Der Wissenschaftsfonds.

3-Class Field Towers of Exact Length 3

Conference:	18 th ÖMG Congress 2013 and
	123 rd Annual DMV Meeting 2013
Place:	University of Innsbruck,
	Technology Campus, 25 b, G
Venue:	Innsbruck, Tyrol, Austria
Date:	Tuesday, September 24, 2013
Time:	17:00 – 17:30, p. m.
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§ 0. Summary of Aims

(1) To provide evidence for an extensive set of complex quadratic fields having a 3-**class field tower** of **exactly three stages**.

(2) To characterize these complex quadratic fields by their **capitulation types** of 3-classes.

(3) To disprove incorrect assertions of **Scholz/Taussky** and **Heider/Schmithals** concerning some pretended two-stage towers which actually turned out to be three-stage towers.

(4) On the one hand, to underpin the caveats of **Brink/Gold**, who had doubts about Scholz and Taussky's claim, but on the other hand, to show that the arguments given by Brink/Gold are unable to invalidate the Scholz/Taussky claim.

(5) To establish items (1) – (4) by proving a single group theoretic theorem which indicates a **new kind of periodicity** of sporadic isolated Schur σ-groups and of subtrees with fixed coclass in descendant trees of finite 3-groups.

Acknowledgement

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§ 1. Status of Previous Research

§ 1.1. Higher p-Class Fields and p-Tower

We use the following notation:
K an algebraic number field,
$p \geq 2$ an arbitrary prime,
$\mathrm{Cl}_p(K)$ the p-class group of K,
$\mathrm{r}_p(K)$ the p-class rank of K.

$m \geq 1$ an integer,
$\mathrm{F}_p^m(K)$ the mth *Hilbert p-class field* of K,
that is, the maximal unramified p-extension of K
with Galois group of derived length at most m.

The Galois group $\mathrm{G}_p^m(K) = \mathrm{Gal}(\mathrm{F}_p^m(K)|K)$
is called the mth *p-class group* of K [8].

$\mathrm{F}_p^\infty(K) = \cup_{m \geq 0} \mathrm{F}_p^m(K)$
the maximal unramified pro-p extension of K,
$\mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$ the *p-tower group* of K,
i. e., the projective limit $\varprojlim_{m \geq 0} \mathrm{G}_p^m(K)$.
$\ell = \ell_p(K)$ *length* of the p-class field tower of K.
$\ell \in \mathbb{N}$ is finite when the p-tower terminates:
$K < \mathrm{F}_p^1(K) < \mathrm{F}_p^2(K) < \ldots < \mathrm{F}_p^{\ell-1}(K) < \mathrm{F}_p^\ell(K) = \mathrm{F}_p^\infty(K)$.

[8] D. C. Mayer, The distribution of second p-class groups on coclass graphs, *J. Théor. Nombres Bordeaux* **25** (2013), no. 2, 401–456.

§ 1.2. Kernels of Artin Transfers

Definition.
G a p-group of generator rank $d(G) = 2$,
$H_1, \ldots, H_{p+1} \lhd G$ its maximal subgroups,
$T_i = T_{G,H_i} : G/G' \longrightarrow H_i/H_i'$, for $1 \le i \le p+1$,
the *Artin transfers* from G to the H_i [1].

The family $\varkappa(G) = (\ker(T_i))_{1 \le i \le p+1}$
is called the *transfer kernel type* (TKT) of G.

[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

Inherited TKTs of Tree Descendants:

Proposition 1.1. (*TKT Singulets*).
Suppose $\tilde{G}$ and G are p-groups, $d(\tilde{G}) = d(G) = 2$, $\tilde{G} = \varphi(G)$ under an epimorphism $\varphi : G \longrightarrow \tilde{G}$, and $\tilde{H} = \varphi(H)$ for a maximal subgroup $H \lhd G$.

(1) If $\ker(\varphi) \le H$ then
$\tilde{H}$ is maximal subgroup of $\tilde{G}$ with $\tilde{H}' = \varphi(H')$,
$\varphi(T_{G,H}(g)) = T_{\tilde{G},\tilde{H}}(\varphi(g))$ for all $g \in G$,
and $\varphi(\ker(T_{G,H})) \le \ker(T_{\tilde{G},\tilde{H}})$.

(2) If $\ker(\varphi) \le H'$ then $\varphi(\ker(T_{G,H})) = \ker(T_{\tilde{G},\tilde{H}})$.

$$\begin{array}{ccccc} & & T_{\tilde{G},\tilde{H}} & & \\ & \tilde{G}/\tilde{G}' & \longrightarrow & \tilde{H}/\tilde{H}' & \\ \varphi & \uparrow & \backslash\backslash\backslash & \uparrow & \varphi \\ & G/G' & \longrightarrow & H/H' & \\ & & T_{G,H} & & \end{array}$$

Corollary 1.1. (*TKT Multiplets*).

(1) If $\ker(\varphi) \le \cap_{i=1}^{p+1} H_i$ then $\varkappa(G) \le \varkappa(\tilde{G})$, i. e., $\varphi(\ker(T_{G,H_i})) \le \ker(T_{\tilde{G},\tilde{H}_i})$ for $1 \le i \le p+1$.

(2) If $\ker(\varphi) \le \cap_{i=1}^{p+1} H_i'$ then $\varkappa(G) = \varkappa(\tilde{G})$, i. e. $\varphi(\ker(T_{G,H_i})) = \ker(T_{\tilde{G},\tilde{H}_i})$ for $1 \le i \le p+1$.

§ 1.3. Capitulation of p-Classes

Definition.
K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa(K) = (\ker(j_i))_{1 \le i \le p+1}$
is called the *p-capitulation type* of K.

Theorem 1.2. (Artin, 1929 [1])
The p-capitulation type $\varkappa(K)$ coincides with the TKT $\varkappa(G)$ of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $n \ge 2$.

$$\begin{array}{ccccc} & & j_{L_i|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L_i) & \\ \text{Artin} & \downarrow & & \downarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H_i/H_i' & \text{isomorphism} \\ & & T_{G,H_i} & & \end{array}$$

[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

§ 1.4. Galois Action on Higher p-Class Groups

Definition. $p \geq 3$ an odd prime.
A pro-p group G is called a *σ-group*,
if it admits an automorphism $\sigma \in \mathrm{Aut}(G)$ acting as inversion $x \mapsto x^{-1}$ on the abelianization G/G'.

