Quadratic p-Ring Spaces for Counting Dihedral Fields

Conference:	Workshop International NTCCCS
Venue:	Université Mohammed Premier
	Faculté des Sciences Oujda (FSO)
Place:	Oujda, Maroc
Date:	26 au 28 Avril 2012
Author:	Daniel C. Mayer

Introduction

In this lecture we present improved theoretical foundations admitting the statement of new formulas for the **number**

$$m_p(d, c)$$

of dihedral fields $N|\mathbb{Q}$ of degree $2p$, $p \geq 3$ prime,
sharing a common conductor c
over their common quadratic subfield K
with discriminant d.

$m_p(d, c)$ is called the *p-multiplicity* of c
with respect to d.

FIGURE 1. **Multiplet** $(N_i)_{1\le i\le m}$ Sharing a Common Conductor c

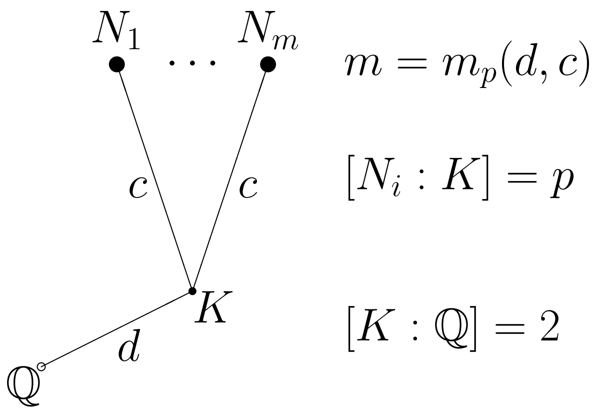


Figure 2. p-**Ring Class Field** $\mathrm{F}_{c,p}(K)$ Modulo c of K

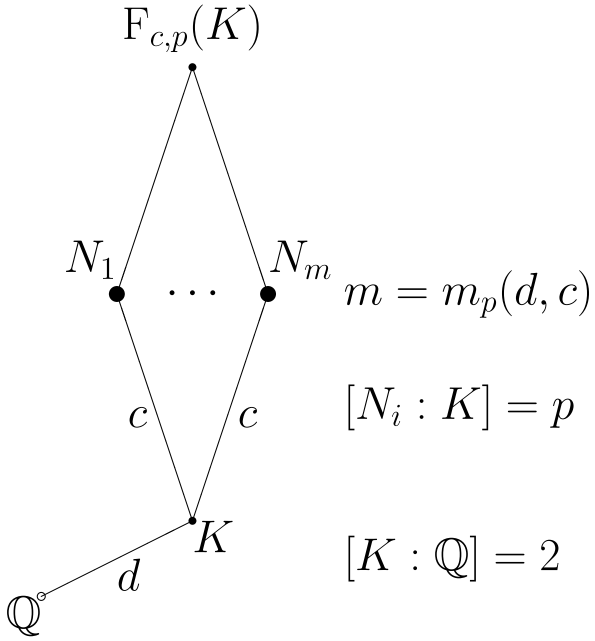
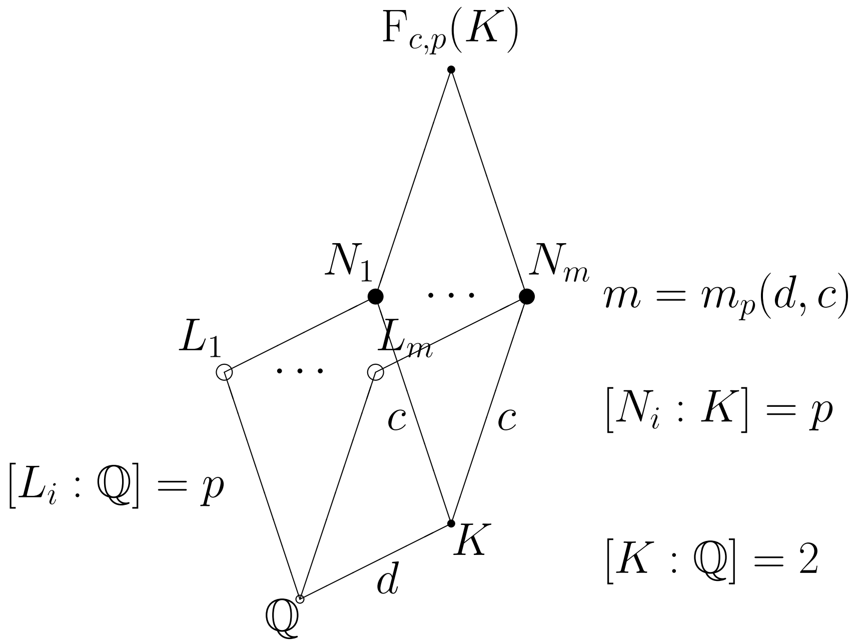


Figure 3. p-Families of Conjugate Non-Galois Subfields $(L_i^{(j)})_{0\le j<p}$, $1 \le i \le m$



1. Quadratic Invariants

Definition 1.1. $K = \mathbb{Q}(\sqrt{d})$ quadratic field, discriminant d, maximal order $\mathcal{O}$, ideal group $\mathcal{I}$, class group Cl, unit group U.

- For a prime p:
$$I_p = \{\alpha \in K \mid \alpha\mathcal{O} = \mathfrak{i}^p \text{ for some } \mathfrak{i} \in \mathcal{I}\}$$
group of *principal pth powers of ideals* of K.
I_p contains the product $U \cdot K^p$ as subgroup and
$I_p/U \cdot K^p \simeq \mathrm{Cl}/\mathrm{Cl}^p$, the p-elementary class group.
- For an integer $c \geq 1$: the subgroup
$$S_c = \{\alpha \in K^\times \mid \alpha \equiv 1 \pmod{c}\}$$
of the multiplicative group $K^\times$
is called *ray* (Strahl) modulo c of K.
- For any system X of numbers or ideals of K, $X(c)$ denotes the elements of X coprime to c.
- The *ring* modulo c of K with generators
$$R_c = \mathbb{Q}(c) \cdot S_c$$
extends the ray S_c.

Definition 1.2. $p \in \mathbb{P}$, $f \in \mathbb{Z}$, $f \geq 1$,
K quadratic field, ϱ_p its p-class rank.

