentations for the 7 isomorphism classes of 3-groups, resp. the 12 isomorphism classes of 5-groups, among the stem groups of Hall's isoclinism family Φ_6 [15] are taken from the listing of James [16] and are used to calculate the kernels of the transfers to the maximal normal subgroups of these metabelian p-groups G of order $|G| = p^5$, nilpotency class $\mathrm{cl}(G) = 3$, and coclass $\mathrm{cc}(G) = 2$, which have first been found by G. Bagnera [4] and O. Schreier [24]. These groups belong to the top vertices of the coclass graph $\mathcal{G}(p, 2)$ and play an important role as starting groups for the p-group generation algorithm [1, 2]. For $p = 3$, the groups B_0, Q_0, U_0 give rise to three coclass trees. The transfer types $\varkappa$ of the 3-groups are due to B. Nebelung [20, 50], but the transfer types $\varkappa$ of the 5-groups have been calculated by D. C. Mayer and are given here for the first time.

Table 4. Transfer types of corresponding p-groups of coclass 2 for $p = 3$ and $p = 5$

$p = 3$			$p = 5$		
type	$\varkappa$	3-group	$\varkappa$	property	5-group
D.5	(4334)	$G_0^{(4,5)}(1, 1, -1, 1)$	(123456)	identity	$\Phi_6(221)_a$
H.4	(4443)	$G_0^{(4,5)}(1, 1, 1, 1)$	(125364)	4-cycle	$\Phi_6(221)_{b_1}$
	no analogon		(126543)	two transpos.	$\Phi_6(221)_{b_2}$
c.21	(1024)	$U_0 = G_0^{(4,5)}(0, 0, 0, 1)$	(612435)	5-cycle	$\Phi_6(221)_{c_1}$
D.10	(4124)	$G_0^{(4,5)}(0, 0, -1, 1)$	(612435)	5-cycle	$\Phi_6(221)_{c_2}$
G.19	(2143)	$G_0^{(4,5)}(0, -1, -1, 0)$	(214365)	three transpos.	$\Phi_6(221)_{d_0}$
c.18	(3023)	$Q_0 = G_0^{(4,5)}(0, -1, 0, 1)$	(512643)	6-cycle	$\Phi_6(221)_{d_1}$
	no analogon		(312564)	two 3-cycles	$\Phi_6(221)_{d_2}$
	no analogon		(022222)	alm.const.with fp.	$\Phi_6(21^3)_a$
	no analogon		(011111)	almost constant	$\Phi_6(21^3)_{b_1}$
	no analogon		(011111)	almost constant	$\Phi_6(21^3)_{b_2}$
b.10	(2100)	$B_0 = G_0^{(4,5)}(0, 0, 0, 0)$	(000000)	constant	$\Phi_6(1^5)$

7. Metabelian 3-groups G with abelianisation of type $(9,3)$

We consider metabelian 3-groups $G = \langle x, y\rangle$ with two generators satisfying $x^9 \in G'$ and $y^3 \in G'$ and commutator quotient group G/G' of type $(9,3)$. Generally, such a group possesses

- four normal subgroups of index 9,

$$\tilde{M}_1 = \langle y, G'\rangle,\ \tilde{M}_2 = \langle x^3 y, G'\rangle,\ \tilde{M}_3 = \langle x^3 y^{-1}, G'\rangle,\ \tilde{M}_4 = \langle x^3, G'\rangle,$$

- and four maximal normal subgroups of index 3,

$$M_1 = \langle x, G'\rangle,\ M_2 = \langle xy, G'\rangle,\ M_3 = \langle xy^{-1}, G'\rangle,\ M_4 = \langle x^3, y, G'\rangle.$$

We use the subscript 4 to indicate that for $M_4 = \prod_{i=1}^{4} \tilde{M}_i$ the factor group $M_4/G' = \langle x^3, y\rangle$ is bicyclic of type $(3,3)$, whereas M_i/G' is cyclic of order 9 for $1 \le i \le 3$, and that $\tilde{M}_4 = \cap_{i=1}^{4} M_i = \Phi(G) = G^3 G'$ coincides with the Frattini subgroup of G, whereas $\tilde{M}_i$ is only contained in M_4 for $1 \le i \le 3$.

