METABELIAN 3-GROUPS WITH ABELIANISATION OF TYPE $(9,3)$

DANIEL C. MAYER

Abstract. Presentations of metabelian 3-groups G with abelianisation G/G' of type $(9,3)$ are used to determine explicit expressions for the transfers V_i from these groups to their maximal normal subgroups M_i, and to calculate the transfer kernels $\ker(\mathrm{V}_i)$ in G/G' and the structure of the transfer targets M_i/M_i', for $1 \le i \le 4$.

1. Introduction

We consider metabelian 3-groups $G = \langle x, y\rangle$ with two generators satisfying $x^9 \in G'$ and $y^3 \in G'$ and commutator quotient group G/G' of type $(9,3)$. Generally, such a group possesses

- four normal subgroups of index 9,

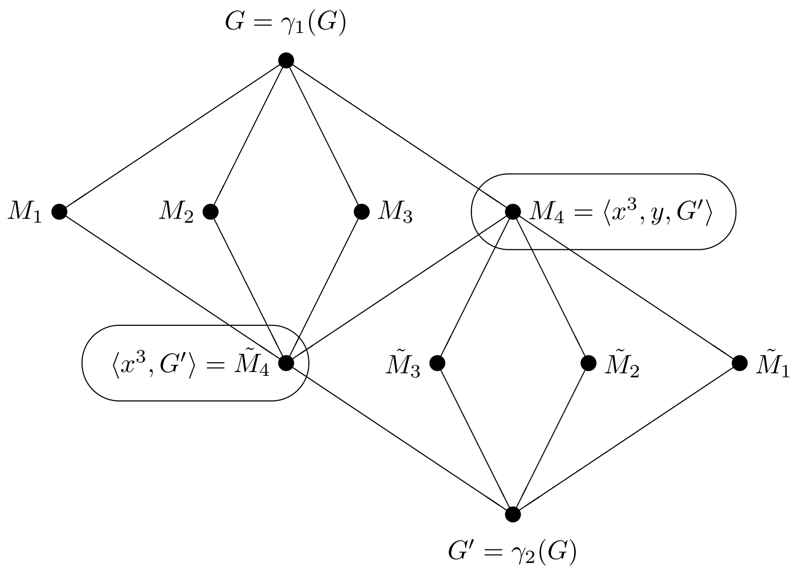
$$\tilde{M}_1 = \langle y, G'\rangle,\ \tilde{M}_2 = \langle x^3 y, G'\rangle,\ \tilde{M}_3 = \langle x^3 y^{-1}, G'\rangle,\ \tilde{M}_4 = \langle x^3, G'\rangle,$$

- and four maximal normal subgroups of index 3,

$$M_1 = \langle x, G'\rangle,\ M_2 = \langle xy, G'\rangle,\ M_3 = \langle xy^{-1}, G'\rangle,\ M_4 = \langle x^3, y, G'\rangle.$$

We use the subscript 4 to indicate that for $M_4 = \prod_{i=1}^{4} \tilde{M}_i$ the factor group $M_4/G' = \langle x^3, y\rangle$ is bicyclic of type $(3,3)$, whereas M_i/G' is cyclic of order 9, for $1 \le i \le 3$, and that $\tilde{M}_4 = \cap_{i=1}^{4} M_i = \Phi(G) = G^3 G'$ coincides with the Frattini subgroup of G, whereas $\tilde{M}_i$ is only contained in M_4, for $1 \le i \le 3$.

Figure 1. Double diamond head of a group G with G/G' of type $(9,3)$



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2. Common formulas for 2-generator groups of small class

Let $G = \langle x, y \rangle$ be a group with two generators x, y. Define the main commutator by $s_2 = [y, x] \in \gamma_2(G)$ and the threefold commutators by $s_3 = [s_2, x], t_3 = [s_2, y] \in \gamma_3(G)$. Then $yx = xy[y, x] = xys_2$, $s_2x = xs_2[s_2, x] = xs_2s_3$, and $s_2y = ys_2[s_2, y] = ys_2t_3$.

If G is metabelian, then
$[s_2^{-1}, x] = [s_2, x]^{-s_2^{-1}} = s_3^{-s_2^{-1}} = s_3^{-1}$, $[s_2^{-1}, y] = [s_2, y]^{-s_2^{-1}} = t_3^{-s_2^{-1}} = t_3^{-1}$,
and $[s_2^{-1}, y^{-1}] = [s_2^{-1}, y]^{-y^{-1}} = (t_3^{-1})^{-y^{-1}} = t_3^{y^{-1}} = t_3$, if t_3 lies in the centre $\zeta_1(G)$.