- The vector space
$$V_p = I_p/K^p \simeq I_p(f)/K(f)^p$$
over $\mathbb{F}_p$ is called vector space of *non-trivial principal pth powers of ideals* of K and $V_p \simeq (\mathrm{Cl}/\mathrm{Cl}^p) \times (U/U^p)$.
- Its dimension over $\mathbb{F}_p$ is called *modified p-class rank* σ_p of K.
$\varrho_p \leq \sigma_p \leq \varrho_p + 1$ if $p \geq 3$.
- The subspace
$$V_p(c) \simeq I_p(f) \cap R_c \cdot K(f)^p/K(f)^p,$$
where $c \in \mathbb{Z}$, $c \geq 1$, $c \mid f$,
is called *p-ring space* modulo c of K.
- Its codimension in V_p is called *p-defect* of c,
$$\delta_p(c) = \mathrm{codim}(V_p(c)) = \dim_{\mathbb{F}_p}(V_p/V_p(c))$$

Algorithms for computing principal cubes $\alpha\mathcal{O} = \mathfrak{i}^3$ of ideals.

Example 1.1. *Complex* quadratic field $K = \mathbb{Q}(\sqrt{d})$, discriminant $d = \mathbf{-3\,321\,607}$

- Torsion unit group: $U = TU$, $E = U/TU = 1$
- 3-**Class group**: $\mathrm{Cl}_3(K) \simeq (9,3,3)$, $\varrho_3 = 3$

Reduced positive definite binary quadratic forms of discriminant d under composition:

$F_1 = (94, 27, 8\,836) \to 9\,601 = r_1$,
$F_2 = (128, 53, 6\,493) \to 8\,329 = r_2$,
$F_3 = (152, 69, 5\,471) \to 6\,217 = r_3$,
3 independent forms of order 3

- **Principal ideal cubes**: $I_3 = \langle \alpha_1, \alpha_2, \alpha_3, K^3 \rangle$

Vector space $V_3 = I_3/K^3 \simeq \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, $\sigma_3 = 3$

$\alpha_1 = 603\,433 + 396\sqrt{d}$,
$\alpha_2 = 636\,499 + 228\sqrt{d}$,
$\alpha_3 = 260\,045 + 228\sqrt{d}$,
where $\alpha_i = \frac{1}{2}(x_i + y_i\sqrt{d})$, $\qquad x_i^2 + y_i^2|d| = 4 \cdot r_i^3$

Example 1.2. *Real* quadratic field $K = \mathbb{Q}(\sqrt{d})$,
discriminant $d = \mathbf{214\,712}$, squarefree radicand $D = 53\,678$

• **Unit group**: $U = \langle -1, \eta \rangle$, $E = U/TU \simeq \langle \eta \rangle$
Continued fraction algorithm:
Period length: $\ell = 16 \Longrightarrow \mathrm{N}_{K|\mathbb{Q}}(\eta) = +1$
Fundamental unit: $\eta = 6\,315\,163\,023 + 27\,257\,524\sqrt{D}$
Regulator: $R = \log(\eta) \approx 23.26$

• 3-**Class group**: $\mathrm{Cl}_3(K) \simeq (3,3)$, $\varrho_3 = 2$
Double cycles of reduced indefinite binary quadratic forms of discriminant d under composition:
$F_1 = (7, 454, -307) \to -7 = r_1$,
$F_2 = (13, 440, -406) \to -13 = r_2$,
2 independent forms of order 3

• **Principal ideal cubes**: $I_3 = \langle \alpha_1, \alpha_2, \eta, K^3 \rangle$
Vector space $V_3 = I_3/K^3 \simeq \langle \alpha_1, \alpha_2, \eta \rangle$, $\sigma_3 = 3$
$\alpha_1 = 46 + \sqrt{d}$, $\qquad \alpha_2 = \frac{1}{2}(759\,658 + 1\,661\sqrt{d})$,
where $\alpha_i = \frac{1}{2}(x_i + y_i\sqrt{d})$, $\qquad x_i^2 - y_i^2 d = 4 \cdot r_i^3$

Proposition 1.1. K quadratic field, $p \geq 3$ prime, c, f positive integers, $c \mid f$.

(1) p-rank of the p-elementary subgroup of
$$U(\mathcal{O}/c\mathcal{O}) \Big/ U(\mathbb{Z}/c\mathbb{Z}) \simeq K(f)/R_c$$
is given by
$$\dim_{\mathbb{F}_p}(K(f)/R_c K(f)^p) = t + w.$$

(2) p-rank of the p-elementary subgroup of the **prime residue class group** modulo c of K,
$$U(\mathcal{O}/c\mathcal{O}) \simeq K(f)/S_c$$
is given by
$$\dim_{\mathbb{F}_p}(K(f)/S_c K(f)^p) = t + \tilde{t} + w + \tilde{w}.$$

$\triangleright$ The numbers t, w, $\tilde{t}$, and $\tilde{w}$ are defined as follows:

$t = \#\{q \in \mathbb{P} \setminus \{p\} \mid v_q(c) \geq 1,\ q \equiv \left(\frac{d}{q}\right) \pmod p\}$
denotes the number of essential prime divisors $q \neq p$ of c:
either $q \equiv +1 \pmod p$ and q splits in K
or $q \equiv -1 \pmod p$ and q remains inert in K.
Contribution of p, dependent on decomposition in K:

$$w = \begin{cases} 0 & \text{if } v_p(c) = 0 \\ & \text{or } v_p(c) = 1,\ p \nmid d, \\ 1 & \text{if } v_p(c) \geq 2,\ p \nmid d \\ & \text{or } v_p(c) \geq 1,\ p \mid d,\ p \geq 5 \\ & \text{or } v_p(c) \geq 1,\ p = 3,\ d \equiv +3 \pmod 9 \\ & \text{or } v_p(c) = 1,\ p = 3,\ d \equiv -3 \pmod 9, \\ 2 & \text{if } v_p(c) \geq 2,\ p = 3,\ d \equiv -3 \pmod 9. \end{cases}$$

On the other hand, independently from K,

$$\tilde{t} = \#\{q \in \mathbb{P} \setminus \{p\} \mid v_q(c) \geq 1,\ q \equiv +1 \pmod p\},$$

$$\tilde{w} = \begin{cases} 0 & \text{if } v_p(c) \leq 1, \\ 1 & \text{if } v_p(c) \geq 2. \end{cases}$$

