MODELING ROOTED IN-TREES BY FINITE p-GROUPS

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Abstract. Graph theoretic foundations for a kind of infinite rooted in-trees $\mathcal{T}(R) = (V, E)$ with root R, weighted vertices $v \in V$, and weighted directed edges $e \in E \subset V \times V$ are described. Vertex degrees $\deg(v)$ are always finite but the trees contain infinite paths $(v_i)_{i \geq 0}$. A concrete group theoretic model of the rooted in-trees $\mathcal{T}(R)$ is introduced by representing vertices by isomorphism classes of finite p-groups G, for a fixed prime p, and directed edges by epimorphisms $\pi : G \to \pi G$ of finite p-groups with characteristic kernels $\ker(\pi)$. The weight of a vertex G is realized by its nuclear rank $n(G)$ and the weight of a directed edge π is realized by its step size $s(\pi) = \log_p(\# \ker(\pi))$. These invariants are essential for understanding the phenomenon of *multifurcation*. Since the structure of the rooted in-trees under investigation is rather complex, pattern recognition methods are used for finding finite subgraphs which repeat indefinitely. Several periodicities admit the reduction of the complete infinite graph to finite patterns. The proof is based on infinite limit groups and successive group extensions. It is underpinned by several explicit algorithms. Additionally, independent component analysis is employed for obtaining a graph dissection into pruned subtrees. As a final application, it is shown that *fork topologies*, arising from repeated multifurcations, provide a convenient description of very complex navigation paths through the trees, which are of the greatest importance for recent progress in determining p-class field towers of algebraic number fields.

1. Introduction

In section 2 we describe the abstract graph theoretic foundations for a kind of infinite rooted in-trees $\mathcal{T}(R) = (V, E)$ with root R, weighted vertices $v \in V$, and weighted directed edges $e \in E \subset V \times V$, which are suited perfectly for describing the crucial phenomenon of *multifurcation* in § 2.3. The vertex degrees $\deg(v)$ are always finite but the trees contain infinite paths $(v_i)_{i \geq 0}$. In section 3 we introduce a group theoretic model of the rooted in-trees $\mathcal{T}(R)$. Vertices are represented by isomorphism classes of finite p-groups G, for a fixed prime number p. Directed edges are represented by epimorphisms $\pi : G \to \pi G$ of finite p-groups with characteristic kernels $\ker(\pi)$. The weight of a vertex G is realized by its *nuclear rank* $n(G)$ and the weight of a directed edge π is realized by its *step size* $s(\pi) = \log_p(\# \ker(\pi))$. Since the structure of our rooted in-trees is rather complex, we use *pattern recognition* methods in § 3.1 for finding finite subgraphs which repeat indefinitely as branches of coclass subtrees, thus giving rise to a *first periodicity*. Additionally, we employ *independent component analysis* for obtaining a graph dissection into pruned subtrees, either by Galois action in § 3.2.1 or by Artin transfers in § 3.2.2. A *second periodicity* of pruned coclass subtrees eventually admits the reduction of the complete infinite graph $\mathcal{T}(R)$ to finite patterns. Evidence of these newly discovered *periodic bifurcations* is provided by a mixture of bottom up techniques, using successive extensions by means of p-covering groups in the *p-group generation algorithm*, and top down techniques using *infinite limits* and their *finite quotients* in §§ 3.4 and 3.5 As a coronation of this chapter, we show in § 3.6 and section 4 that *fork topologies* provide a convenient description of very complex navigation paths through the trees, arising from repeated multifurcations, which are of the greatest importance for recent progress in determining *p-class field towers* $F_p^{(\infty)}$ of algebraic number fields F.

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2. Underlying abstract graph theory

Let $\mathcal{G} = (V, E)$ be a *graph* with set of *vertices* V and set of *edges* E. We expressly admit infinite sets V and E, but we assume that the in- and out-degree of each vertex is finite.

2.1. **Directed edges and paths.** In this chapter, we shall be concerned with *directed graphs* (digraphs) whose edges are rather ordered pairs $(v_1, v_2) \in V \times V$ than only subsets $\{v_1, v_2\} \subset V$ with two elements. Such a *directed edge* $e = (v_1, v_2)$ is also denoted by an arrow $v_1 \to v_2$ with starting vertex v_1 and ending vertex v_2. Thus, we have $E \subset V \times V$. Now, infinitude comes in.

Definition 2.1. (Finite and infinite paths.)
A *finite path* of *length* $\ell \geq 0$ in $\mathcal{G}$ is a finite sequence $(v_i)_{0\leq i\leq \ell}$ of vertices $v_i \in V$ such that $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq \ell - 1$. We call v_0, resp. v_ℓ, the starting vertex, resp. ending vertex, of the path. The degenerate case of a single vertex (v_0) is called a *point path* of length $\ell = 0$.

An *infinite path* in $\mathcal{G}$ is an infinite sequence $(v_i)_{i\geq 0}$ of vertices $v_i \in V$ such that $(v_{i+1}, v_i) \in E$ for all $i \geq 0$. In this case, v_0 is the ending vertex of the path, and there is no starting vertex.

2.2. **Rooted in-trees with parent operator.** Our attention will even be restricted to *rooted in-trees* $\mathcal{G} = \mathcal{T}(R)$, that is, connected digraphs without cycles such that the *root vertex* R has out-degree 0 whereas any other vertex $v \in V \setminus \{R\}$ has out-degree 1. A vertex with in-degree at least 1 is called *capable* whereas a vertex with in-degree 0 is called a *leaf*. For a rooted in-tree we can define the parent operator as follows.

Definition 2.2. Let $\mathcal{T}(R) = (V, E)$ be a rooted in-tree. Then the mapping $\pi : V \setminus \{R\} \to V$, $v \mapsto \pi v$, where $(v, \pi v) \in E$ is the unique edge with starting vertex v, is called the *parent operator* of $\mathcal{T}(R)$. For each vertex $v \in V$, there exists a unique finite *root path* from v to the root R,

$$v = \pi^0 v \to \pi^1 v \to \pi^2 v \to \ldots \to \pi^{\ell-1} v \to \pi^\ell v = R,$$

expressed by iterated applications of the parent operator, and with some *length* $\ell \geq 0$. Each vertex in the root path of v is called an *ancestor* of v.

The *descendant tree* $\mathcal{T}(a) = (V(a), E(a))$ of a vertex $a \in V$ is the subtree of $\mathcal{T}(R) = (V, E)$ consisting of vertices v with ancestor a, that is $v \in V(a) := \{u \in V \mid (\exists\, j \geq 0)\ \pi^j u = a\}$, and edges $e \in E(a) := E \bigcap (V(a) \times V(a))$.

A vertex $u \in V$ is called an *immediate descendant* (or *child*) of a vertex $a \in V$, if there exists a directed edge $(u, a) \in E$. In this case, $a = \pi u$ is necessarily the *parent* of u .