Theorem 1.3. (Artin, 1928 [4])
For any *quadratic* field $K = \mathbb{Q}(\sqrt{D})$,
the p-tower group $\mathrm{G}_p^\infty(K)$ and the higher p-class groups $\mathrm{G}_p^n(K)$, for $n \geq 2$, are σ-groups.

[4] G. Frei, P. Roquette, and F. Lemmermeyer, *Emil Artin and Helmut Hasse. Their Correspondence 1923–1934*, Universitätsverlag Göttingen, 2008.

§ 1.5. Cohomology of p-Tower Groups

$p \geq 3$ an odd prime,
G a pro-p group,
$d(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(G, \mathbb{F}_p))$ the *generator rank* of G,
$r(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^2(G, \mathbb{F}_p))$ the *relation rank* of G.

Definition. A pro-p group G which satisfies the equation $r(G) = d(G)$ is said to have a *balanced presentation*, or to be a *Schur group*.

Theorem 1.4. (Shafarevich, 1963 [10])
The p-tower group $\mathrm{G}_p^\infty(K)$ of a *complex quadratic* field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, is a Schur group.

Remarks.

- In view of Theorem 1.3, $\mathrm{G}_p^\infty(K)$ is even a Schur σ-group.
- For any $m \in \mathbb{N} \cup \{\infty\}$, the generator rank $d(\mathrm{G}_p^m(K))$ equals the p-class rank $\mathrm{r}_p(K)$.

[10] I. R. Shafarevich, Extensions with prescribed ramification points, *Publ. Math., Inst. Hautes Études Sci.* **18** (1963), 71–95 (Russian). English transl. by J. W. S. Cassels: *Am. Math. Soc. Transl.*, II. Ser., **59** (1966), 128–149.

§ 2. Breaking Through beyond Current Limits

§ 2.1. Cover and Balanced Cover

To get a complete overview of all possible higher p-class groups $H = \mathrm{G}_p^n(K)$ of derived length $n \geq 3$ for a given metabelian 2nd p-class group $G = \mathrm{G}_p^2(K)$, we define two fundamental new concepts.

Definition.
The *cover* $\mathrm{cov}(G)$ of a finite metabelian p-group G is defined as the set of all (isomorphism classes of) finite non-metabelian p-groups H whose metabelianization, that is the second derived quotient H/H'', is isomorphic to G.

The subset of the cover $\mathrm{cov}(G)$ consisting of Schur groups is called the *balanced cover* $\mathrm{cov}_*(G)$ of G.

In the sequel, let $p = 3$. We are going to focus on 3-groups G with abelianization G/G' of type $(3,3)$.

§ 2.2. Strategy for Finding Three-Stage Towers

- According to the Shafarevich Theorem 1.5, sharp statements concerning the length $\ell_3(K)$ of 3-towers are expected for **complex quadratic base fields** K.
- To discourage two-stage towers, $\ell_3(K) = 2$, we have to find a set of complex quadratic fields K whose second 3-class groups $G = \mathrm{G}_3^2(K)$ are **unbalanced** metabelian 3-groups.
- To make sure that the tower has length exactly
$$\ell_3(K) = 3,$$
we must show that the balanced cover $\mathrm{cov}_*(G)$ of any G, mentioned before, contains at least one Schur σ-group and that **none** of the groups $H \in \mathrm{cov}(G)$ has **derived length** $\mathrm{dl}(H) \geq 4$.

§ 2.3. Sieving 3-Groups of Given TKT

G a finite 3-group,
$c = \mathrm{cl}(G)$ the nilpotency class of G,
$M \simeq G/\gamma_c(G)$ the parent of G, where
$\gamma_c(G)$ the last non-trivial lower central of G.

In view of Proposition 1.1 and Corollary 1.1, we have:

Theorem 2.1.
If the TKT of M is c.21, $\varkappa(M) = (2034)$, then the immediate descendant G of M has one of the following TKTs $\varkappa(G) \le \varkappa(M)$:
c.21, $\varkappa(G) = (2034)$, or
G.16, $\varkappa(G) = (2134)$, or
E.8, $\varkappa(G) = (2234)$, or
E.9, $\varkappa(G) = (2334) \sim (2434)$ [7].
(See Figure 3.)

[7] D. C. Mayer, Transfers of metabelian p-groups, *Monatsh. Math.* **166** (2012), no. 3–4, 467–495.

Remark.

The 3-group M of minimal order having TKT c.21 is $\langle 243, 8\rangle$ in the SmallGroups library.
(See Figure 2.)

Definition.

The *TKT-pruned descendant tree* $\mathcal{T}^*(\langle 243, 8\rangle)$ consists of all descendants G of the root $\langle 243, 8\rangle$ such that