Definition 1.3. c is *p-admissible conductor* over K if

$$c = p^e q_1 \cdots q_t$$

with $t \geq 0$, pairwise distinct primes $q_1, \ldots, q_t \in \mathbb{P} \setminus \{p\}$ such that $q_i \equiv \left(\frac{d}{q_i}\right) \pmod p$, and

$$e \in \begin{cases} \{0,2\} & \text{if } \left(\frac{d}{p}\right) = \pm 1, \\ \{0,1\} & \text{if } p \geq 5,\ p \mid d \\ & \text{or } p = 3,\ d \equiv +3 \pmod 9, \\ \{0,1,2\} & \text{if } p = 3,\ d \equiv -3 \pmod 9). \end{cases}$$

Formally, we write $c = q_1 \cdots q_\tau$,
where $\tau = t$ if $e = 0$
and $\tau = t + 1$, $q_{t+1} = p^e$ if $e \geq 1$.

Theorem 1.1. K quadratic field,
p-class rank ϱ_p, modified p-class rank σ_p,
$p \geq 3$ prime, c, f positive integers, $c \mid f$.

(1) The p-rank of the Galois group $\mathrm{Gal}(\mathrm{F}_{c,p}(K)|K)$ of the p-**ring class field** modulo c of K is given by
$\varrho_{c,p} = \dim_{\mathbb{F}_p}(\mathcal{I}(f)/\mathcal{R}_c\mathcal{I}(f)^p) = \varrho_p + t + w - \delta_p(c)$,
where $\delta_p(c) \leq \min\{t + w, \sigma_p\}$.

(2) The p-rank of the Galois group $\mathrm{Gal}(\tilde{\mathrm{F}}_{c,p}(K)|K)$ of the p-**ray class field** modulo c of K is given by
$\tilde{\varrho}_{c,p} = \dim_{\mathbb{F}_p}(\mathcal{I}(f)/\mathcal{S}_c\mathcal{I}(f)^p) = \varrho_p + t + \tilde{t} + w + \tilde{w} - \tilde{\delta}_p(c)$,
where $\tilde{\delta}_p(c) \leq \min\{t + \tilde{t} + w + \tilde{w}, \sigma_p\}$.

$\triangleright$ Visualization of 2 **Estimates** for $\delta_p(c)$

FIGURE 4. **Defect** $\delta_p(c)$ Connecting V_p and $K(f)/R_cK(f)^p$

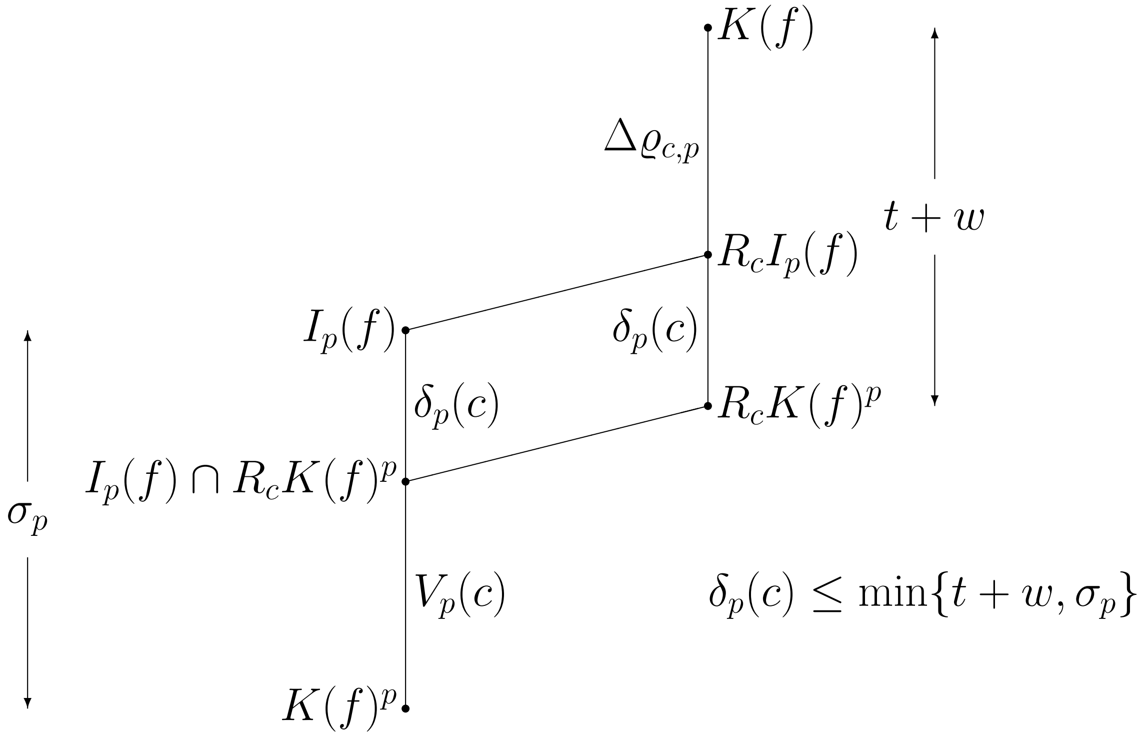
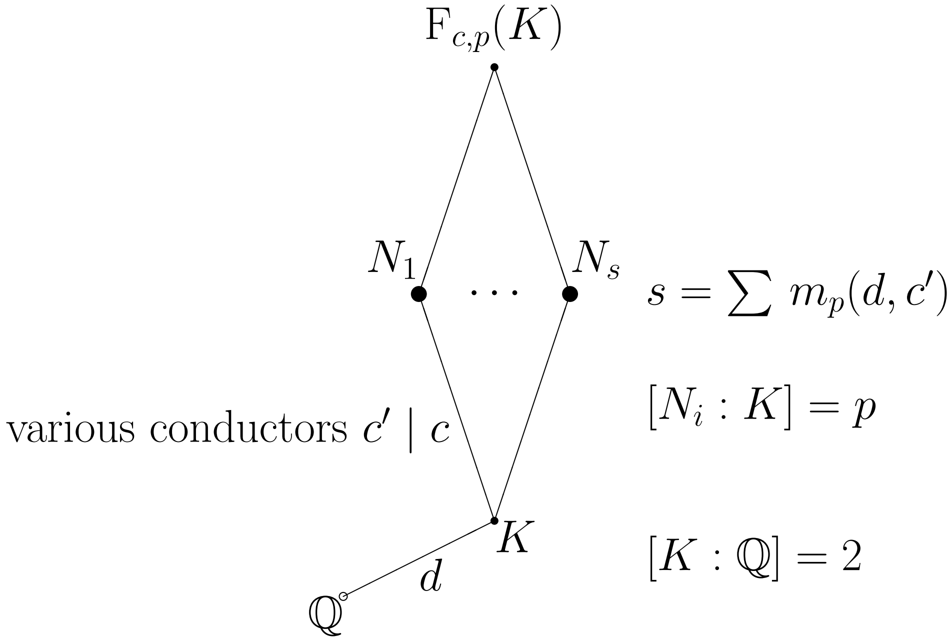