The transfers V_i from G to its maximal subgroups M_i are given by

$$\mathrm{V}_i = \mathrm{V}_{G,M_i} : G/G' \to M_i/M_i',\ g \mapsto \begin{cases} g^3, & \text{if } g \in G \setminus M_i, \\ g^{\mathrm{S}_3(h)}, & \text{if } g \in M_i, \end{cases}$$

where $\mathrm{S}_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$, with an arbitrary element $h \in G \setminus M_i$, denotes the third trace element (Spur) in the group ring, acting as a symbolic exponent.

There are five possibilities for the kernel of V_i for each $1 \le i \le 4$. Either $\mathrm{Ker}(\mathrm{V}_i) = \tilde{M}_j/G'$ for some $1 \le j \le 4$ and we denote the *partial* transfer by the singulet $\varkappa(i) = j$ or $\mathrm{Ker}(\mathrm{V}_i) = M_4/G'$ and we denote the *total* transfer by $\varkappa(i) = 0$. Due to the distinguished role of the subscript 4, we combine the singulets to form a multiplet

$$\varkappa = (\varkappa(1), \varkappa(2), \varkappa(3); \varkappa(4)) \in [0,4]^3 \times [0,4]$$

which we call the *punctured transfer type* of the group G with respect to the selected generators.

To be independent from the choice of generators and the ordering of M_1, M_2, M_3 and $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$, we define the double orbit

$$\varkappa^{S_3 \times S_3} = \{\tilde{\sigma} \circ \varkappa \circ \hat{\tau} \mid \sigma, \tau \in S_3\}$$

of $\varkappa$ under the operation of $S_3 \times S_3$ as an isomorphism invariant $\varkappa(G)$ of G. Here, $\tilde{\sigma}$ denotes the extension of σ from $[1,3]$ to $[0,4]$ which fixes 0 and 4 and $\hat{\tau}$ denotes the extension of τ from $[1,3]$ to $[1,4]$ which fixes 4.

Two further isomorphism invariants of G are $\mu = \mu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 4\}$ and the number of total transfers $\nu = \nu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 0\}$.

7.1. **Combinatorially possible punctured transfer types.** In this subsection, we arrange all combinatorially possible S_3-double orbits of the 5^4 punctured quadruplets $\varkappa \in [0,4]^3 \times [0,4]$ by increasing invariant $0 \le \mu \le 4$ and cardinality of the image. Table 5 shows the partial punctured quadruplets with invariant $\nu = 0$ and table 6 the total punctured quadruplets with invariant $1 \le \nu \le 4$ as possible punctured transfer types of 3-groups G with $G/\gamma_2(G)$ of type $(9,3)$, resp. punctured principalisation types of base fields K with 3-class group $\mathrm{Cl}_3(K)$ of type $(9,3)$. The double orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\varkappa) = (|\varkappa^{-1}\{i\}|)_{0\le i\le 4}$ the family of occupation numbers of the selected double orbit representative $\varkappa$ and by κ the quadruplet of Taussky's conditions associated with $\varkappa$.

If a double orbit $\varkappa^{S_3\times S_3}$ can be realised as a transfer type $\varkappa(G)$, then a suitable 3-group G is given in the notation of James [16], using Hall's isoclinism families [15].

Table 5 gives a coarse classification into sections A to E, an identification by ordinal numbers 1 to 20, and a set theoretical characterisation.