Consequently, $[y^{-1}, x] = [y, x]^{-y^{-1}} = s_2^{-y^{-1}} = (s_2^{-1})^{y^{-1}} = s_2^{-1}[s_2^{-1}, y^{-1}] = s_2^{-1}t_3$.

After this preliminary commutator calculus, we prove two formulas for 3rd powers of products of the generators of a metabelian 2-generator group G, now assuming that s_3, t_3 belong to the centre $\zeta_1(G)$.

$$(xy)^3 = x^3y^3s_2^3s_3t_3^5, \tag{1}$$

$$(xy^{-1})^3 = x^3y^{-3}s_2^{-3}s_3^{-1}t_3^8. \tag{2}$$

Proof. $(xy)^3 = xyxyxy = xxys_2xys_2y = x^2yxs_2s_3ys_2y = x^2xys_2s_2ys_2ys_3 = x^3ys_2ys_2t_3ys_2t_3s_3 =$
$= x^3yys_2t_3ys_2t_3s_2s_3t_3^2 = x^3y^2s_2ys_2^2s_3t_3^4 = x^3y^2ys_2t_3s_2^2s_3t_3^4 = x^3y^3s_2^3s_3t_3^5$
and $(xy^{-1})^3 = xy^{-1}xy^{-1}xy^{-1} = xxy^{-1}[y^{-1}, x]xy^{-1}[y^{-1}, x]y^{-1} = x^2y^{-1}s_2^{-1}t_3xy^{-1}s_2^{-1}t_3y^{-1} =$
$= x^2y^{-1}xs_2^{-1}[s_2^{-1}, x]y^{-1}y^{-1}s_2^{-1}[s_2^{-1}, y^{-1}]t_3^2 = x^2xy^{-1}s_2^{-1}t_3s_2^{-1}s_3^{-1}y^{-1}y^{-1}s_2^{-1}t_3t_3^2 =$
$= x^3y^{-1}s_2^{-1}y^{-1}s_2^{-1}t_3s_3^{-1}y^{-1}s_2^{-1}t_3^4 = x^3y^{-1}y^{-1}s_2^{-1}t_3s_2^{-1}y^{-1}s_2^{-1}s_3^{-1}t_3^5 =$
$= x^3y^{-2}s_2^{-1}y^{-1}s_2^{-1}t_3s_2^{-1}s_3^{-1}t_3^6 = x^3y^{-2}y^{-1}s_2^{-1}t_3s_2^{-2}s_3^{-1}t_3^7 = x^3y^{-3}s_2^{-3}s_3^{-1}t_3^8.$ □

3. S_3-double orbits of punctured transfer kernel types

The *transfer* V_i (Verlagerung) from G to its maximal subgroup M_i is given by

$$\mathrm{V}_i = \mathrm{V}_{G,M_i} : G/G' \to M_i/M_i',\ g \mapsto \begin{cases} g^3, & \text{if } g \in G \setminus M_i\,, \\ g^{\mathrm{S}_3(h)}, & \text{if } g \in M_i\,, \end{cases} \tag{3}$$

where $\mathrm{S}_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$, with an arbitrary element $h \in G \setminus M_i$, denotes the third *trace element* (Spur) in the group ring, acting as a symbolic exponent.

There are five possibilities for the kernel of V_i, for each $1 \le i \le 4$. Either $\ker(\mathrm{V}_i) = \tilde{M}_j/G'$, for some $1 \le j \le 4$, and we denote the *one-dimensional* transfer by the singulet $\varkappa(i) = j$, or $\ker(\mathrm{V}_i) = M_4/G'$, and we denote the *two-dimensional* transfer by $\varkappa(i) = 0$. Due to the distinguished role of the subscript 4, we combine the singulets to form a multiplet

$$\varkappa = (\,(\varkappa(1), \varkappa(2), \varkappa(3));\ \varkappa(4)) \in [0, 4]^3 \times [0, 4]$$

which we call the *punctured transfer kernel type* (TKT) of the group G with respect to the selected generators.