We can define a *partial order* on the vertices $u, a \in V$ of the tree $\mathcal{T}(R)$ by putting $u \geq a$ if $u \in \mathcal{T}(a)$, that is, if u is descendant of a, and a is ancestor of u. The root R is the minimum.

The root R is always a common ancestor of two vertices $u, v \in V$. By the *fork* of u and v we understand their biggest common ancestor, denoted by $\mathrm{Fork}(u, v)$, which admits a measure.

Definition 2.3. (Vertex distance.) The sum $\ell_u + \ell_v$ of the path lengths from two vertices $u, v \in V$ to their fork is called the *distance* $d(u, v)$ of the vertices.

2.3. **Mainlines and multifurcation.** We shall also need *weight functions* with non-negative integer values for vertices $w_V : V \to \mathbb{N}_0$, and with positive integer values for edges $w_E : E \to \mathbb{N}$. In particular, the sets of vertices and edges have disjoint partitions

$$\begin{aligned} V &= \dot{\bigcup}_{n\geq 0} V_n \text{ with } V_n := \{v \in V \mid w_V(v) = n\} \text{ for } n \geq 0, \\ E &= \dot{\bigcup}_{s\geq 1} E_s \text{ with } E_s := \{e \in E \mid w_E(e) = s\} \text{ for } s \geq 1, \end{aligned} \tag{2.1}$$

such that V_0 is precisely the *set of leaves* of the tree $\mathcal{T}(R)$. Thus, there arise weighted measures.

Definition 2.4. (Path weight and weighted distance.)
By the *path weight* of a finite path $(v_i)_{0\leq i\leq \ell}$ with length $\ell \geq 0$ in $\mathcal{T}(R)$ such that $(v_i, v_{i+1}) \in E_{s_i}$ for $0 \leq i \leq \ell - 1$ we understand the sum $\sum_{i=0}^{\ell-1} s_i$. The sum $w_u + w_v$ of the path weights from two vertices $u, v \in V$ to their fork is called the *weighted distance* $w(u, v)$ of the vertices.

In Definition 2.5 and 2.6, some concepts are introduced using the minimal possible weight.

Definition 2.5. (Mainlines and minimal trees.) An infinite path $(v_i)_{i\ge 0}$ in $\mathcal{T}(R)$ with edges of weight 1, that is, such that $(v_{i+1}, v_i) \in E_1$ for all $i \ge 0$, is called a *mainline* in $\mathcal{T}(R)$.

The *minimal tree* $\mathcal{T}_1(a) = (V_1(a), E_1(a))$ of a vertex $a \in V$ is the subtree of the descendant tree $\mathcal{T}(a) = (V(a), E(a))$ consisting of vertices v, whose root path in $\mathcal{T}(a)$ possesses edges e of weight 1 only, that is $v \in V_1(a) := \{u \in V(a) \mid (\forall\, 0 \le j < \ell)\ (\pi^j u, \pi^{j+1} u) \in E_1\}$, and edges $e \in E_1(a) := E(a) \bigcap (V_1(a) \times V_1(a))$.

Definition 2.6. (Branches.) Let $(v_i)_{i\ge 0}$ be a mainline in $\mathcal{T}(R)$. For $i \ge 0$, the difference set $\mathcal{B}(v_i) := \mathcal{T}_1(v_i) \setminus \mathcal{T}_1(v_{i+1})$ of minimal trees is called the *branch* with root v_i of the minimal tree $\mathcal{T}_1(v_0)$. The branches give rise to a disjoint partition $\mathcal{T}_1(v_0) = \dot{\bigcup}_{i\ge 0} \mathcal{B}(v_i)$.

Finally, we complete our abstract graph theoretic language by considering arbitrary weights.

Definition 2.7. (Multifurcation.)
Let $n \ge 2$ be a positive integer. A vertex $a \in V_n$ has an n-fold *multifurcation* if its in-degree is an n-fold sum $N_1 + N_2 + \ldots + N_n$ due to $N_s \ge 1$ incoming edges of weight s, for each $1 \le s \le n$. That is, we define counters N_s of all incoming edges of weight s, and additionally, we have counters C_s of all incoming edges of weight s with capable starting vertex,

$$\begin{aligned} N_s &:= N_s(a) := \#\{e \in E_s \mid e = (u, a) \text{ for some } u \in V\}, \\ C_s &:= C_s(a) := \#\{e \in E_s \mid e = (u, a) \text{ for some } u \in V \text{ with } w_V(u) \ge 1\}. \end{aligned} \tag{2.2}$$

We also define an ordering and a notation [1] for immediate descendants of a by writing $a - \#s; i$ for the ith immediate descendant with edge of weight s, where $1 \le s \le n$ and $1 \le i \le N_s$.

3. Concrete model in p-group theory

Now we introduce a group theoretic model of the rooted in-trees $\mathcal{T}(R) = (V, E)$ in § 2. Vertices $v \in V$ are represented by isomorphism classes of finite p-groups G, for a fixed prime number p. Directed edges $e \in E$ are represented by epimorphisms $\pi : G \to \pi G$ of finite p-groups with characteristic kernels $\ker(\pi) = \gamma_c G$, where $c := \mathrm{cl}(G)$ denotes the nilpotency class of G and $(\gamma_i G)_{i\ge 1}$ is the lower central series of G.

We emphasize that the symbol π is used now intentionally for two distinct mappings, the abstract parent operator $\pi : V \setminus \{R\} \to V$, $v \mapsto \pi v$, in Definition 2.2, and the concrete natural projection onto the quotient $\pi : G \to \pi G \simeq G/\gamma_c G$, $g \mapsto \pi(g) = g \cdot \gamma_c G$, for each individual vertex $G = v \in V \setminus \{R\}$, which should precisely be denoted by $\pi = \pi_G$, but we omit the subscript, since there is no danger of misinterpretation. In both views, πG is the parent of G.

The weight of a vertex G is realized by its *nuclear rank* $n(G)$ [2, § 14, eqn. (28), p. 178] and the weight of a directed edge $\pi : G \to \pi G$ is realized by its *step size* $s(\pi) = \log_p(\#\gamma_c G)$ [2, § 17, eqn. (33), p. 179]. These invariants are essential for understanding the phenomenon of *multifurcation* in Definition 2.7. In particular, we can hide multifurcation by restricting all edges π to step size $s(\pi) = 1$, that is, by considering the minimal tree $\mathcal{T}_1(v)$ instead of the entire descendant tree $\mathcal{T}(v)$ of a vertex $v \in V$. In our concrete p-group theoretic model, all vertices G of a minimal tree share a common *coclass*, which is the additive complement $\mathrm{cc}(G) := \mathrm{lo}(G) - \mathrm{cl}(G)$ of the (nilpotency) class $c = \mathrm{cl}(G)$ with respect to the *logarithmic order* $\mathrm{lo}(G) := \log_p(\mathrm{ord}(G))$ of G. Generally, the logarithmic order of an immediate descendant G with parent πG increases by the step size, $\mathrm{lo}(G) = \mathrm{lo}(\pi G) + s(\pi)$, since $\log_p(\#\pi G) = \log_p(\#(G/\ker \pi)) = \log_p(\#G) - \log_p(\#\ker \pi)$. Consequently, the coclass remains fixed in a minimal tree with $s(\pi) = 1$, since

$$\mathrm{cc}(G) = \mathrm{lo}(G) - \mathrm{cl}(G) = \mathrm{lo}(\pi G) + 1 - (\mathrm{cl}(\pi G) + 1) = \mathrm{lo}(\pi G) - \mathrm{cl}(\pi G) = \mathrm{cc}(\pi G).$$

A minimal tree $\mathcal{T}_1(G)$ which contains a unique infinite mainline is called a *coclass tree*. It is denoted by $\mathcal{T}^{(r)}(G) := \mathcal{T}_1(G)$ when its root G is of coclass $r := \mathrm{cc}(G)$. For further details, see [2, § 5, p. 164].