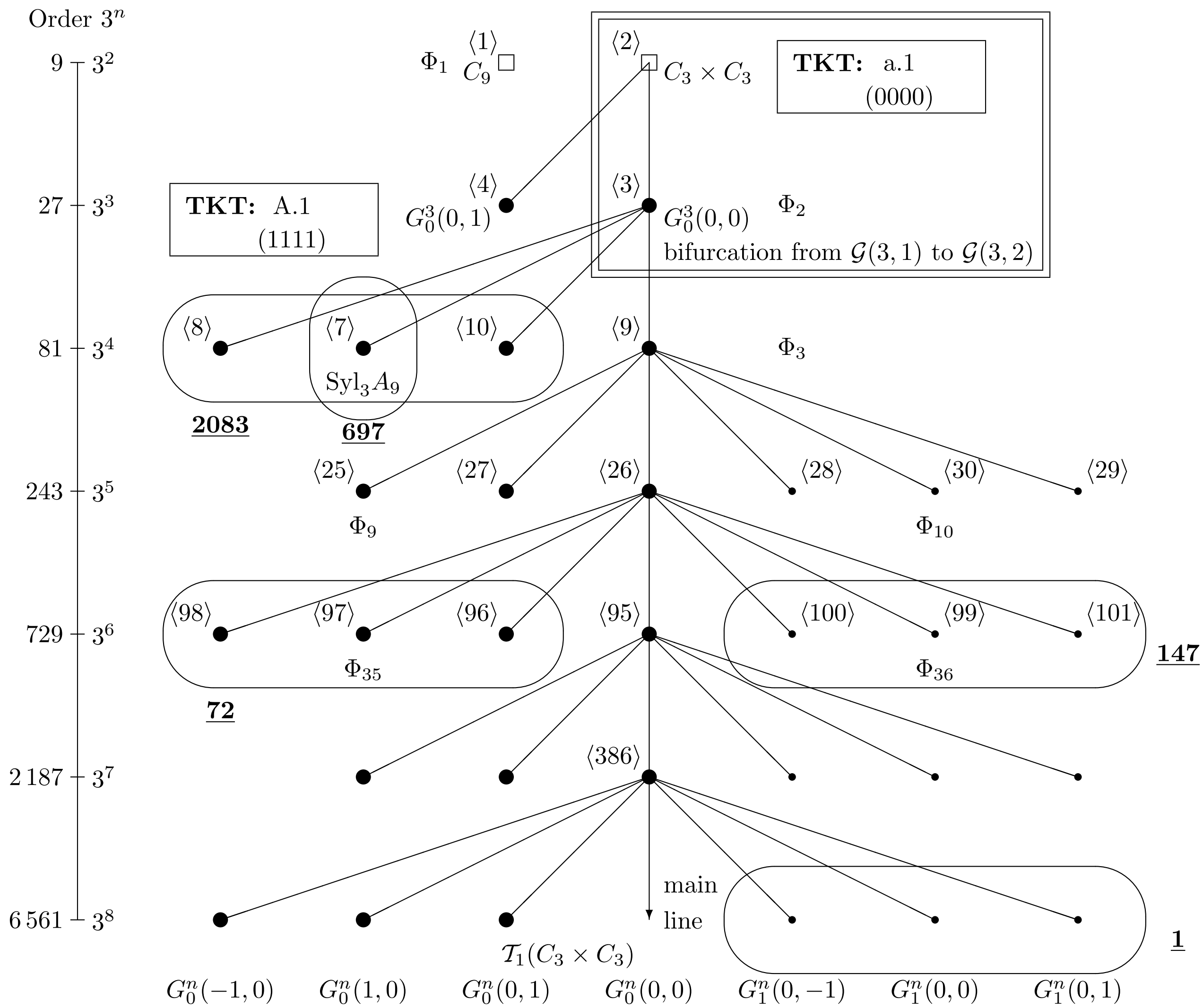
(1) $\varkappa(G)$ is one of the TKTs c.21 or E.8 or E.9 (that is, we cancel all the trash with TKT G.16),
(2) if $\varkappa(G)$ is of TKT c.21 then G has descendants, (i.e., we omit terminal vertices with TKT c.21),
(3) G is a σ-group.

(See Figure 3.)

Remark.

The motivation for defining $\mathcal{T}^*(\langle 243, 8\rangle)$ is that Brink and Gold indicated a possible length $\ell_3(K) \geq 3$ for the field $K = \mathbb{Q}(\sqrt{-9748})$ with TKT E.9 for which Scholz and Taussky had claimed $\ell_3(K) = 2$.
(See [2], [3], and page 41 in [9].)

FIGURE 1. Starting 3-group generation at the top of coclass graph $\mathcal{G}(3,1)$

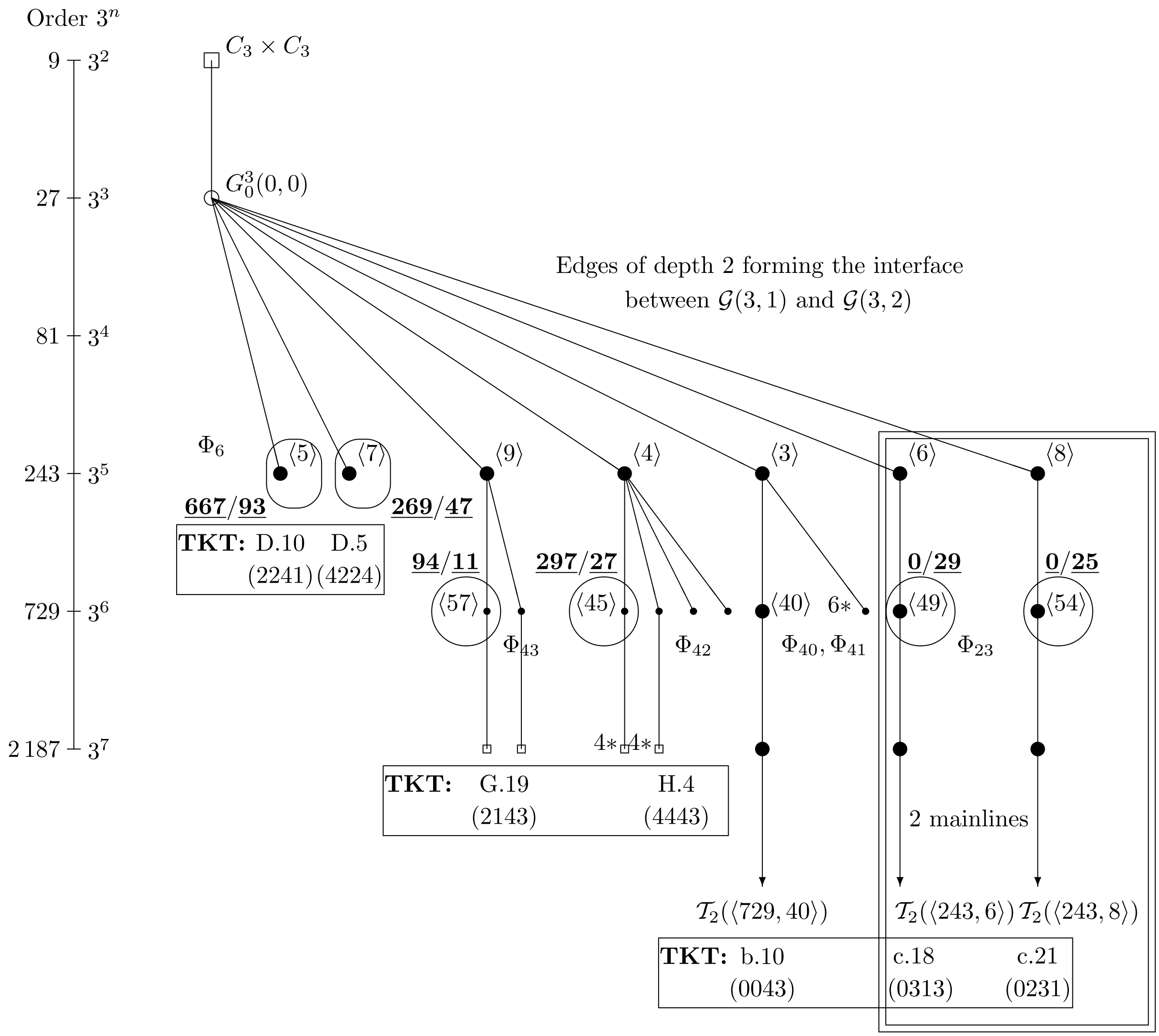


TKT:	a.3	a.3	a.2	a.1	a.1	a.1	a.1
	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

We start our search for 3-groups with TKT in section E at the abelian root $C_3 \times C_3 \simeq \langle 9, 2 \rangle$ of the unique coclass tree $\mathcal{T}_1$ in coclass graph $\mathcal{G}(3,1)$. However, we leave this graph very quickly, since all 3-groups of maximal class have TKTs in section a or A.