FIGURE 5. Ring Class Field $\mathrm{F}_{c,p}(K)$ as **Inhomogeneous Split Extension**



Additive multiplicity formula

$$s = \sum_{c'|c} m_p(d, c') = \frac{p^{\varrho_{c,p}} - 1}{p - 1}, \text{ where}$$

$$\varrho_{c,p} = \varrho_p + t + w - \delta_p(c).$$

2. Multiplicity formulas

2.1. **Unramified extensions.** Formula (0.0)

Theorem 2.1. If $N|K$ is *unramified* with conductor $c = 1$, then

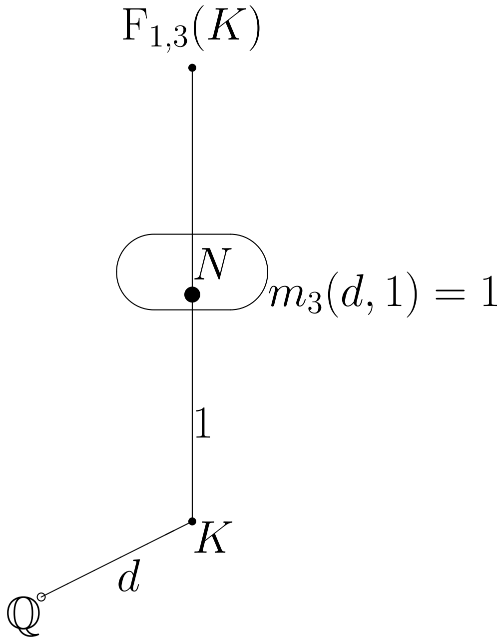
$$m_p(d,1) = \frac{p^{\varrho} - 1}{p - 1} \qquad (0.0)$$

where we denote by
$\varrho = \varrho_p$ the p-class rank of K.

In particular, $m_p(d,1) = 0$ for $\varrho_p = 0$ only.

$\triangleright$ **Frequency** and **Minimal Occurrence**
for
159 682 simply real cubic discriminants $-10^6 < d_L < 0$ (SRC)
and
9 842 totally real cubic discriminants $0 < d_L < 2 \cdot 10^5$ (TRC)

FIGURE 6. **Non-Split** Extension: 1 Singulet

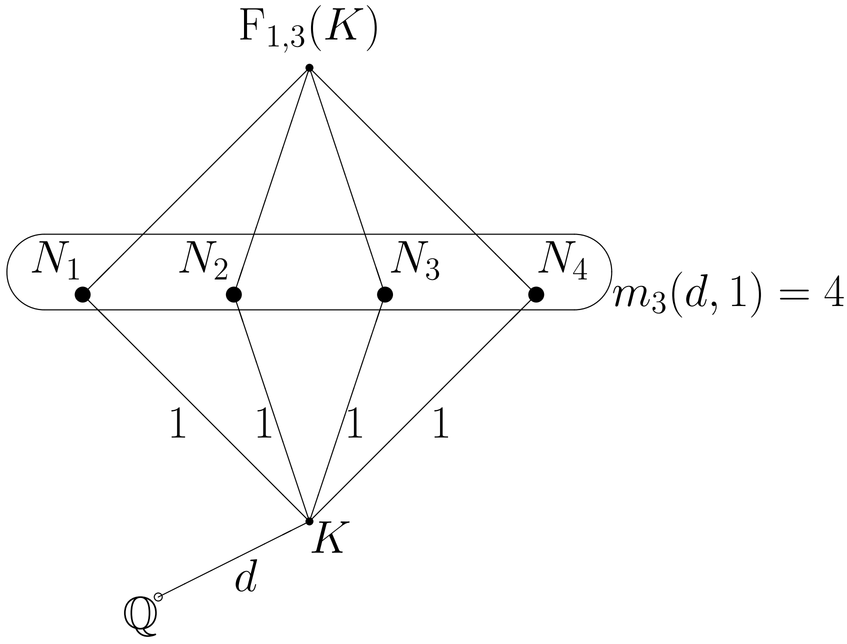


Invariants: $p = 3$, $\varrho_3 = 1$, $1 \le \sigma_3 \le 2$, $c = 1$

Frequency: $\frac{118\,455}{159\,682} \approx 74\%$ complex, $\frac{6\,924}{9\,842} \approx 70\%$ real

Minima: $d = -23$ complex, $d = 229$ real

Figure 7. **Homogeneous Split** Extension: 1 Quartet



Invariants: $p = 3, \quad \varrho_3 = 2, \quad 2 \le \sigma_3 \le 3, \quad c = 1$
Frequency: $\frac{3\,190}{159\,682} \approx 2\%$ complex, $\frac{16}{9\,842} \approx 0.2\%$ real
Minima: $d = -3\,299$ complex, $d = 32\,009$ real

2.2. **Special types of conductors.**

Definition 2.1. Let $c = p^e q_1 \cdots q_t$ be p-admissible over K.