In view of the principal goals of this chapter, we must specify our intended situation even more concretely. We put $p := 3$, the smallest odd prime number, and we select as the root either $R := \langle 243, 6\rangle$ or $R := \langle 243, 8\rangle$, characterized by its SmallGroup identifier [3]. These are metabelian 3-groups of order $\#R = 243 = 3^5$, logarithmic order $\mathrm{lo}(R) = 5$, class $c = 3$, and coclass $r = 2$.

3.1. **Periodicity of finite patterns.** Within the frame of the above-mentioned model with $p = 3$ for the theory of rooted in-trees as developed in § 2, the following finiteness and periodicity statement becomes provable.

The *virtual periodicity* of depth-pruned branches of coclass trees has been proven rigorously with analytic methods (using zeta functions and cone integrals) by du Sautoy [4] in 2000, and with algebraic methods (using cohomology groups) by Eick and Leedham-Green [5] in 2008. We recall that a coclass tree contains a unique infinite path of edges π with uniform step size $s(\pi) = 1$, the so-called mainline. Pattern recognition and pattern classification concerns the branches.

Theorem 3.1. *(A finite periodically repeating pattern.)*
Among the vertices of any mainline $(v_i)_{i\ge 0}$ in $\mathcal{T}(R)$, there exists a ***periodic root*** *v_ϱ with $\varrho \ge 0$ and a* ***period length*** *$\lambda \ge 1$ such that the branches*

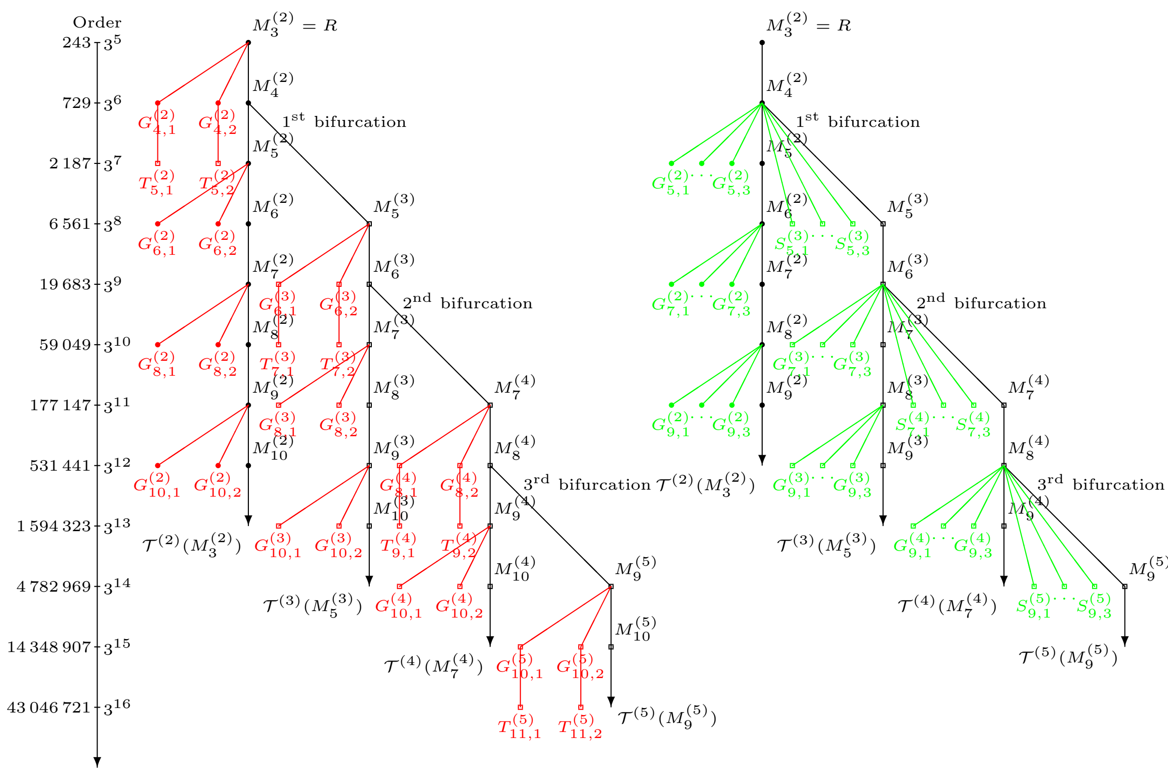
$$\mathcal{B}(v_{i+\lambda}) \simeq \mathcal{B}(v_i)$$

are isomorphic ***finite*** *graphs, for all $i \ge \varrho$. Up to a finite pre-periodic component, the minimal tree $\mathcal{T}_1(v_0)$ consists of periodically repeating copies of the finite pattern $\dot{\bigcup}_{i=0}^{\lambda-1} \mathcal{B}(v_{\varrho+i})$.*

Proof. According to [4] and [5], the claims are true for pruned branches with any fixed depth. However, for $p = 3$ and under the pruning operation on $\mathcal{T}(R)$ described in § 3.2.2, the virtual periodicity becomes a strict periodicity, since the depth is bounded uniformly for all branches. $\square$

Before we visualize a particular instance of Theorem 3.1 in the diagram of Figure 1, we have to establish techniques for disentangling dense branches of high complexity.

FIGURE 1. Graph dissection into pruned branches connected by the mainline skeleton



3.2. **Graph dissection by independent component analysis.**

3.2.1. *Dissection by Galois action.* Figure 1 visualizes a graph dissection of the tree $\mathcal{T}(R)$ by independent component analysis. This technique drastically reduces the complexity of visual representations and avoids overlaps of dense subgraphs. The left hand scale gives the order of groups whose isomorphism classes are represented by vertices of the graph. The mainline skeleton (black) connects branches of non-σ groups (red) in the left subfigure and branches of σ-groups (green) in the right subfigure. This terminology has its origin in the action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ on the abelianization $\mathfrak{M}/\mathfrak{M}'$, when a vertex of $\mathcal{T}(R)$ is realized as second 3-class group $\mathfrak{M} := \mathrm{Gal}\left(F_3^{(2)}/F\right)$ of an algebraic number field F. For quadratic fields F, we obtain σ-groups.