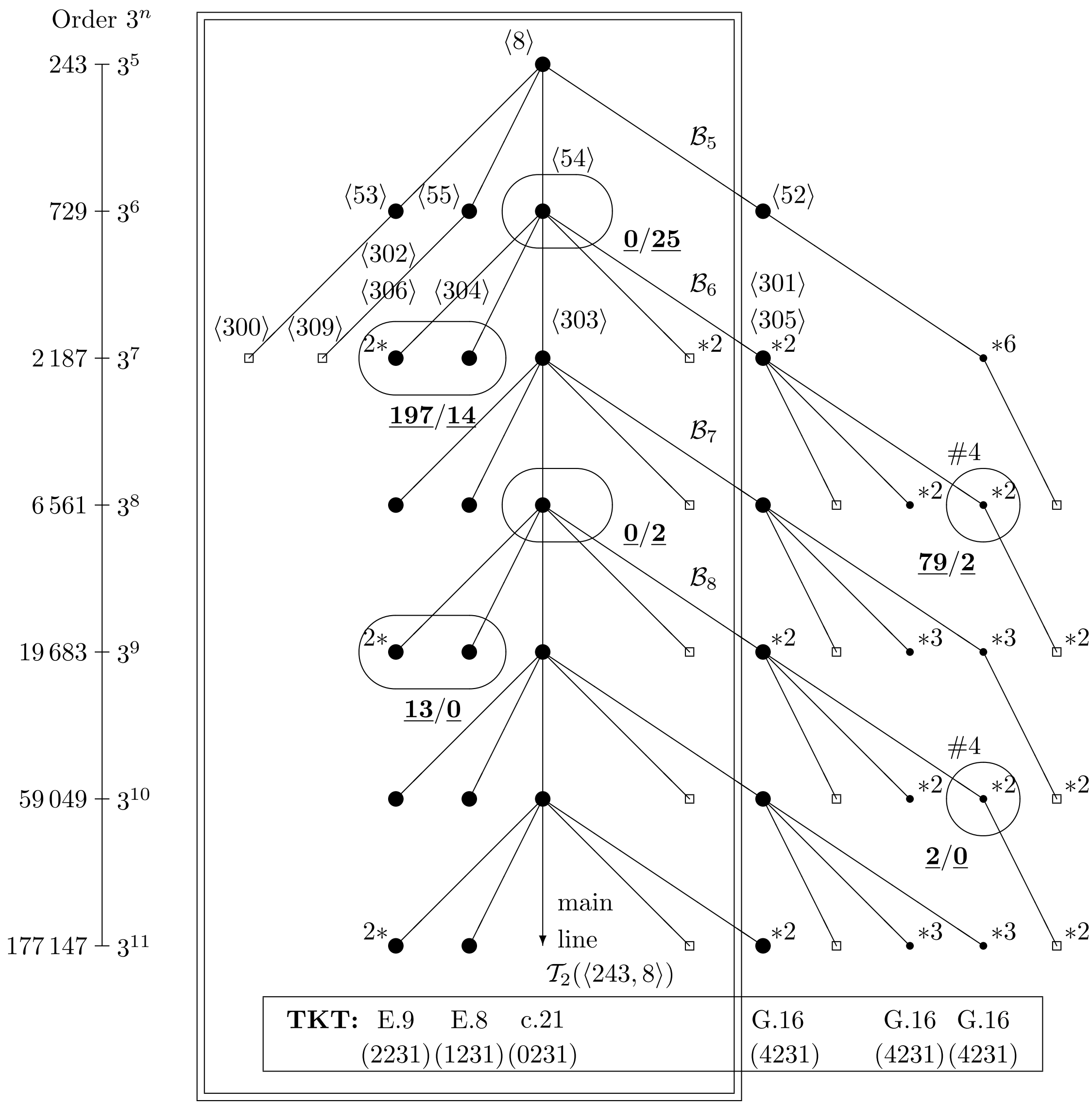
The immediate descendant $G_0^3(0,0) \simeq \langle 27, 3 \rangle$ gives rise to a bifurcation from $\mathcal{G}(3,1)$ to $\mathcal{G}(3,2)$, but the following mainline vertex $G_0^4(0,0) \simeq \langle 81, 9 \rangle$ is coclass-settled and no further bifurcations can occur.

Figure 2. TKT-pruning the top of coclass graph $\mathcal{G}(3,2)$



The top vertices $\langle 243, 5 \rangle$ and $\langle 243, 7 \rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243, 9 \rangle$, resp. $\langle 243, 4 \rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243, 3 \rangle$ must contain a transposition, which is not the case for sections c and E. Therefore, only descendants of $\langle 243, 6 \rangle$ and $\langle 243, 8 \rangle$ can have TKTs in sections c and E.

Figure 3. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$



The bifurcation at $\langle 729, 54\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

§ 2.4. Biperiodic Structure of $\mathcal{T}^*(\langle 243, 8\rangle)$

We consider the intersections of $\mathcal{T}^*(\langle 243, 8\rangle)$ with coclass graphs $\mathcal{G}(3, r)$. We put

$$\mathcal{T}_2^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, 2)$$

and, for all $r \geq 3$,

$$\mathcal{G}_r^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, r).$$

Theorem 2.2. (*First periodicity*).
(See Figures 3 and 4.)

(1) $\mathcal{T}_2^*(\langle 243, 8\rangle)$ is a subtree of $\mathcal{T}^*(\langle 243, 8\rangle)$.
(2) All vertices are metabelian and unbalanced.
(3) Vertices of TKT c.21 form an infinite mainline with unique group $M_n^{(2)}$ of each order 3^n, $n \geq 5$.
(4) Every branch is of depth 1 and contains two groups $G_{n,1}^{(2)}, G_{n,2}^{(2)}$ of TKT E.9 and a single group $G_{n,3}^{(2)}$ of TKT E.8, each of order 3^n with odd $n \geq 7$.