To be independent from the choice of generators and the order of M_1, M_2, M_3 and $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$, we define the *double orbit*

$$\varkappa^{S_3 \times S_3} = \{\tilde{\sigma} \circ \varkappa \circ \hat{\tau} \mid \sigma, \tau \in S_3\}$$

of $\varkappa$ under the operation of $S_3 \times S_3$ as an isomorphism invariant $\varkappa(G)$ of G. Here, $\tilde{\sigma}$ denotes the extension of σ from $[1, 3]$ to $[0, 4]$ which fixes 0 and 4 and $\hat{\tau}$ denotes the extension of τ from $[1, 3]$ to $[1, 4]$ which fixes 4.

Two further *isomorphism invariants* of G are $\mu = \mu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 4\}$ and the number of two-dimensional transfers $\nu = \nu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 0\}$.

4. Combinatorially possible punctured transfer kernel types

In this section, we arrange all combinatorially possible S_3-double orbits of the 5^4 punctured quadruplets $\varkappa \in [0,4]^3 \times [0,4]$ by increasing invariant $0 \le \mu \le 4$ and cardinality of the image. Table 1 shows the punctured quadruplets with invariant $\nu = 0$ and Table 2 the punctured quadruplets with invariant $1 \le \nu \le 4$ as possible punctured transfer kernel types of 3-groups G with G/G' of type $(9,3)$, resp. *punctured principalisation types* of number fields K with 3-class group $\mathrm{Cl}_3(K)$ of type $(9,3)$, according to Artin's reciprocity law [15]. The double orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\varkappa) = (|\varkappa^{-1}\{i\}|)_{0 \le i \le 4}$ the family of occupation numbers of the selected double orbit representative $\varkappa$ and by κ the quadruplet of Taussky's conditions [26] associated with $\varkappa$.

If a double orbit $\varkappa^{S_3 \times S_3}$ can be realised as a punctured transfer kernel type $\varkappa(G)$, then a suitable 3-group G is given in the notation of James [12], using Hall's isoclinism families [11].

Table 1 gives a coarse classification into sections A to E, an identification by ordinal numbers 1 to 20, and a set theoretical characterisation.

Table 1. The 20 S_3-double orbits of $\varkappa \in [1,4]^4$ with $\nu = 0$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
A	1	(1111)	(04000)	$(BBBA)$	constant	3	$\Phi_2(31)$
B	2	(1112)	(03100)	$(BBBA)$	nearly	6	???
B	3	(1121)	(03100)	$(BBBA)$	constant	18	
C	4	(1122)	(02200)	$(BBBA)$		18	???
D	5	(1123)	(02110)	$(BBBA)$		18	
D	6	(1231)	(02110)	$(BBBA)$		18	???
B	7	(1114)	(03001)	$(BBBA)$	nearly	3	$\Phi_6(321)_{b_{1,1}}, \Phi_6(321)_{b_{1,2}}$
B	8	(1141)	(03001)	$(BBAA)$	constant	9	
D	9	(1124)	(02101)	$(BBBA)$		18	???
D	10	(1142)	(02101)	$(BBAA)$		18	???
D	11	(1241)	(02101)	$(BBAA)$		36	$\Phi_6(321)_{a_1}, \Phi_6(321)_{a_2}$
E	12	(1234)	(01111)	$(BBBA)$	per-	6	$\Phi_6(321)_{b_{2,1}}, \Phi_6(321)_{b_{2,2}}$
E	13	(1243)	(01111)	$(BBAA)$	mutation	18	
C	14	(1144)	(02002)	$(BBAA)$		9	
C	15	(1441)	(02002)	$(BAAA)$		9	
D	16	(1244)	(01102)	$(BBAA)$		18	???
D	17	(1442)	(01102)	$(BAAA)$		18	???
B	18	(1444)	(01003)	$(BAAA)$	nearly	9	???
B	19	(4441)	(01003)	$(AAAA)$	constant	3	???
A	20	(4444)	(00004)	$(AAAA)$	constant	1	$\Phi_6(2^2 1^2)_g, \Phi_2(2^2), \Phi_8(32)$
					Total number:	256	

Table 2 gives a coarse classification into sections a to e, an identification by ordinal numbers 1 to 32, and a set theoretical characterisation.