- With respect to the p-contribution:
  c is *irregular* if $p = 3$, $e = 2$, and $d \equiv -3 \pmod 9$,
  marked by an irregularity flag $\omega = 1$.
  Otherwise, c is *regular* and we put $\omega = 0$.
- With respect to the p-defect:
  c is *free* if $\delta_p(c) = 0$ (in the sense of restriction-free)
  and *restrictive* if $\delta_p(c) \geq 1$.
  An irregular c is *tamely irregular* if $\delta_3(3) = 0$,
  and *wildly irregular* if $\delta_3(3) = 1$.
- With respect to the p-multiplicity:
  c is *capable* if $m_p(d, c) \geq 1$ (gives rise to extensions),
  and *disabled* if $m_p(d, c) = 0$.

2.3. **Free conductors.** Formula (0.1)

Theorem 2.2. If c is *free* with p-defect $\delta_p(c) = 0$, then

$$m_p(d, c) = p^{\varrho} \cdot p^{\omega} \cdot (p - 1)^{\tau - 1} \qquad (0.1)$$

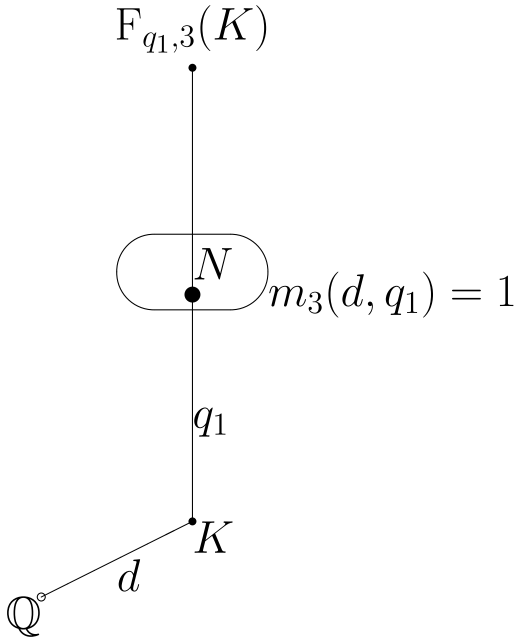
where we denote by
ϱ the p-class rank of K,
ω the irregularity flag of c,
$\tau \geq 1$ the number of all prime divisors of c.

In particular, a free conductor is certainly capable.

Corollary 2.1. $p \geq 3$ odd prime,
$K = \mathbb{Q}(\sqrt{d})$ complex quadratic field, discriminant $d < 0$,
modified p-class rank $\sigma_p = 0$,
$c = p^e q_1 \cdots q_t$ p-admissible.
Then c is free, $\delta_p(c) = 0$ and $m_p(d, c)$ is given by formula (0.1) in Theorem 2.2.

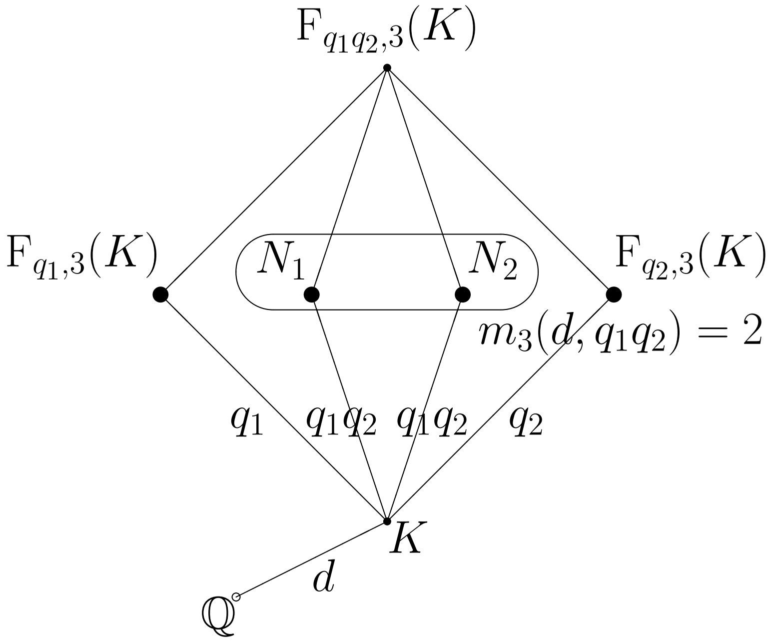
$\triangleright$ **Frequency** and **Minimal Occurrence**
for
159 682 simply real cubic discriminants $-10^6 < d_L < 0$ (SRC)
and
9 842 totally real cubic discriminants $0 < d_L < 2 \cdot 10^5$ (TRC)

FIGURE 8. Ramified **Non-Split** Extension: 1 Singulet



Invariants: $p = 3$, $\varrho_3 = 0$, $0 \le \sigma_3 \le 1$, $c = q_1$, $\delta_3(q_1) = 0$
Frequency: $\frac{30\,593}{159\,682} \approx 19\%$ complex, $\frac{2\,440}{9\,842} \approx 25\%$ real
Minima: $d = -44$ complex, $d = 148$ real

Figure 9. **Inhomogeneous Split** Extension: 2 Singulets, 1 Doublet



Invariants: $p = 3$, $\varrho_3 = 0$, $0 \le \sigma_3 \le 1$, $c = q_1q_2$, $\delta_3(q_i) = 0$

Frequency: $\frac{1\,640}{159\,682} \approx 1\%$ complex, $\frac{12}{9\,842} \approx 0.1\%$ real

Minima: $d = -1\,836$ complex, $d = 37\,300$ real

m	ρ_K	τ	u	v	ω	$\delta(3)$	complex	pure	subtotal
1									149 204
	1	0					118 455	—	
	0	1	1	0	0		30 559	34	
	0	2	0	2	0		—	111	
	0	3	0	3	0		—	44	
	0	1	0	1	1	1	—	1	
2									1 683
	0	2	2	0	0		1 639	1	
	0	3	1	2	0		—	24	
	0	4	1	3	0		—	2	
	0	2		≥ 1	1	1	—	17	
3									5 510
	0	4	0	4	0		—	5	
	0	1	1	0	1	0	311	—	
	0	2	0	2	1	0	—	—	
	0	3	0	3	1	0	—	—	
	1	1	1	0	0		4 608	—	
	1	2	0	2	0		464	—	
	1	3	0	3	0		2	—	
	1	1	0	1	1	1	120	—	

TABLE 1. Count of various multiplets of dihedral fields with $-10^6 < c^2 d < 0$

$p = 3$				$p = 5$		
$m_3(d,c)$	$c = 1$	$c > 1$	$d = -3$	$m_5(d,c)$	$c = 1$	$c > 1$
1	118455	30559	190	1	69365	3887
2	0	1639	44	4	0	20
4	3190	15	11	6	398	0

The count of 149204 cubic singulets revealed the omission of 495 complex cubic fields of multiplicity 1 in the table of Fung/Williams (1990), corrected later (1994) by Williams. These 149204 cubic fields form 81.79% among a total of 182417.