Definition 3.1. A *σ-group* G admits an automorphism $\sigma \in \mathrm{Aut}(G)$ acting as inversion $\sigma(x) = x^{-1}$ on the commutator quotient G/G'.

The actual graph $\mathcal{T}(R)$ consists of the overlay (superposition) of both subfigures in Figure 1. Infinite mainlines are indicated by arrows. The periodic bifurcations form an infinite path with edges of alternating step sizes 1 and 2, according to Theorem 3.2. We call it the *maintrunk*.

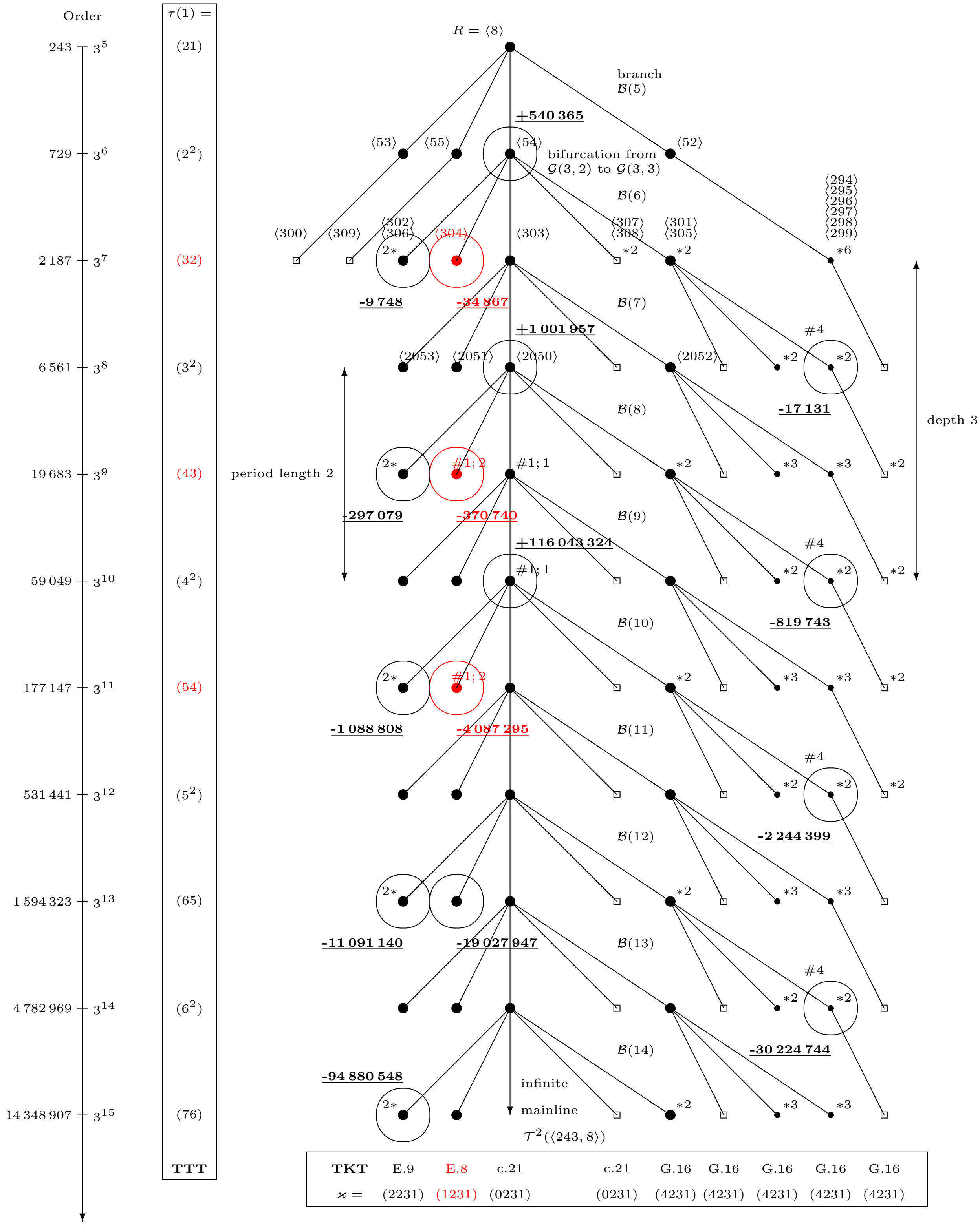
With the aid of Figure 1, a particular instance of Theorem 3.1 can be expressed in a more concrete and ostensive way by taking the tree root as the ending vertex $v_0 := R$ of the mainline $(v_i)_{i\geq 0}$, and by using the variable class $c \geq 3$ and the fixed coclass $r = 2$ as parameters describing all mainline vertices $M_c^{(r)} := v_{c-3}$. The periodic root is $M_5^{(2)} = v_2$ with $\varrho = 2$ and the period length is $\lambda = 2$. The finite periodic pattern consists of the two branches $\mathcal{B}(M_5^{(2)}) = \{M_5^{(2)}, G_{6,1}^{(2)}, G_{6,2}^{(2)}\}$ (red) and $\mathcal{B}(M_6^{(2)}) = \{M_6^{(2)}, G_{7,1}^{(2)}, G_{7,2}^{(2)}, G_{7,3}^{(2)}\}$ (green). The pre-period is irregular and consists of the two branches $\mathcal{B}(M_3^{(2)}) = \{M_3^{(2)}, G_{4,1}^{(2)}, G_{4,2}^{(2)}, T_{5,1}^{(2)}, T_{5,2}^{(2)}\}$ (red) and $\mathcal{B}(M_4^{(2)}) = \{M_4^{(2)}, G_{5,1}^{(2)}, G_{5,2}^{(2)}, G_{5,3}^{(2)}\}$ (green). But $M_4^{(2)}$ is not coclass-settled, has nuclear rank $n = 2$, and gives rise to a bifurcation with immediate descendants $S_{5,1}^{(3)}, S_{5,2}^{(3)}, S_{5,3}^{(3)}, M_5^{(3)}$ (green) of step size $s = 2$.

3.2.2. *Dissection by Artin transfers.* In Figure 1, we have tacitly used a second technique of graph dissection by independent component analysis. Figure 2 is restricted to the coclass tree $\mathcal{T}^{(2)}(R)$ with exemplary root $R = \langle 243, 8\rangle$, which is the leftmost coclass tree in both subfigures of Figure 1. However, now this coclass tree is drawn completely up to logarithmic order 15, containing both, non-σ branches and σ-branches. The tree is embedded in a kind of coordinate system having the transfer kernel type (TKT) $\varkappa$ as its horizontal axis and the first component $\tau(1)$ of the transfer target type (TTT) τ as its vertical axis [6, Dfn. 4.2, p. 27]. It is convenient to employ a second graph dissection, according to three fundamental types of transfer kernels:

- the vertices with *simple* types E.8, $\varkappa = (1231)$, and E.9, $\varkappa = (2231)$, which are leaves (left of the mainline), except those of order 3^6,
- the vertices with *scaffold* (or skeleton) type c.21, $\varkappa = (0231)$, which are either infinitely capable mainline vertices or non-metabelian leaves (immediately right of the mainline),
- the vertices with *complex* type G.16, $\varkappa = (4231)$, which are capable at depth 1 and give rise to a complicated brushwood of various descendants (right of the mainline).