TABLE 2. The 32 S_3-double orbits of $\varkappa \in [0,4]^4 \setminus [1,4]^4$ with $1 \le \nu \le 4$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
a	1	(0000)	(40000)	$(AAAA)$	constant	1	$\Phi_2(21^2)_c, \Phi_3(21^3)_d, \Phi_3(21^3)_e$
b	2	(0001)	(31000)	$(AAAA)$	nearly	3	$\Phi_3(31^2)_a$
b	3	(0010)	(31000)	$(AABA)$	constant	9	$\Phi_3(31^2)_{b_1}, \Phi_3(31^2)_{b_2}$
c	4	(0011)	(22000)	$(AABA)$		9	
c	5	(0110)	(22000)	$(ABBA)$		9	
d	6	(0012)	(21100)	$(AABA)$		18	
d	7	(0120)	(21100)	$(ABBA)$		18	
b	8	(0111)	(13000)	$(ABBA)$	nearly	9	
b	9	(1110)	(13000)	$(BBBA)$	constant	3	
d	10	(0112)	(12100)	$(ABBA)$		18	$\Phi_6(31^3)_a$
d	11	(0121)	(12100)	$(ABBA)$		36	
d	12	(1120)	(12100)	$(BBBA)$		18	
e	13	(0123)	(11110)	$(ABBA)$	per-	18	
e	14	(1230)	(11110)	$(BBBA)$	mutation	6	$\Phi_6(31^3)_{b_1}, \Phi_6(31^3)_{b_2}$
b	15	(0004)	(30001)	$(AAAA)$	nearly	1	$\Phi_3(2^21)_{b_1}, \Phi_3(2^21)_{b_2}, \Phi_6(21^4)_d$
b	16	(0040)	(30001)	$(AAAA)$	constant	3	$\Phi_3(2^21)_a$
d	17	(0014)	(21001)	$(AABA)$		9	
d	18	(0041)	(21001)	$(AAAA)$		9	
d	19	(0140)	(21001)	$(ABAA)$		18	
d	20	(0114)	(12001)	$(ABBA)$		9	
d	21	(0141)	(12001)	$(ABAA)$		18	
d	22	(1140)	(12001)	$(BBAA)$		9	
e	23	(0124)	(11101)	$(ABBA)$	per-	18	
e	24	(0142)	(11101)	$(ABAA)$	muta-	36	
e	25	(1240)	(11101)	$(BBAA)$	tion	18	
c	26	(0044)	(20002)	$(AAAA)$		3	
c	27	(0440)	(20002)	$(AAAA)$		3	$\Phi_6(2^21^2)_{h_1}$
d	28	(0144)	(11002)	$(ABAA)$		18	
d	29	(0441)	(11002)	$(AAAA)$		9	
d	30	(1440)	(11002)	$(BAAA)$		9	
b	31	(0444)	(10003)	$(AAAA)$	nearly	3	$\Phi_6(2^21^2)_{h_2}$
b	32	(4440)	(10003)	$(AAAA)$	constant	1	
				Total number:	$625 - 256 =$	369	

5. Actual realisation of punctured transfer kernel types

In this section, we characterise all punctured quadruplets $\varkappa \in [0,4]^4$ which can be realised as punctured transfer kernel types of metabelian 3-groups G with abelianisation G/G' of type $(9,3)$. For this purpose we assume that G occurs as the *second 3-class group* $\mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ [16] of an algebraic number field K with 3-class group $\mathrm{Cl}_3(K)$ of type $(9,3)$. Then the structure of the abelianisations M_i/M_i' of the maximal normal subgroups M_i of G, which we call the *transfer target type* (TTT) τ of G, is identical with the structure of the 3-class groups $\mathrm{Cl}_3(N_i)$ of the unramified cyclic cubic extensions $N_i|K$, for $1 \le i \le 4$. Further, the structure of the abelianisation $\tilde{M}_4/\tilde{M}_4'$ of the distinguished normal subgroup $\tilde{M}_4 = \Phi(G)$ of index 9 in G is identical with the structure of the 3-class group $\mathrm{Cl}_3(\tilde{N}_4)$ of the *Frattini extension*, the unique unramified bicyclic bicubic extension $\tilde{N}_4|K$. The *isomorphism invariant* $\varepsilon = \varepsilon(G)$ denotes the number of 3-class groups $\mathrm{Cl}_3(N_i)$ of 3-rank at least 3. In the case of a quadratic base field $K = \mathbb{Q}(\sqrt{D})$ with discriminant D, the 3-class numbers $\mathrm{h}_i = \mathrm{h}_3(L_i)$ of the non-Galois absolutely cubic subfields L_i of the N_i can be used additionally for the characterisation.