TABLE 2. Seven fake quintuplets in the table of Fung/Williams – actually sextuplets

d	ϱ_3	c	$d_L = c^2 d$	$m_3(d,c)$	missing regulator R_L
-291	0	18	-94284	6	25.64
-1299	0	18	-420876	6	20.54
-1659	0	18	-537516	6	24.58
-1947	0	18	-630828	6	38.72
-2307	0	18	-747468	6	17.96
-2667	0	18	-864108	6	31.18
-1371	1	18	-444204	6	25.93

7 sextuplets of complex cubic fields L, discriminant $d_L = c^2 d$, erroneously announced as quintuplets by Fung /Williams.
Discriminants of the complex quadratic base fields K:
$d \equiv 5 \pmod 8$ and $d \equiv -3 \pmod 9$.
All L share the common irregular conductor $c = 2 \cdot 3^2 = 18$.
Multiplicity $m_3(d,c) = 3 \cdot 2 = 6$ of the first 6 cases, where K has 3-class rank $\varrho = 0$, simply by Hasse's formula (0.1), putting $\omega = 1$ and $\tau = 2$.
The single case $d = -1371$ with $\varrho = 1$ and $\delta_3(3) = 1$ is wildly irregular and thus accessible by formula (1.2).

2.4. **Restrictive conductors.** Formula (1.1)

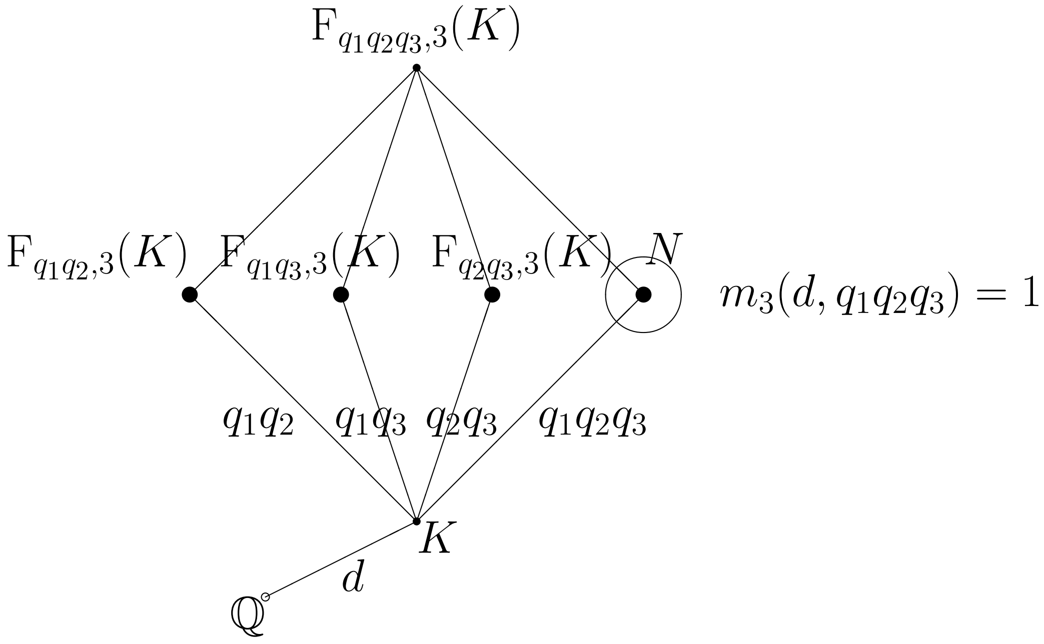
Theorem 2.3. If c is *restrictive* with p-defect $\delta_p(c) = 1$, then

$$m_p(d,c) = p^{\varrho} \cdot p^{\omega} \cdot (p-1)^{u} \cdot \frac{1}{p}[(p-1)^{v-1} - (-1)^{v-1}] \qquad (1.1)$$

where we denote by
ϱ the p-class rank of K,
ω the irregularity flag of c,
u the number of free prime divisors of c,
v the number of restrictive prime divisors of c.

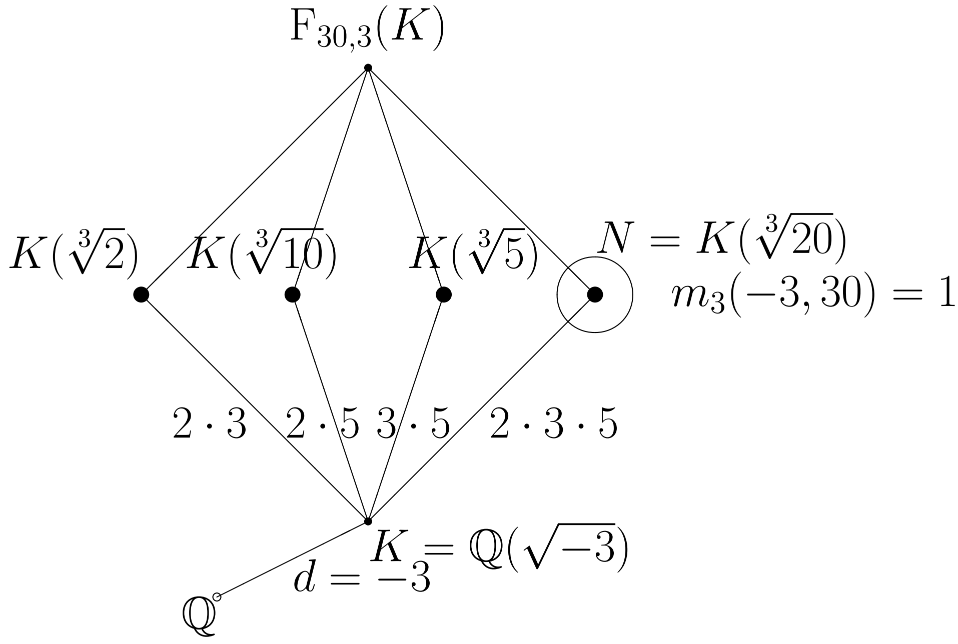
In particular, a disabled conductor must be restrictive, however, a restrictive conductor may be capable.