Theorem 2.3. (*Second periodicity*).
(See Figure 4.)
For $3 \le r \le 5$,

(1) the graph $\mathcal{G}_r^\ast(\langle 243,8\rangle)$ consists of
3 isolated vertices $S_k^{(r)}$, $1 \le k \le 3$,
and a subtree $\mathcal{T}_r^\ast(M_{3r-1}^{(r)})$ of $\mathcal{T}^\ast(\langle 243,8\rangle)$,

(2) $\mathcal{T}_r^\ast(M_{3r-1}^{(r)})$ is isomorphic to $\mathcal{T}_2^\ast(\langle 243,8\rangle)$ as a graph, and additionally, the two trees share the same distribution of TKTs,

(3) all vertices G of $\mathcal{G}_r^\ast(\langle 243,8\rangle)$ are
non-metabelian of derived length $\mathrm{dl}(G) = 3$ with
cyclic second derived subgroup G'' of order 3^{r-2}
contained in the centre $\zeta_1(G)$ of type $(3,3^{r-1})$,

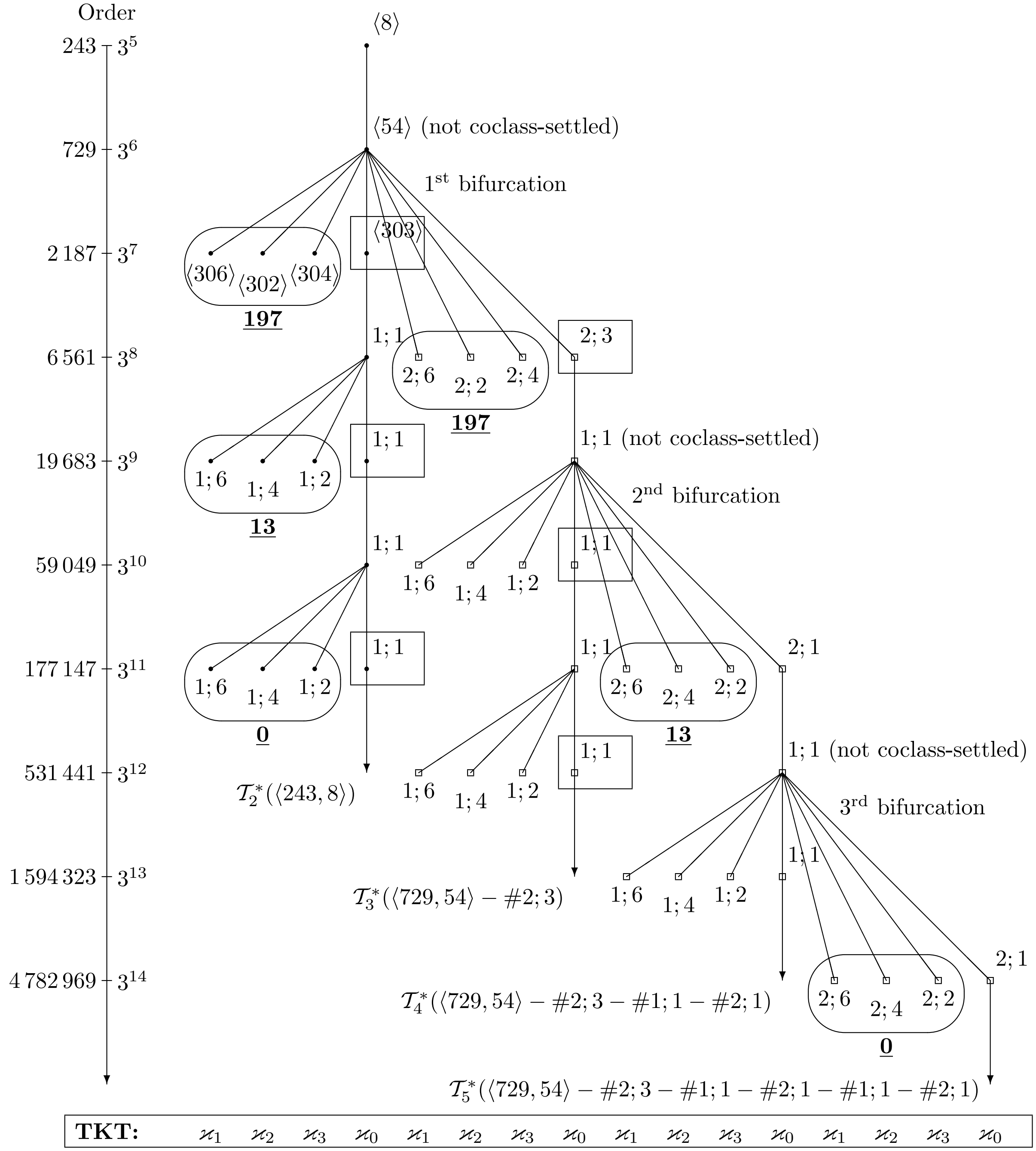
(4) the tree root $M_{3r-1}^{(r)}$ and the isolated vertices $S_k^{(r)}$ are of order 3^{3r-1},

(5) only the isolated vertices $S_k^{(r)}$ are Schur σ-groups,
two of them $S_1^{(r)}, S_2^{(r)}$ have TKT E.9,
and the remaining one $S_3^{(r)}$ has TKT E.8,

(6) each $S_k^{(r)}$ is the unique element in the balanced cover $\mathrm{cov}_\ast(G_{2r+1,k}^{(2)})$ of the branch group $G_{2r+1,k}^{(2)}$ of order 3^{2r+1} on the tree $\mathcal{T}_2^\ast(\langle 243,8\rangle)$.

Conjecture 2.3.
Theorem 2.3 is also correct for any $r \ge 6$.

FIGURE 4. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups



Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

Theorem 2.4. (Brink and Gold – tidy !)
The 3-groups with parametrized presentation $M_2(\beta) = \langle x, y, s_2, s_3, t_3 \mid [y, x] = s_2, [s_2, x] = s_3, [s_2, y] = t_3, [s_3, x] = s_3^{-1} s_2^{-3} s_3^{-2} t_3^6,\ [s_3, y] = t_3^6,\ [s_3, s_2] = t_3^3, [t_3, x] = [t_3, y] = [t_3, s_2] = [t_3, s_3] = 1, x^3 = t_3^{-1},\ y^3 = s_2^{-3} s_3^{-1},\ s_2^{3^\beta} = s_3^{3^\beta} = t_3^9 = 1 \rangle$,
which were constructed by Brink and Gold for all parameter values $\beta \geq 2$, are non-metabelian with derived length 3 and cyclic second derived subgroup of order 3. They are of order $3^{2\beta+4}$, class $2\beta+1$ and fixed coclass 3.
The annihilator ideal $\mathfrak{A} = \{f(X, Y) \in \mathbb{Z}[X, Y] \mid s_2^{f(x-1,y-1)} = 1\}$ of the group $M_2(\beta)'/M_2(\beta)''$ is $\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3\rangle$ with $\alpha = 2\beta$.
However, none of these groups has a balanced presentation and furthermore they are all of TKT c.21. Their metabelianizations $M_2(\beta)/M_2(\beta)''$ are the mainline groups of order $3^{2\beta+3}$ on the tree $\mathcal{T}_2^*(\langle 243, 8\rangle)$.