Table 3 lists the 13 isomorphism classes of 3-groups G with abelianisation G/G' of type $(9,3)$ in the isoclinism family Φ_6 [19]. They form branch 1 of this family, whence their order, nilpotency class, and coclass [14] are given by $|G| = 3^6$, $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 3$, whereas the stem groups of Φ_6 have $|G| = 3^5$, $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 2$. Generally, the nilpotency class $\mathrm{cl}(G) = 3$ is a family invariant of Φ_6. $\downarrow$ denotes a descendant.

Table 3. TKT and TTT of 3-groups in branch 1 of isoclinism family Φ_6 or descendants

type	$\varkappa$	h_1	h_2	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε	$\mathrm{Cl}_3(\tilde{N}_4)$	min. $\lvert D\rvert$	group
D.11	(4232)	3	3	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$	$\lvert -3\,299\rvert$	$\Phi_6(321)_{a_1}$
D.11	(4322)	3	3	$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$	255 973	$\Phi_6(321)_{a_2}$
B.7	(1114)	3	3	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1	$(9,3,3,3)$	$\lvert -54\,695\rvert$	$\Phi_6(321)_{b_{1,1}} \downarrow$
B.7	(1114)	3	3	$(27,3)$	$(27,3)$	$(27,3)$	$(3,3,3,3)$	1	$(9,3,3,3)$	1 664 444	$\Phi_6(321)_{b_{1,2}} \downarrow$
E.12	(1234)	3	3	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,9,3)$	$\lvert -5\,703\rvert$	$\Phi_6(321)_{b_{2,1}} \downarrow$
E.12	(1324)	3	3	$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,9,3)$	1 893 032	$\Phi_6(321)_{b_{2,2}} \downarrow$
d.10	(0112)			$(9,3,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	2	$(9,3,3)$		$\Phi_6(31^3)_a$
e.14	(1320)			$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$		$\Phi_6(31^3)_{b_1}$
e.14	(1230)			$(27,3)$	$(27,3)$	$(27,3)$	$(9,3,3)$	1	$(9,3,3)$		$\Phi_6(31^3)_{b_2}$
A.20	(4444)	3	3	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4	$(9,9,3,3,3)$	$\lvert -289\,704\rvert$	$\Phi_6(2^21^2)_g \downarrow$
c.27	(0440)			$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	4	$(3,3,3,3)$		$\Phi_6(2^21^2)_{h_1}$
b.31	(0444)			$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	4	$(3,3,3,3)$		$\Phi_6(2^21^2)_{h_2}$
b.15	(0004)			$(9,3,3)$	$(9,3,3)$	$(9,3,3)$	$(3,3,3,3)$	4	$(3,3,3,3)$		$\Phi_6(21^4)_d$

In Table 4 we give the 12 isomorphism classes of 3-groups G with abelianisation G/G' of type $(9,3)$ in the isoclinism families Φ_2, Φ_3, and Φ_8. For Φ_2, they form branch 1 of this family, whence their order and coclass are given by $|G| = 3^4$, $\mathrm{cc}(G) = 2$, whereas the stem groups of Φ_2 have $|G| = 3^3$, $\mathrm{cc}(G) = 1$. The class $\mathrm{cl}(G) = 2$ is a family invariant of Φ_2. For Φ_3, they form branch 1 of this family, whence their order and coclass are given by $|G| = 3^5$, $\mathrm{cc}(G) = 2$, whereas the stem groups of Φ_3 have $|G| = 3^4$, $\mathrm{cc}(G) = 1$. The class $\mathrm{cl}(G) = 3$ is a family invariant of Φ_3. Finally, the stem of Φ_8 consists of a unique isomorphism class with $|G| = 3^5$, $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 2$.
$\downarrow$ denotes a descendant.

TABLE 4. TKT and TTT of 3-groups in isoclinism families Φ_2, Φ_3, Φ_8 or descendants