FIGURE 10. **Inhomogeneous** Split Extension: $3 + 1$ Singulets



Invariants: $p = 3$, $\varrho_3 = 0$, $\sigma_3 = 1$, $c = q_1q_2q_3$, $\delta_3(q_i) = 1$
Frequency: $\frac{44}{159\,682}$ complex, $\frac{2}{9\,842}$ real
Minima: $d = -2\,700$ complex, $d = 91\,476$ real

FIGURE 11. Realization by **Pure Cubic** Fields



Invariants: $\varrho_3 = 0$, $\sigma_3 = 1$, $c = q_1 q_2 q_3 = 2 \cdot 3 \cdot 5$, $\delta_3(q_i) = 1$
Discriminants: $d = -108, -300, -675, -2\,700$
Dedekind Types: I, II, I, I

2.5. **Regular and tamely irregular conductors.**

Theorem 2.4. If c is **restrictive** with p-defect $\delta_p(c) = 2$, then $m_p(d, c) =$

$$p^{\varrho} \cdot p^{\omega} \cdot (p-1)^{u} \cdot \frac{1}{p^2}[(p-1)^{v-1} - \sum_{i=1}^{p+1} (-1)^{v-n_i}(p-1)^{n_i}] \qquad (2.1)$$

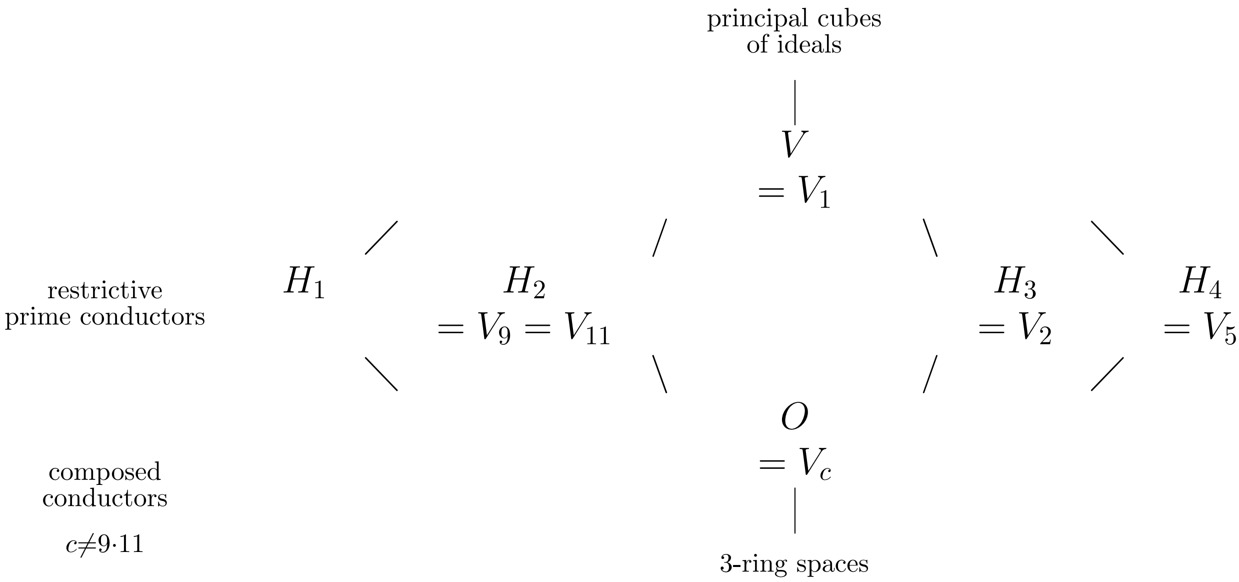
where we denote by
ϱ the p-class rank of K,
ω the irregularity flag of c,
u the number of free prime divisors of c,
v the number of restrictive prime divisors of c,
n_i the occupation number of the hyperplane
$V_p(c) < H_i < V_p$, $1 \leq i \leq p+1$,
by restrictive prime divisors of c.

Actual occurrence of a multiplet of 9 complex cubic fields L with discriminant $d_L = c^2 d$ and 3-defect $\delta_3(c) = 2$:
the smallest case we found (not necessarily minimal) is $d_L = -32\,618\,700$ with associated $d = -4027$, $\varrho_3 = 2$. Obviously, $\alpha_2 \in \mathcal{O}_5$, $\alpha_3 \in \mathcal{O}_9$, $\alpha_4 \in \mathcal{O}_2$, and the occupation numbers of the four hyperplanes $H_i < V_3$ with respect to the capable restrictive 3-admissible conductor $c = 3^2 \cdot 2 \cdot 5 = 90$ are $(n_1, n_2, n_3, n_4) = (0, 1, 1, 1)$, whence formula (2.1) yields $m_3(-4027, 90) = 9$.

TABLE 3. Small example of a capable restrictive conductor c with $\delta_3(c) = 2$

d	$F_i = (A, B, C)$	r_i	(x_i, y_i)	c	$d_L = c^2 d$	$m_3(d, c)$
-4027	$(13, 9, 79)$	13	$(69, 1)$	90	$-32\,618\,700$	9
	$(17, 11, 61)$	61	$(43, 15)$			
	$(19, 1, 53)$	19	$(153, 1)$			
	$(29, -27, 41)$	43	$(416, 6)$			