The tacit omission in Figure 1 concerns all vertices with complex type and the leaves with scaffold type. Our main results in this chapter will shed complete light on all mainline vertices and the vertices with simple types. Underlined boldface integers in Figure 2 indicate the minimal discriminants d of (real and imaginary) quadratic fields $F = \mathbb{Q}(\sqrt{d})$ whose second 3-class group $\mathrm{G}_3^{(2)}F := \mathrm{Gal}\left(F_3^{(2)}/F\right)$ realizes the vertex surrounded by the adjacent oval. Three leaves of type E.8 are drawn with red color, because they will be referred to in Theorem 4.1 on 3-class towers.

FIGURE 2. Coclass tree $\mathcal{T}^{(2)}(\langle 243, 8\rangle)$ with simple, scaffold and complex types



3.3. **Periodicity of infinite patterns.** With the aid of a combination of top down and bottom up techniques, we are now going to provide evidence of a new kind of *periodic bifurcations* in pruned descendant trees which contain a unique infinite path of edges π with strictly alternating step sizes $s(\pi) = 1$ and $s(\pi) = 2$, the so-called *maintrunk*. It is very important that the trees are *pruned* in the sense explained at the end of the preceding section § 3.2.2, for otherwise the maintrunk will not be unique. In fact, each of our pruned descendant trees $\mathcal{T}(R)$ is a countable disjoint union of pruned coclass trees $\mathcal{T}^{(r)}$, $r \geq 2$, which are isomorphic as infinite graphs and connected by edges of weight 2, and finite batches $\mathcal{T}_0^{(r)}$, $r \geq 3$, of sporadic vertices outside of coclass trees. The top down and bottom up techniques are implemented simultaneously in two recursive Algorithms 3.1 and 3.2.

The first Algorithm 3.1 recursively constructs the mainline vertices $M_c^{(r)}$, with class $c \geq 2r-1$, of the coclass tree $\mathcal{T}^{(r)} \subset \mathcal{T}(R)$, for an assigned value $r \geq 2$, by means of the *bottom up* technique. In each recursion step, the *top down* technique is used for constructing the class-c quotient $\mathcal{L}_c^{(r)}$ of an infinite limit group $\mathcal{L}^{(r)}$. Finally, the isomorphism $M_c^{(r)} \simeq \mathcal{L}_c^{(r)}$ is proved.

Theorem 3.2. *(An infinite periodically repeating pattern.)* *Let $u_r = 30$ be an upper bound. An infinite path is generated recursively, since for each $2 \leq r < u_r$, the immediate descendant $M_{2r+1}^{(r+1)} = M_{2r}^{(r)} - \#2;1$ of step size 2 of the second mainline vertex $M_{2r}^{(r)}$ of the coclass tree $\mathcal{T}^{(r)}(M_{2r-1}^{(r)})$, is root of a new coclass tree $\mathcal{T}^{(r+1)}(M_{2(r+1)-1}^{(r+1)})$. The pruned coclass trees*

$$\mathcal{T}^{(r)}(M_{2r-1}^{(r)}) \simeq \mathcal{T}^{(2)}(M_3^{(2)})$$

are isomorphic ***infinite*** *graphs, for each $2 \leq r \leq u_r$. Note that the nuclear rank $n(M_{2r}^{(r)}) = 2$.*

This is the first main theorem of the present chapter. The proof will be conducted with the aid of an infinite limit group $\mathcal{L}_\pm$, due to M. F. Newman. Certain quotients of $\mathcal{L}_\pm$ give precisely the mainline vertices $M_c^{(r)}$ with $r \geq 2$ and $c \geq 2r-1$, as will be shown in Theorem 3.3, Remark 3.2.

Conjecture 3.1. Theorem 3.2 remains true for any upper bound $u_r > 30$.

3.4. **Mainlines of the pruned descendant tree $\mathcal{T}(R)$.**

Definition 3.2. The complete theory of the mainlines in $\mathcal{T}(R)$ is based on the group

$$\mathcal{L}_\pm := \langle\ a, t \ \mid\ (at)^3 = a^3,\ [[t,a],t] = a^{\pm 3}\ \rangle. \tag{3.1}$$

For each $r \geq 2$, quotients of $\mathcal{L}_\pm$ are defined by

$$\mathcal{L}_\pm^{(r)} := \mathcal{L}_\pm \ /\ \langle\ a^{3^r}\ \rangle. \tag{3.2}$$

For each $r \geq 2$, and for each $c \geq 2r-1$, quotients of $\mathcal{L}_\pm^{(r)}$ are defined by

$$\mathcal{L}_{\pm,c}^{(r)} := \begin{cases} \mathcal{L}_\pm^{(r)} \ /\ \langle\ [t,a]^{3^\ell}\ \rangle & \text{if } c = 2\ell+1 \text{ odd },\ \ell \geq r-1, \\ \mathcal{L}_\pm^{(r)} \ /\ \langle\ t^{3^\ell}\ \rangle & \text{if } c = 2\ell \text{ even },\ \ell \geq r. \end{cases} \tag{3.3}$$

The following Algorithm 3.1 is based on iterated applications of the p-group generation algorithm by Newman [7] and O'Brien [8]. It starts with the root R, given by its compact presentation, and constructs an initial section of the unique infinite maintrunk with strictly alternating step sizes 1 and 2 in the pruned descendant tree $\mathcal{T}(R)$. In each step, the required selection of the child with appropriate transfer kernel type (TKT) is achieved with the aid of our own subroutine `IsAdmissible()`, which is an elaborate version of [9, § 4.1, p. 76]. After reaching an assigned coclass `r`=`hb`+2, our algorithm navigates along the mainline of the coclass tree $\mathcal{T}^{(r)} \subset \mathcal{T}(R)$ and tests each vertex for isomorphism to the corresponding quotient $\mathcal{L}_{\pm,c}^{(r)}$ of class `c`≤`2r`−`1`+`vb`.