type	$\varkappa$	h_1	h_2	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε	$\mathrm{Cl}_3(\tilde{N}_4)$	min. $\lvert D\rvert$	group
A.1	(1111)			(27)	(27)	(27)	(9,3)	0	(9)		$\Phi_2(31)$
A.20	(4444)			(9,3)	(9,3)	(9,3)	(9,3)	0	(3,3)		$\Phi_2(2^2)$
a.1	(0000)			(9,3)	(9,3)	(9,3)	(3,3,3)	1	(3,3)		$\Phi_2(21^2)_c$
b.2	(0001)	3	3	(9,3)	(9,3)	(9,3)	(9,3,3)	1	(9,3)	529 393	$\Phi_3(31^2)_a$
b.3	(1000)	3	3	(27,3)	(9,3)	(9,3)	(3,3,3)	1	(9,3)	635 909	$\Phi_3(31^2)_{b_1}$
b.3	(1000)	3	3	(27,3)	(9,3)	(9,3)	(3,3,3)	1	(9,3)	946 733	$\Phi_3(31^2)_{b_2}$
b.16	(4000)	3	3	(9,3,3)	(9,3)	(9,3)	(3,3,3)	2	(3,3,3)	282 461	$\Phi_3(2^21)_a$
b.15	(0004)	3	3	(9,3)	(9,3)	(9,3)	(3,3,3,3)	1	(3,3,3,3)	3 763 580	$\Phi_3(2^21)_{b_1}\downarrow$
b.15	(0004)	3	3	(9,3)	(9,3)	(9,3)	(9,3,3)	1	(9,3,3)	700 313	$\Phi_3(2^21)_{b_2}\downarrow$
a.1	(0000)	9	3	(9,3)	(9,3)	(9,3)	(9,9,3)	1	(9,9,3)	783 689	$\Phi_3(21^3)_d\downarrow$
a.1	(0000)	9	3	(9,9,3)	(9,3)	(9,3)	(3,3,3)	2	(9,9,3)	626 264	$\Phi_3(21^3)_e\downarrow$
A.20	(4444)			(9,3)	(9,3)	(9,3)	(9,3)	0	(9,3)		$\Phi_8(32)$

6. 3-groups of the first branch of isoclinism family Φ_3

Generally, the p-groups G of isoclinism family Φ_3 are characterized by the nilpotency class $\mathrm{cl}(G)=3$ [12, p.618, 4.1]. Their common central quotient $G/\zeta_1(G)$ is the extra special p-group $G_0^3(0,0)$ of order p^3 and of exponent p [15, Thm.2.5]. For the 2-generator groups $G=\langle x,y\rangle$ in Φ_3, the structure of their lower central series $(\gamma_j(G))_{j\ge 1}$ can be expressed by means of the main commutator, $s_2=[y,x]\in\gamma_2(G)=[G,G]$, and the threefold commutator in $\gamma_3(G)=[\gamma_2(G),G]$,

$$s_3=\begin{cases}[s_2,x], & \text{if } [s_2,y]=1,\\ [s_2,y], & \text{if } [s_2,x]=1.\end{cases}$$

The groups are metabelian with $\gamma_2(G)=\langle s_2,s_3\rangle$ of type (p,p) and $\gamma_3(G)=\langle s_3\rangle$ cyclic of order p.

The 2-generator groups in the first branch of Φ_3 have order $|G|=p^5$, coclass $\mathrm{cc}(G)=2$ and abelianization G/G' of type (p^2,p). If we select the generators of $G=\langle x,y\rangle$ such that $x^{p^2}\in G'$ and $y^p\in G'$.

In the special case $p=3$, the 4 maximal subgroups of G are given by

$$M_1=\langle x,G'\rangle,\ M_2=\langle xy,G'\rangle,\ M_3=\langle xy^{-1},G'\rangle,\ M_4=\langle x^3,y,G'\rangle.$$

To calculate the transfer target type (TTT) $\tau(G)$, we need generators for the commutator quotients of the maximal subgroups. According to [5, p.52, Lem.2.1], we have

$$M_1'=[G',M_1]=(G')^{x-1}=\langle s_2^{x-1}\rangle=\langle[s_2,x]\rangle=\begin{cases}\langle s_3\rangle, & \text{if } [s_2,y]=1,\\ 1, & \text{if } [s_2,x]=1,\end{cases}$$

and $M_1/M_1'=\langle x,s_2,s_3\rangle/\langle s_3\rangle=\langle x,s_2\rangle/\langle s_3\rangle$, if $[s_2,y]=1$,
but $M_1/M_1'\simeq M_1=\langle x,s_2,s_3\rangle$, if $[s_2,x]=1$.

Since

$$s_2^{xy-1}=[s_2,xy]=[s_2,y][s_2,x]^y=\begin{cases}1\cdot s_3^y=s_3, & \text{if } [s_2,y]=1,\\ s_3\cdot 1^y=s_3, & \text{if } [s_2,x]=1,\end{cases}$$

i.e. $s_2^{xy-1}=s_3$ in any case, we have $M_2'=[G',M_2]=(G')^{xy-1}=\langle s_2^{xy-1}\rangle=\langle s_3\rangle$
and $M_2/M_2'=\langle xy,s_2,s_3\rangle/\langle s_3\rangle=\langle xy,s_2\rangle/\langle s_3\rangle$.