Remark.
The root of the subtree $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ of $\mathcal{G}_r^*(\langle 243, 8\rangle)$ is given by $M_{3r-1}^{(r)} =$

$$= \begin{cases} \langle 729, 54\rangle \#2; 3 & \text{for } r = 3, \\ \langle 729, 54\rangle \#2; 3\#1; 1\#2; 1 & \text{for } r = 4, \\ \langle 729, 54\rangle \#2; 3\#1; 1\#2; 1\#1; 1\#2; 1 & \text{for } r = 5. \end{cases}$$

§ 2.5. Two Conclusive Main Results

1. Group Theoretic Main Result.

All metabelian 3-groups with TKT in section E share the common coclass 2.

They are vertices $G_{n,k}^{(2)}$ with $n \geq 6$, $1 \leq k \leq 3$ for odd n, $2 \leq k \leq 3$ for even n, of depth 1 on the coclass tree $\mathcal{T}_2^*(M_5^{(2)})$ with root $M_5^{(2)}$ either $\langle 243, 6\rangle$ or $\langle 243, 8\rangle$.

None of these groups has a balanced presentation. The cardinalities of the covers and balanced covers of these groups are finite. They are given by

(1) $\#\mathrm{cov}(G_{2j+1,k}^{(2)}) = j+1$ and $\#\mathrm{cov}_*(G_{2j+1,k}^{(2)}) = 1$, for $j \geq 3$ and $1 \leq k \leq 3$, that is, σ-groups have a unique Schur σ-group as their balanced cover,

(2) $\#\mathrm{cov}(G_{2\ell,k}^{(2)}) = j + 1$ and $\#\mathrm{cov}_*(G_{2\ell,k}^{(2)}) = 0$, for $j \geq 3$ and $2 \leq k \leq 3$, that is, the balanced cover of non-σ groups is empty.

Remark. The Result is a Theorem for $j \leq 10$ and a Conjecture for $j \geq 11$.

2. Number Theoretic Main Result.

If the TKT $\varkappa(G)$ of the second 3-class group $G = \mathrm{G}_3^2(K)$ of a *complex quadratic* number field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with $\mathrm{Cl}_3(K) \simeq (3,3)$ belongs to the four types of section E,
then the 3-tower of K has exactly three stages, that is, $\ell_3(K) = 3$.

Remark.

The Result is a Theorem for odd $\mathrm{cl}(G) \leq 19$ and a Conjecture for odd $\mathrm{cl}(G) \geq 21$.
This is more than enough to disprove the claims of Scholz and Taussky on page 41 in [9] and of Heider and Schmithals on page 20 in [5], since for this purpose $\mathrm{cl}(G) = 5$ suffices already.

Among complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3,3)$, those with TKT in section E occur with relative frequency $\frac{411}{2020} \approx 20.3\%$.

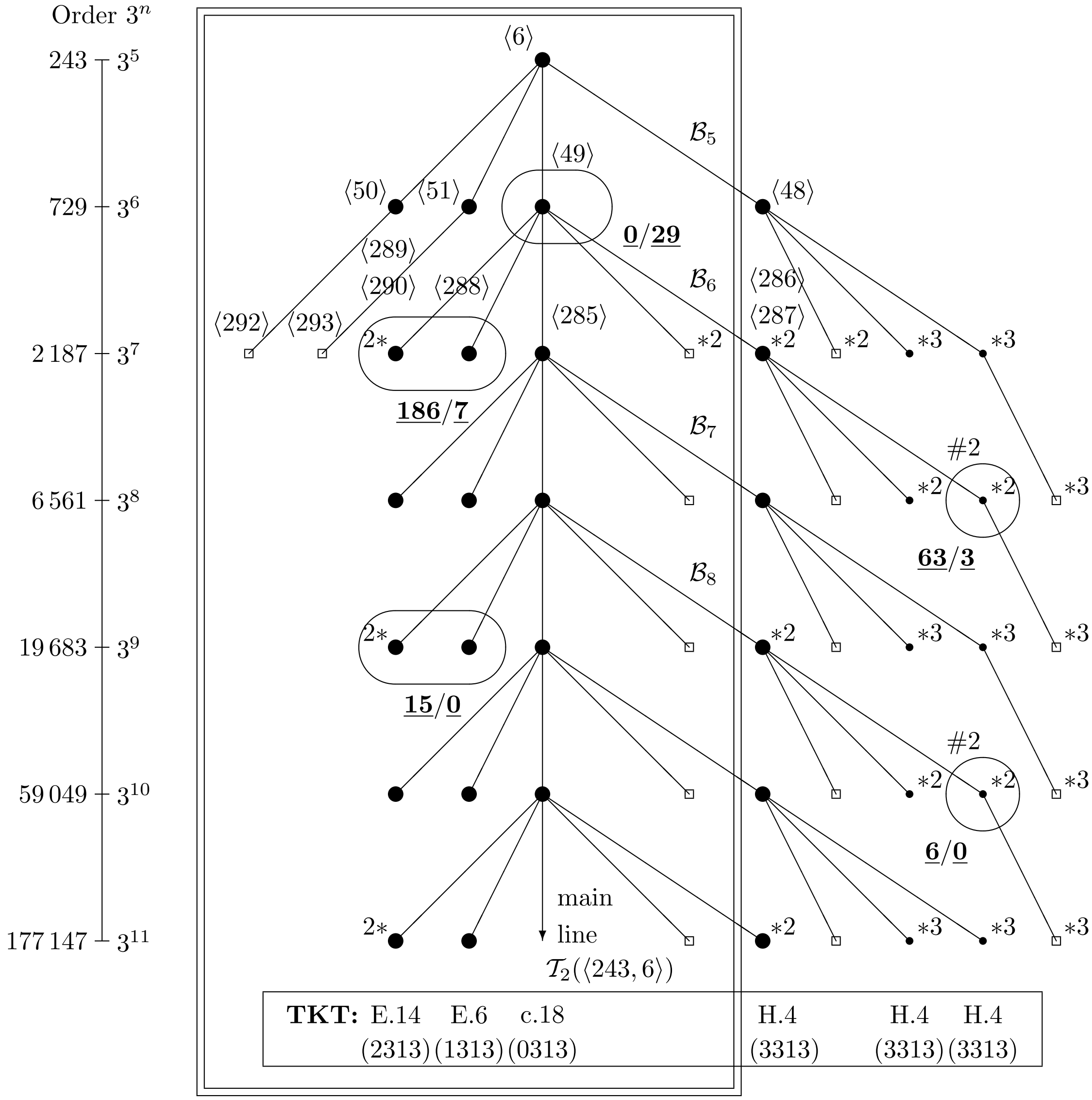
Corresponding complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3,9)$ and TKT in sections C and D occur with relative frequency $\frac{182}{875} \approx 20.8\%$.