$d=-4\,027$	q	F_1	$F_2=F_1F_4$	$F_3=F_1F_4^2$	F_4	i
$(F_1,\ldots,F_4)$		$(13,9,79)$	$(19,1,53)$	$(29,-27,41)$	$(17,11,61)$	
$(r_1,\ldots,r_4)$		13	19	43	61	
$(\alpha_1,\ldots,\alpha_4)$		$(69,1)$	$(153,1)$	$(416,6)$	$(43,15)$	
		local matrix				
$d\equiv 5\pmod 8$	2	0	0	+1	0	3
$(d/5)=-1$	5	0	0	0	+1	4
	11	0	+1	0	0	2
$d\equiv -1\pmod 3$	3^2	0	+1	0	0	2


principal cubes of ideals
V
$= V_1$
restrictive prime conductors
H_1
H_2
$= V_9 = V_{11}$
H_3
$= V_2$
H_4
$= V_5$
O
$= V_c$
composed conductors
$c \neq 9 \cdot 11$
3-ring spaces

c	τ	$\delta_3(c)$	$m_3(d,c)$	$(n_1,\dots,n_4)$	c^2d
1	0	0	4	$(0,0,0,0)$	$-4\,027$
$2, 5, 3^2, 11$	1	1	0	$(1,0,0,0)$	
$2\cdot 5, 2\cdot 3^2, 2\cdot 11, 5\cdot 3^2, 5\cdot 11$	2	2	0	$(1,1,0,0)$	
$3^2\cdot 11$	2	1	9	$(2,0,0,0)$	$-39\,468\,627$
$2\cdot 3^2\cdot 11, 5\cdot 3^2\cdot 11$	3	2	0	$(2,1,0,0)$	
$2\cdot 5\cdot 3^2$	3	2	9	$(1,1,1,0)$	$-32\,618\,700$
$2\cdot 5\cdot 11$	3	2	9	$(1,1,1,0)$	$-48\,726\,700$
$2\cdot 5\cdot 3^2\cdot 11$	4	2	9	$(2,1,1,0)$	$-3\,946\,862\,700$

2.6. **Wildly irregular conductors.**

Theorem 2.5. If $p = 3$, $d \equiv -3 \pmod 9$, $\delta_3(3) = 1$, and $c = 3^2 q_1 \cdots q_t$ is **restrictive** with 3-defect $\delta_3(c) = 2$, then $m_3(d, c) =$

$$= 3^{\varrho} \cdot 2^{u+n} \cdot \frac{1}{3}\left(2^{v-n-1} - (-1)^{v-n-1}\right) \qquad (2.2)$$

where we denote by
ϱ the 3-class rank of K,
u the number of free prime divisors of c,
v the number of restrictive prime divisors of c,
n the occupation number of the hyperplane
$V_3(c) < H = V_3(3) < V_3$
by restrictive prime divisors of c.

Survey of minimal discriminants with irregular prime power conductor $c = 3^2$ in the case $p = 3$, $d \equiv -3 \pmod 9$.

Table 4. Complex and real quadratic fields admitting an irregular conductor $c = 9$

d	ϱ_3	σ_3	$\delta_3(3)$	$\delta_3(9)$	type	c	$d_L = c^2 d$	$m_3(d,c)$
-3	0	1	1	1	wild	9	-243	1
-39	0	0	0	0	free	9	-3159	3
-255	1	1	1	1	wild	9	-20655	3
-687	1	1	0	1	tame	9	—	0
-3387	1	1	0	0	free	9	-274347	9
-8751	2	2	1	2	rst.	9	—	0
-42591	2	2	1	1	wild	9	-3449871	9
-128451	2	2	0	1	tame	9	—	0
-2069688	2	2	0	0	free	9	-167644728	27
-4447704	3	3	1	2	rst.	9	—	0
24	0	1	1	1	wild	9	1944	1
69	0	1	0	1	tame	9	—	0
717	0	1	0	0	free	9	58077	3

2.7. **Regular and tamely irregular conductors.**

Theorem 2.6. If c is **restrictive** with p-defect $\delta_p(c) = 3$, then $m_p(d,c) =$

$$p^{\varrho}\cdot p^{\omega}\cdot(p-1)^{u}\cdot\frac{1}{p^3}[(p-1)^{v-1}-\sum_{i=1}^{p^2+p+1}(-1)^{v-b_i}(p-1)^{b_i}] \qquad (3.1)$$

where we denote by
ϱ the p-class rank of K,
ω the irregularity flag of c,
u the number of free prime divisors of c,
v the number of restrictive prime divisors of c,
b_i the occupation number of the hyperplane bundle
over the subspace $V_p(c) < G_i < V_p$, $1 \le i \le p^2+p+1$,
by restrictive prime divisors of c.

2.8. **Trichotomy of multiplicity formulas.** A common feature of all multiplicity formulas, except formula (0.0), is the **Trichotomy**

$$m_p(d,c) = U(\varrho) \cdot F(\omega, u) \cdot R(v, \ldots)$$

into a product of three components,

(1) **unramified** contribution $U(\varrho) = p^{\varrho}$,
(2) **free** contribution $F(\omega, u) = p^{\omega} \cdot (p-1)^u$, and
(3) **restrictive** contribution (dependent on $0 \le \delta_p(c) \le 3$)

$$R() = \frac{1}{p-1} \qquad (0.1)$$

$$R(v) = \frac{1}{p}[(p-1)^{v-1} - (-1)^{v-1}] \qquad (1.1)$$

$$R(v, n_i) = \frac{1}{p^2}[(p-1)^{v-1} - \sum_{i=1}^{p+1} (-1)^{v-n_i}(p-1)^{n_i}] \qquad (2.1)$$

$$R(v, b_i) = \frac{1}{p^3}[(p-1)^{v-1} - \sum_{i=1}^{p^2+p+1} (-1)^{v-b_i}(p-1)^{b_i}] \qquad (3.1)$$

References.

[1] D. C. Mayer,
Lattice minima and units in real quadratic number fields,
Publ. Math. Debrecen **39** (1991), no. 1–2, 19–86.

[2] D. C. Mayer,
Multiplicities of dihedral discriminants
(1990 Westcoast Number Theory Conference,
Asilomar Conference Grounds, Pacific Grove, Monterey, California, USA, Dec. 1990).

[3] D. C. Mayer,
Multiplicities of dihedral discriminants,
Math. Comp. **58** (1992), no.198, 831–847 and S55–S58.

[4] D. C. Mayer,
Dihedral fields from quadratic invariants
(1992 Westcoast Number Theory Conference,
Oregon State University, Corvallis, Portland, Oregon, USA, Dec. 1992).

[5] D. C. Mayer,
Diskriminanten-Vielfachheiten von p-Ring-Klassenkörpern über quadratischen Körpern mit modifiziertem p-Klassenrang $\sigma \geq 2$
(Fünfzehnter Kongress der ÖMG und der DMV,
Univ. Wien, Vienna, Austria, Sep. 2001).