Algorithm 3.1. (Mainline vertices.)
Input: prime `p`, compact presentation `cp` of the root, bounds `hb`,`vb`, sign `s`.
Code: uses the subroutine `IsAdmissible()`.

```
r := 2; // initial coclass
Root := PCGroup(cp);
for i in [1..hb] do // bottom up in double steps along the maintrunk
    Des := Descendants(Root,NilpotencyClass(Root)+1:  StepSizes:=[1]);
    for j in [1..#Des] do
        if IsAdmissible(Des[j],p,0) then
            Root := Des[j];
        end if;
    end for;
    r := r + 1; // coclass recursion
    Des := Descendants(Root,NilpotencyClass(Root)+1:  StepSizes:=[2]);
    for j in [1..#Des] do
        if IsAdmissible(Des[j],p,0) then
            Root := Des[j];
        end if;
    end for;
end for;
c := 2*r - 1; // starting class c in dependence on the coclass r
er := p^r; l := (c - 1) div 2; ec := p^l;
M<a,t> := Group<a,t|(a*t)^p=a^p,((t,a),t)=a^(s*p),a^er=1,(t,a)^ec=1>;
QM,pM := pQuotient(M,p,c); // top down construction
if IsIsomorphic(Root,QM) then // identification
    printf "Isomorphism for cc=%o, cl=%o.\n",r,c;
end if;
for i in [1..vb] do // bottom up in single steps along a mainline
    c := c + 1; // nilpotency class recursion
    if (0 eq c mod 2) then // even nilpotency class
        l := c div 2; ec := p^l;
        M<a,t> := Group<a,t|(a*t)^p=a^p,((t,a),t)=a^(s*p),a^er=1,t^ec=1>;
    else // odd nilpotency class
        l := (c - 1) div 2; ec := p^l;
        M<a,t> := Group<a,t|(a*t)^p=a^p,((t,a),t)=a^(s*p),a^er=1,(t,a)^ec=1>;
    end if;
    QM,pM := pQuotient(M,p,c); // top down construction
    Des := Descendants(Root,NilpotencyClass(Root)+1:  StepSizes:=[1]);
    for j in [1..#Des] do
        if IsAdmissible(Des[j],p,0) then
            Root := Des[j];
        end if;
    end for;
    if IsIsomorphic(Root,QM) then // identification
        printf "Isomorphism for cc=%o, cl=%o.\n",r,c;
    end if;
end for;
```

Output: coclass `r` and class `c` in each case of an isomorphism.

Remark 3.1. Algorithm 3.1 is designed to be called with input parameters the prime `p=3` and `cp` the compact presentation of either the root $\langle 243, 6\rangle$ with sign `s=-1` or the root $\langle 243, 8\rangle$ with sign `s=+1`. In the current version V2.22-7 of the computational algebra system MAGMA [10], the bounds are restricted to `r`$=$`hb`$+2\le 8$ and `c`$=$`vb`$+2$`r`$-1\le 35$, since otherwise the maximal possible internal word length of relators in MAGMA is surpassed. Close to these limits, the required random access memory increases to a considerable value of approximately 8 GB RAM.

Theorem 3.3. *(Mainline vertices as quotients of the limit group $\mathcal{L}_\pm$.) Let $u_r := 8$, $u_c := 35$.*

(1) *For each $2 \le r \le u_r$, and for each $2r-1 \le c \le u_c$, the mainline vertex $M_c^{(r)}$ of coclass r and nilpotency class c in the tree $\mathcal{T}(R)$ is isomorphic to $\mathcal{L}_{\pm,c}^{(r)}$.*
(2) *For each $2 \le r \le u_r$, the projective limit of the mainline $\left(M_c^{(r)}\right)_{c \ge 2r-1}$ with vertices of coclass r in the tree $\mathcal{T}(R)$ is isomorphic to $\mathcal{L}_{\pm}^{(r)}$.*
(3) *$\mathcal{L}_{\pm}$ is an infinite non-nilpotent profinite limit group.*

Proof. (1) The repeated execution of Algorithm 3.1 for successive values from `hb:=0` to `hb:=6`, with input data `p:=3`, `cp:=CompactPresentation(SmallGroup(243,i))`, $i \in \{6, 8\}$, $s \in \{-1, +1\}$, and `vb:=32`, proves the isomorphisms $M_c^{(r)} \simeq \mathcal{L}_{\pm,c}^{(r)}$ for $2 \le r \le u_r = 8$ and $2r-1 \le c \le u_c = 35$. The algorithm is initialized by the starting group $R = M_3^{(2)} = \langle 243, i \rangle$ of coclass `r:=2`. The first loop moves along the maintrunk recursively with strictly alternating step sizes 1 and 2 until the root $M_{2r-1}^{(r)}$ of the coclass tree $\mathcal{T}^{(r)}$ with `r=2+hb` is reached. The second loop iterates through the mainline vertices $M_c^{(r)}$, $c \ge 2r-1$, of the coclass tree $\mathcal{T}^{(r)}\left(M_{2r-1}^{(r)}\right)$, always checking for isomorphism to the appropriate quotient $\mathcal{L}_{\pm,c}^{(r)}$. The subroutine `IsAdmissible()` tests the transfer kernel type of all descendants and selects the unique capable descendant with type c.18 resp. c.21. (2) Since periodicity sets in for $2u_r - 1 = 17 \le c \le u_c = 35$, the claim is a consequence of Theorem 3.1. (3) The quotient $\mathcal{L}_{\pm}^{(1)}$ is already infinite and non-nilpotent. Adding the relation $[t, t^a, t] = 1$ suffices to give $[t, a, t]$ central and $\mathcal{L}_{\pm}$ profinite. □

Conjecture 3.2. Theorem 3.3 remains true for arbitrary upper bounds $u_r > 8$, $u_c > 35$.

Remark 3.2. When the top down constructions in Algorithm 3.1 are cancelled, the bottom up operations are still able to establish much bigger initial sections of the infinite maintrunk and of the infinite coclass tree with fixed coclass $r \ge 2$. Admitting an increasing amount of CPU time, we can easily reach astronomic values of the coclass, $r = 32$, and the nilpotency class, $c = 63$, that is a logarithmic order of $r + c = 95$, without surpassing any internal limitations of MAGMA, and the required storage capacity remains quite modest, i.e., clearly below 1 GB RAM. This remarkable stability underpins Conjecture 3.2 with additional support from the bottom up point of view.

3.5. **Covers of metabelian 3-groups.** Only one of the coclass subtrees $\mathcal{T}^{(r)}$, $r \ge 2$, of the entire rooted in-tree $\mathcal{T}(R)$ contains metabelian vertices, namely the first subtree $\mathcal{T}^{(2)}$. The following theorem shows how transfer kernel types are distributed among metabelian vertices G of depth $\mathrm{dp}(G) \le 1$ on the tree $\mathcal{T}^{(2)}$, as partially illustrated by the Figures 1 and 2.

Theorem 3.4. *(Metabelian vertices of the coclass tree $\mathcal{T}^{(2)}R$.)*
For each finite 3-group G, we denote by $c := \mathrm{cl}(G)$ the nilpotency class, by $r := \mathrm{cc}(G)$ the coclass, and by $\varkappa$ the transfer kernel type of G. More explicitly, such a group is also denoted by $G = G_c^{(r)}$. The following statements describe the structure of the metabelian skeleton of the coclass tree $\mathcal{T}^{(2)}R$ with root $R := \langle 243, 6 \rangle$, resp. $R := \langle 243, 8 \rangle$, down to depth 1.