§ 3. Appendix

In Figures 5 and 6 we indicate how the TKT-pruned descendant tree for $\langle 243, 6\rangle$ can be constructed in a completely similar manner as for $\langle 243, 8\rangle$ in Figures 3 and 4.

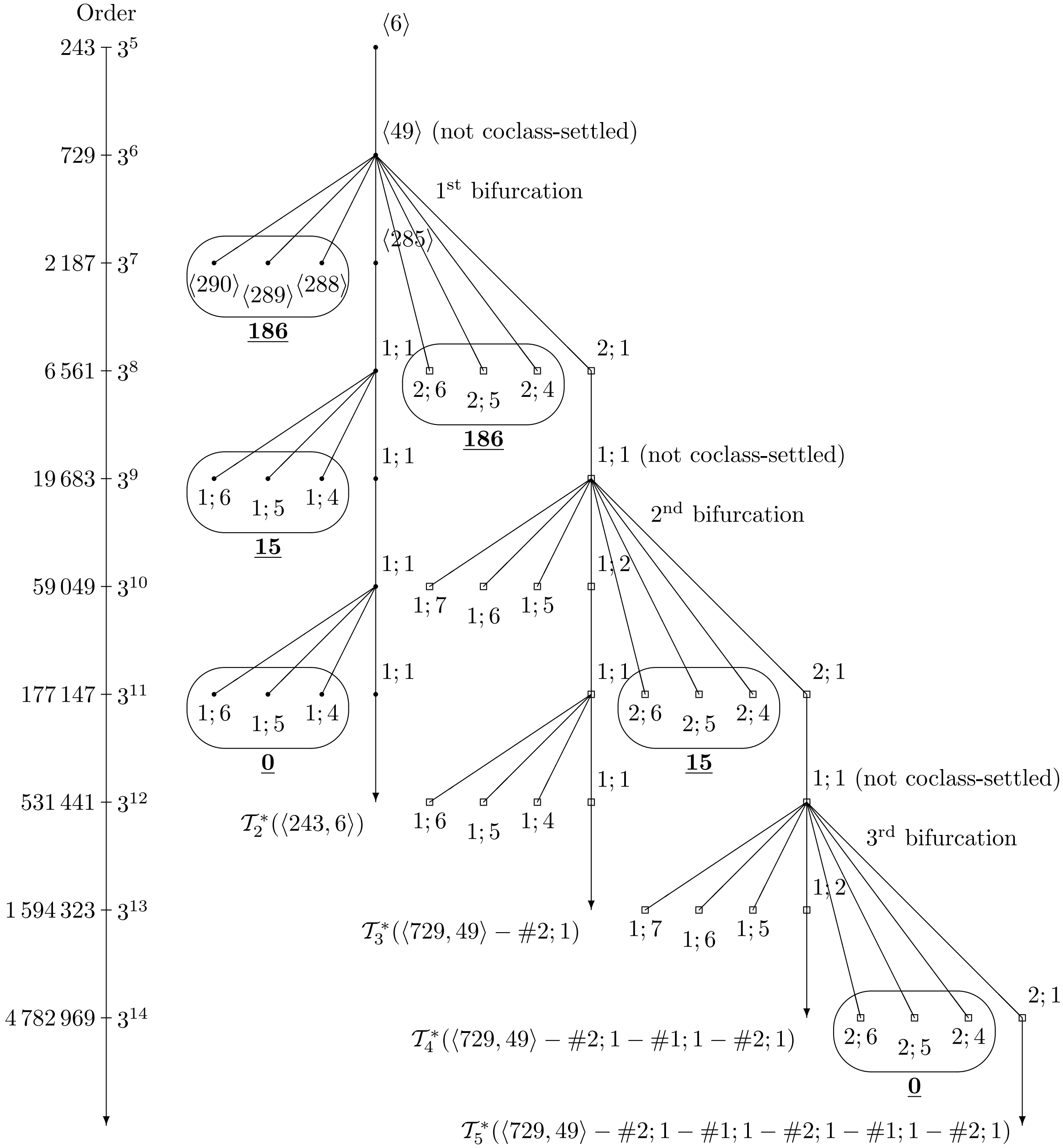
The structure of the smallest non-metabelian Schur σ-groups with TKT in section E is shown for class 5 in Figure 7 and for class 7 in Figure 8.
The smallest group $M_2(\beta)$ with $\beta = 2$ constructed by Brink and Gold has also the structure in Figure 7 although it is not a Schur group and has different TKT in section c.

FIGURE 5. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$



The bifurcation at $\langle 729, 49\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

FIGURE 6. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 6\rangle)$ restricted to σ-groups



TKT:	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$

Here we also prune the tree from vertices with TKT c.18 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (3122) \sim \varkappa_2 = (4122)$ E.14,
$\varkappa_3 = (1122)$ E.6,
$\varkappa_0 = (0122)$ c.18.

FIGURE 7. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G with TKT E, class 5.