Since

$$s_2^{xy^{-1}-1}=[s_2,xy^{-1}]=[s_2,y^{-1}][s_2,x]^{y^{-1}}=[s_2,y]^{-y^{-1}}[s_2,x]^{y^{-1}}=\begin{cases}1^{-y^{-1}}\cdot s_3^{y^{-1}}=s_3, & \text{if } [s_2,y]=1,\\ s_3^{-y^{-1}}\cdot 1^{y^{-1}}=s_3^{-1}, & \text{if } [s_2,x]=1,\end{cases}$$

we have $M_3'=[G',M_3]=(G')^{xy^{-1}-1}=\langle s_2^{xy^{-1}-1}\rangle=\langle s_3\rangle$
and $M_3/M_3'=\langle xy^{-1},s_2,s_3\rangle/\langle s_3\rangle=\langle xy^{-1},s_2\rangle/\langle s_3\rangle$, in any case.

Since $M_4/\Phi(G)$ is cyclic and $x^3\in\zeta_1(G)$, we have

$$M_4'=[\Phi(G),M_4]=[G',M_4]=(G')^{y-1}=\langle s_2^{y-1}\rangle=\langle[s_2,y]\rangle=\begin{cases}1, & \text{if } [s_2,y]=1,\\ \langle s_3\rangle, & \text{if } [s_2,x]=1,\end{cases}$$

and $M_4/M_4'=\langle x^3,y,s_2,s_3\rangle/\langle s_3\rangle=\langle x^3,y,s_2\rangle/\langle s_3\rangle$, if $[s_2,x]=1$,
but $M_4/M_4'\simeq M_4=\langle x^3,y,s_2,s_3\rangle$, if $[s_2,y]=1$.

These formulas admit to give upper bounds for the 3-rank of the abelianisations. Whereas M_2/M_2' and M_3/M_3' are at most of 3-rank 2, the 3-rank of M_1/M_1' is bounded by 2, if $[s_2,y]=1$, and by 3, if $[s_2,x]=1$. The biggest 3-rank 4 can occur for M_4/M_4', if $[s_2,y]=1$, and is bounded by 3, if $[s_2,x]=1$.

Since the source of all transfers $\mathrm{V}_i : G/G' \to M_i/M_i'$ can be represented by the generators as $G/G' = \{x^j y^\ell G' \mid 0 \le j < 9,\ 0 \le \ell < 3\}$, the possible transfer kernels $\ker(\mathrm{V}_i)$ are either of dimension 1 (partial), $\tilde{M}_1/G' = \{y^\ell G' \mid 0 \le \ell < 3\}$, $\varkappa(i) = 1$, or $\tilde{M}_2/G' = \{x^{3\ell} y^\ell G' \mid 0 \le \ell < 3\}$, $\varkappa(i) = 2$, or $\tilde{M}_3/G' = \{x^{-3\ell} y^\ell G' \mid 0 \le \ell < 3\}$, $\varkappa(i) = 3$, or $\tilde{M}_4/G' = \{x^j G' \mid j = 0, 3, 6\}$, $\varkappa(i) = 4$, or of dimension 2 (total), $M_4/G' = \{x^j y^\ell G' \mid j = 0, 3, 6,\ 0 \le \ell < 3\}$, $\varkappa(i) = 0$.

To calculate the punctured transfer kernel type (TKT) $\varkappa(G)$, we need explicit expressions for the transfers $\mathrm{V}_i = \mathrm{V}_{G,M_i}$ from G/G' to the abelianisations of the maximal subgroups M_i/M_i', based on equation (3).

For our fixed arrangement of the maximal subgroups of $G = \langle x, y\rangle$, we have $x \in M_1$ but $x \notin M_2, M_3, M_4$ and $y \in M_4$ but $y \notin M_1, M_2, M_3$. Consequently, the following transfer images are powers, $\mathrm{V}_i(xG') = x^3 M_i'$ for $2 \le i \le 4$ and $\mathrm{V}_i(yG') = y^3 M_i'$ for $1 \le i \le 3$. However, for the remaining transfer images we need a formula for the action of third trace elements as symbolic exponents. According to [15, Thm.3.1,(6)], we have

$$\mathrm{V}_1(xG') = x^{\mathrm{S}_3(y)} M_1' = x^{1+y+y^2} M_1' = x^3 [x,y]^3 [[x,y],y] M_1' = x^3 s_2^{-3} [s_2^{-1}, y] M_1' = x^3 s_2^{-3} [s_2, y]^{-s_2^{-1}} M_1' =$$