TABLE 5. The 20 S_3-double orbits of punctured quadruplets $\varkappa \in [1,4]^4$ with $\nu = 0$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
A	1	(1111)	(04000)	$(BBBA)$	constant	3	$\Phi_2(31)$
B	2	(1112)	(03100)	$(BBBA)$	almost	6	
B	3	(1121)	(03100)	$(BBBA)$	constant	18	
C	4	(1122)	(02200)	$(BBBA)$		18	
D	5	(1123)	(02110)	$(BBBA)$		18	
D	6	(1231)	(02110)	$(BBBA)$		18	
B	7	(1114)	(03001)	$(BBBA)$	almost	3	
B	8	(1141)	(03001)	$(BBAA)$	constant	9	
D	9	(1124)	(02101)	$(BBBA)$		18	
D	10	(1142)	(02101)	$(BBAA)$		18	
D	11	(1241)	(02101)	$(BBAA)$		36	
E	12	(1234)	(01111)	$(BBBA)$	per-	6	
E	13	(1243)	(01111)	$(BBAA)$	mutation	18	
C	14	(1144)	(02002)	$(BBAA)$		9	
C	15	(1441)	(02002)	$(BAAA)$		9	
D	16	(1244)	(01102)	$(BBAA)$		18	
D	17	(1442)	(01102)	$(BAAA)$		18	
B	18	(1444)	(01003)	$(BAAA)$	almost	9	
B	19	(4441)	(01003)	$(AAAA)$	constant	3	
A	20	(4444)	(00004)	$(AAAA)$	constant	1	$\Phi_2(2^2), \Phi_8(32)$
					Total number:	256	

Table 6 gives a coarse classification into sections a to e, an identification by ordinal numbers 1 to 32, and a set theoretical characterisation.

Table 6. The 32 S_3-double orbits of punctured quadruplets $\varkappa \in [0,4]^4 \setminus [1,4]^4$ with $1 \le \nu \le 4$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
a	1	(0000)	(40000)	$(AAAA)$	constant	1	$\Phi_2(21^2)_c, \Phi_3(31^2)_a$
b	2	(0001)	(31000)	$(AAAA)$	almost	3	
b	3	(0010)	(31000)	$(AABA)$	constant	9	
c	4	(0011)	(22000)	$(AABA)$		9	
c	5	(0110)	(22000)	$(ABBA)$		9	
d	6	(0012)	(21100)	$(AABA)$		18	
d	7	(0120)	(21100)	$(ABBA)$		18	
b	8	(0111)	(13000)	$(ABBA)$	almost	9	
b	9	(1110)	(13000)	$(BBBA)$	constant	3	
d	10	(0112)	(12100)	$(ABBA)$		18	
d	11	(0121)	(12100)	$(ABBA)$		36	
d	12	(1120)	(12100)	$(BBBA)$		18	
e	13	(0123)	(11110)	$(ABBA)$	per-	18	
e	14	(1230)	(11110)	$(BBBA)$	mutation	6	
b	15	(0004)	(30001)	$(AAAA)$	almost	1	$\Phi_3(2^21)_{b_1}, \Phi_3(2^21)_{b_2}$
b	16	(0040)	(30001)	$(AAAA)$	constant	3	$\Phi_3(2^21)_a$
d	17	(0014)	(21001)	$(AABA)$		9	
d	18	(0041)	(21001)	$(AAAA)$		9	
d	19	(0140)	(21001)	$(ABAA)$		18	
d	20	(0114)	(12001)	$(ABBA)$		9	
d	21	(0141)	(12001)	$(ABAA)$		18	
d	22	(1140)	(12001)	$(BBAA)$		9	
e	23	(0124)	(11101)	$(ABBA)$	per-	18	
e	24	(0142)	(11101)	$(ABAA)$	muta-	36	
e	25	(1240)	(11101)	$(BBAA)$	tion	18	
c	26	(0044)	(20002)	$(AAAA)$		3	
c	27	(0440)	(20002)	$(AAAA)$		3	
d	28	(0144)	(11002)	$(ABAA)$		18	
d	29	(0441)	(11002)	$(AAAA)$		9	
d	30	(1440)	(11002)	$(BAAA)$		9	
b	31	(0444)	(10003)	$(AAAA)$	almost	3	
b	32	(4440)	(10003)	$(AAAA)$	constant	1	
					Total number:	$625-256=369$	

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