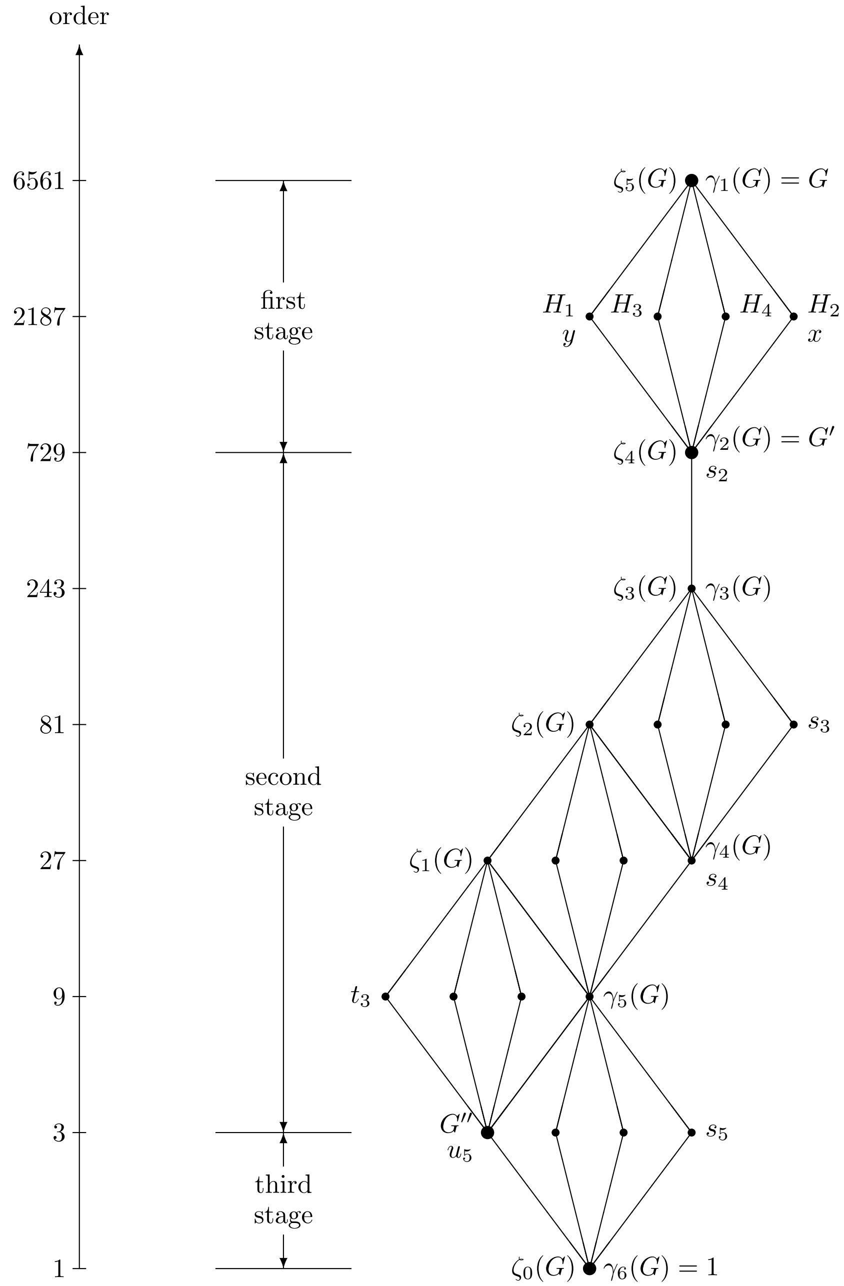
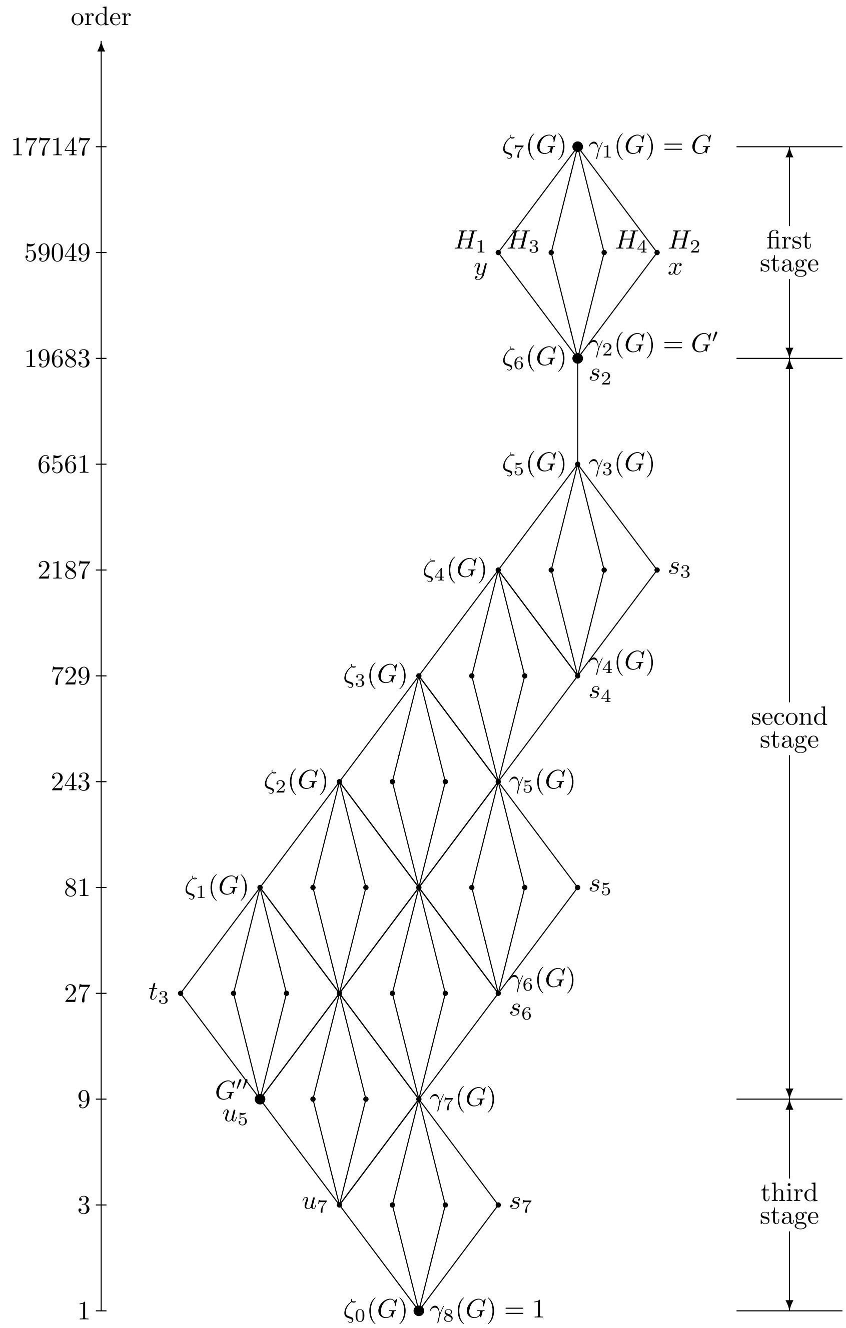


FIGURE 8. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G with TKT E, class 7.



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[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

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