$$= \begin{cases} x^3 s_2^{-3} M_1', & \text{if } [s_2, y] = 1, \\ x^3 s_2^{-3} s_3^{-1} M_1', & \text{if } [s_2, x] = 1, \end{cases}$$

and $\mathrm{V}_4(yG') = y^{\mathrm{S}_3(x)} M_4' = y^{1+x+x^2} M_4' = y^3 [y,x]^3 [[y,x],x] M_4' = y^3 s_2^3 [s_2, x] M_4' =$

$$= \begin{cases} y^3 s_2^3 s_3 M_4', & \text{if } [s_2, y] = 1, \\ y^3 s_2^3 M_4', & \text{if } [s_2, x] = 1. \end{cases}$$

Summarised, $\mathrm{V}_i(x^j y^\ell G') = x^{3j} y^{3\ell} M_i'$, if either $2 \le i \le 3$ or $i = 1$, $[s_2, y] = 1$ or $i = 4$, $[s_2, x] = 1$, but exceptionally $\mathrm{V}_1(x^j y^\ell G') = x^{3j} s_3^{-j} y^{3\ell}$, if $[s_2, x] = 1$ and thus $M_1' = 1$, and $\mathrm{V}_4(x^j y^\ell G') = x^{3j} y^{3\ell} s_3^\ell$, if $[s_2, y] = 1$ and thus $M_4' = 1$.

To determine the transfer kernel we have to solve the equation $\mathrm{V}_i(x^j y^\ell G') = 1 \cdot M_i'$ with respect to j and ℓ.

For the standard case this can be done independently from the details of the presentation of the group G. If either $2 \le i \le 3$ or $i = 1$, $[s_2, y] = 1$ or $i = 4$, $[s_2, x] = 1$, then we have uniformly $M_i' = \langle s_3 \rangle = \gamma_3(G)$ and $\mathrm{V}_i(x^j y^\ell G') = x^{3j} y^{3\ell} M_i' = M_i'$, i.e. $x^{3j} y^{3\ell} \in \langle s_3 \rangle$, implies $3 \mid j$ but admits arbitrary ℓ, since $x^9, y^3 \in \langle s_3 \rangle$, in any case. Consequently, $\varkappa(i) = 0$, generally in the standard case.

The exceptional cases, however, depend on the isomorphism class of the group G.

There are 8 isomorphism classes of 2-generator groups $G = \langle x, y \rangle$ in the first branch of Φ_3 and table 5 gives 3 representatives for each isomorphism class in the notation of GAP 4.4 [10], James [12, p.620, 4.5], and Ascione, Havas, Leedham-Green [3, p.272, 7] resp. [1, p.79, Fig.5.4]. A common feature of all 8 isomorphism classes are the relations $s_2 = [y, x]$, $s_2^3 = 1$, $s_3^3 = 1$ and we only give the remaining relations for $[s_2, x]$, $[s_2, y]$, x^9, and y^3.

Table 5. Representatives of the 8 isomorphism classes in branch 1 of Φ_3

GAP 4.4	James	Ascione	$[s_2, x]$	$[s_2, y]$	x^9	y^3
$\langle 243, 20\rangle$	$\Phi_3(31^2)_{b_1}$	B	1	s_3	s_3^{-1}	1
$\langle 243, 19\rangle$	$\Phi_3(31^2)_{b_2}$	C	1	s_3	s_3	1
$\langle 243, 16\rangle$	$\Phi_3(31^2)_a$	F	s_3	1	s_3	s_3^{-1}
$\langle 243, 18\rangle$	$\Phi_3(2^2 1)_a$	D	1	s_3	1	s_3^{-1}
$\langle 243, 14\rangle$	$\Phi_3(2^2 1)_{b_2}$	H	s_3	1	1	s_3
$\langle 243, 13\rangle$	$\Phi_3(2^2 1)_{b_1}$	E	s_3	1	1	1
$\langle 243, 15\rangle$	$\Phi_3(21^3)_d$	G	s_3	1	1	s_3^{-1}
$\langle 243, 17\rangle$	$\Phi_3(21^3)_e$	A	1	s_3	1	1

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Naglergasse 53, 8010 Graz, Austria
E-mail address: `algebraic.number.theory@algebra.at`
URL: `http://www.algebra.at`