(1) *For each $c \ge 3$, the mainline vertex $M_c^{(2)}$ of the coclass tree possesses type* c.18, $\varkappa = (0122)$, *resp.* c.21, $\varkappa = (0231)$.
(2) *For each $c \ge 4$, there exists a unique child $G_{c,1}^{(2)}$ of $M_{c-1}^{(2)}$ with type* E.6, $\varkappa = (1122)$, *resp.* E.8, $\varkappa = (1231)$.
(3) *For even $c \ge 4$, there exists a unique child $G_{c,2}^{(2)}$ of $M_{c-1}^{(2)}$ with type* E.14, $\varkappa = (3122)$, *resp.* E.9, $\varkappa = (2231)$. *Thus, $N_1(M_{c-1}^{(2)}) = 3$ and $C_1(M_{c-1}^{(2)}) = 1$, in the pruned tree.*
(4) *For odd $c \ge 5$, there exist two children $G_{c,2}^{(2)}$, $G_{c,3}^{(2)}$ of $M_{c-1}^{(2)}$ with type* E.14, $\varkappa = (3122) \sim (4122)$, *resp.* E.9, $\varkappa = (2231) \sim (3231)$. *Thus, $N_1(M_{c-1}^{(2)}) = 4$ and $C_1(M_{c-1}^{(2)}) = 1$.*
(5) *For even $c \ge 4$, there exists a unique child $G_{c,4}^{(2)}$ of $M_{c-1}^{(2)}$ with type* H.4, $\varkappa = (2122)$, *resp.* G.16, $\varkappa = (4231)$. *It is removed from the pruned tree.*
(6) *For odd $c \ge 5$, there exist two children $G_{c,4}^{(2)}$, $G_{c,5}^{(2)}$ of $M_{c-1}^{(2)}$ with type* H.4, $\varkappa = (2122)$, *resp.* G.16, $\varkappa = (4231)$. *They are removed from the pruned tree.*

Proof. See Nebelung [11, Lemma 5.2.6, p. 183, Figures, p. 189 f., Satz 6.14, p. 208]. □

Definition 3.3. For $e \in \{0, 1\}$, we define the *cover limit*, due to M. F. Newman, to be the group

$$\begin{aligned}\mathcal{C}^{(e)} := \langle\ a, t, u, y, z \ \mid\ t^a = u,\ u^a tuy = [u, t]^e,\ a^3[t, a, t] = z,\ [u, t, t] = [u, t, u] = 1, \\ y^3 = 1,\ [a, y] = [t, y] = [u, y] = [z, y] = 1,\ z^3 = 1,\ [t, z] = [u, z] = 1\ \rangle,\end{aligned} \tag{3.4}$$

which was introduced in [12]. For each $k \in \{-1, 0, 1\}$ and for each integer $c \geq 4$, let

$$\mathcal{Q}_c^{(e,k)} := \mathcal{C}^{(e)}\ /\ \langle\ yw_c^k v_c,\ zw_c\ \rangle \tag{3.5}$$

be the *class-c quotient with parameter k* of $\mathcal{C}^{(e)}$, where $w_c := [t,\ \overbrace{a, \ldots, a}^{(c-1)\text{ times}}\ ]$ and $v_c := [w_{c-2}, [t, a]]$.

In each step, $i \geq 1$, of the second Algorithm 3.2, the *top down* technique constructs a certain class-c quotient $\mathcal{Q}_c$, $c = i + 3$, of a fixed infinite pro-3 group $\mathcal{C}$, the *cover limit*, and the *bottom up* technique constructs all metabelian children of a certain vertex M_{i-1} on the mainline of the first coclass tree $\mathcal{T}^{(2)}(R)$, and selects, firstly, the next vertex M_i of depth $\mathrm{dp}(M_i) = 0$ on the mainline of $\mathcal{T}^{(2)}(R)$ for continuing the recursion, secondly, a vertex G_i of depth $\mathrm{dp}(G_i) = 1$ with assigned transfer kernel type $\varkappa(G_i)$. Each recursion step is completed by proving that G_i is isomorphic to the second derived quotient $\mathcal{Q}_c/\mathcal{Q}_c''$, that is, $\mathcal{Q}_c \in \mathrm{cov}(G_i)$ belongs to the *cover* of G_i in the sense of [13, § 1.3, Dfn. 1.1, p. 75]. More precisely, we have $M_i = M_{i+3}^{(2)}$ and $G_i = G_{i+3,j}^{(2)}$ with some j.

Algorithm 3.2. (Shafarevich cover.)
Input: prime `p`, compact presentation `cp` of the root, bound `vb`, parameters `e` and `k`.
Code: uses the subroutine `IsAdmissible()`.

```
C<a,t,u,y,z> := Group< a,t,u,y,z |
    y^p, (a,y), (t,y), (u,y), (y,z), (t,z), (u,z), z^p,
    (u,t,t), (u,t,u), t^a = u, u^a*t*u*y*(u,t)^-e, a^p*(t,a,t) = z >;
Root := PCGroup(cp);
Leaf := Root;
for i in [1..vb] do // bottom up along the mainline of coclass 2
    c := i + 3; // nilpotency class
    w := [t];
    for j in [1..c] do // construction of iterated commutator
        s := (w[j],a);
        Append(~w,s);
    end for;
    w1 := w[c-2]^-1*(a,t)*w[c-2]*(t,a);
    H := quo<C | y*w[c]^k*w1, z*w[c]>;
    Q,pQ := pQuotient(H,p,c); // top down construction of Shafarevich cover
    Des := Descendants(Root,NilpotencyClass(Root)+1);
    m := 0;
    for cnt in [1..#Des] do
        if IsAdmissible(Des[cnt],p,0) then
            Root := Des[cnt]; // next mainline vertex
        elif IsAdmissible(Des[cnt],p,2) then
            m := m + 1;
            if (1 eq m) then
                Leaf := Des[cnt]; // first leaf with assigned TKT
            end if;
        end if;
    end for;
    DQ := DerivedSubgroup(Q);
    D2Q := DerivedSubgroup(DQ);
    Q2Q := Q/D2Q; // metabelianization
    if IsIsomorphic(Leaf,Q2Q) then // identification
```

```
        printf "Dsc.cl.%o isomorphic to 2nd drv.qtn. of Cov.cl.%o.\n",c,c;
    end if;
end for;
```

Output: nilpotency class `c` in each case of an isomorphism.

The next theorem is the second main result of this chapter, establishing the finiteness and structure of the *cover* for each metabelian 3-group with transfer kernel type in section E.

Theorem 3.5. *(Explicit covers of metabelian 3-groups.) Let $u := 8$ be an upper bound and $G_{c,j}^{(2)}$ in $\mathcal{T}^{(2)}\left(M_3^{(2)}\right)$ be the metabelian 3-group of nilpotency class $c \geq 4$ with transfer kernel type*

$$\varkappa = \begin{cases} (1122),\ \mathrm{E.6},\ \textit{resp.}\ (1231),\ \mathrm{E.8} & \textit{if } j = 1, \\ (3122),\ \mathrm{E.14},\ \textit{resp.}\ (2231),\ \mathrm{E.9} & \textit{if } j = 2 \textit{ or } (j = 3 \textit{ and } c \textit{ odd}). \end{cases}$$

(1) *The **cover** of $G_{c,j}^{(2)}$ is given by*

$$\operatorname{cov}\left(G_{c,j}^{(2)}\right) = \begin{cases} \left\{G_{c,j}^{(2)}; G_{c,j}^{(3)}, \ldots, G_{c,j}^{(\ell+1)}, G_{c,j}^{(\ell+2)}, T_{c+1,j}^{(\ell+2)}\right\} & \textit{if } c = 2\ell+4,\ 1 \leq j \leq 2, \\ \left\{G_{c,j}^{(2)}; G_{c,j}^{(3)}, \ldots, G_{c,j}^{(\ell+1)}, G_{c,j}^{(\ell+2)}, S_{c,j}^{(\ell+3)}\right\} & \textit{if } c = 2\ell+5,\ 1 \leq j \leq 3. \end{cases} \tag{3.6}$$

where $0 \leq \ell \leq u$. In particular, the cover is a finite set with $\ell+2$ elements ($\ell+1$ of them non-trivial), which are non-σ groups for even $c \geq 4$, and σ-groups for odd $c \geq 5$.

(2) *The **Shafarevich cover** of $G_{c,j}^{(2)}$ with respect to imaginary quadratic fields F is given by*

$$\operatorname{cov}\left(G_{c,j}^{(2)}, F\right) = \begin{cases} \emptyset & \textit{if } c = 2\ell+4,\ 0 \leq \ell \leq u,\ 1 \leq j \leq 2, \\ \left\{S_{c,j}^{(\ell+3)}\right\} & \textit{if } c = 2\ell+5,\ 0 \leq \ell \leq u,\ 1 \leq j \leq 3. \end{cases} \tag{3.7}$$

In particular, the Shafarevich cover contains a unique Schur σ-group, if $c \geq 5$ is odd.

(3) *The class-c quotient with parameter k of the cover limit $\mathcal{C}^{(e)}$ is isomorphic to a Schur σ-group $\mathcal{Q}_c^{(e,k)} \simeq S_{c,j}^{(\ell+3)}$, for $c = 2\ell+5$, or to a non-σ group $\mathcal{Q}_c^{(e,k)} \simeq G_{c,j}^{(\ell+2)}$, for $c = 2\ell+4$. The precise correspondence between the parameters k and j is given in the following way.*

$$\begin{aligned} \textit{Types } \mathrm{E.6}, \mathrm{E.8}:\ \mathcal{Q}_c^{(e,0)} &\simeq \begin{cases} S_{c,1}^{(\ell+3)} & \textit{for odd class } c = 2\ell+5,\ 0 \leq \ell \leq u, \\ G_{c,1}^{(\ell+2)} & \textit{for even class } c = 2\ell+4,\ 0 \leq \ell \leq u, \end{cases} \\ \textit{type } \mathrm{E.9}:\ \mathcal{Q}_c^{(+1,-1)} &\simeq \begin{cases} S_{c,2}^{(\ell+3)} & \textit{for odd class } c = 2\ell+5,\ 0 \leq \ell \leq u, \\ G_{c,2}^{(\ell+2)} & \textit{for even class } c = 2\ell+4,\ 0 \leq \ell \leq u, \end{cases} \\ \textit{type } \mathrm{E.9}:\ \mathcal{Q}_c^{(+1,+1)} &\simeq \begin{cases} S_{c,3}^{(\ell+3)} & \textit{for odd class } c = 2\ell+5,\ 0 \leq \ell \leq u, \\ G_{c,2}^{(\ell+2)} & \textit{for even class } c = 2\ell+4,\ 0 \leq \ell \leq u. \end{cases} \end{aligned} \tag{3.8}$$

In particular, $\mathcal{Q}_c^{(+1,-1)} \simeq \mathcal{Q}_c^{(+1,+1)}$ for even class $c = 2\ell+4$, $0 \leq \ell \leq u$.
The variant $e = 0$, resp. $e = 1$, is associated to the root $R = \langle 243, 6\rangle$, resp. $R = \langle 243, 8\rangle$.

(4) *A parametrized family of **fork topologies** for second 3-class groups $\mathrm{Gal}\left(F_3^{(2)}/F\right)$ of imaginary quadratic fields F is given uniformly for the states $\uparrow^\ell$ (ground state for $\ell = 0$, excited state for $1 \leq \ell \leq u$) of transfer kernel types in section* E *by the symmetric topology symbol*

$$P = \overbrace{\mathrm{E} \xrightarrow{1}}^{\textit{Leaf}} \ \overbrace{\left\{\mathrm{c} \xrightarrow{1}\right\}^{2\ell}}^{\textit{Mainline}} \ \overbrace{\mathrm{c}}^{\textit{Fork}} \ \overbrace{\left\{\xleftarrow{2} \mathrm{c} \xleftarrow{1} \mathrm{c}\right\}^{\ell}}^{\textit{Maintrunk}} \ \overbrace{\xleftarrow{2} \mathrm{E}}^{\textit{Leaf}} \tag{3.9}$$

with scaffold type c *and the following invariants:*
distance $d = 4\ell+2$ (Dfn. 2.3), weighted distance $w = 5\ell+3$ (Dfn. 2.4),
class increment $\Delta\mathrm{cl} = (2\ell+5)-(2\ell+5) = 0$, coclass increment $\Delta\mathrm{cc} = (\ell+3)-2 = \ell+1$,
logarithmic order increment $\Delta\mathrm{lo} = (3\ell+8)-(2\ell+7) = \ell+1$ [13, Dfn. 5.1, p. 89].

Proof. We compare the uniform generator rank $d_1 = 2$ of all involved groups $G_{c,j}^{(r)}$, $c \geq 4$, $r \geq 2$, $1 \leq j \leq 3$, with their relation rank d_2. Since $d_2 = \mu$ and the p-multiplicator rank is $\mu = 2$ for $S_{c,j}^{(r)}$ with odd $c = 2\ell + 5 \geq 5$ and $r = \ell + 3 \geq 3$, but $\mu = 3$ otherwise, only the groups $S_{c,j}^{(r)}$ are Schur σ-groups with balanced presentation $d_2 = 2 = d_1$, and are therefore admissible as 3-tower groups of imaginary quadratic fields F, according to our corrected version [6, § 5, Thm. 5.1, pp. 28–29] of the Shafarevich Theorem [14, Thm. 6, (18′)]. Finally we remark that the nuclear rank is $\nu = 1$ for $G_{c,j}^{(r)}$ with even $c = 2\ell + 4$, $r = \ell + 2$, and child $T_{c+1,j}^{(r)}$, but $\nu = 0$ otherwise.

The execution of Algorithm 3.2 with input data `p:=3`, `vb:=25`, either `i:=6`, `e:=0`, or `i:=8`, `e:=1`, and `cp:=CompactPresentation(SmallGroup(243,i))`, proves the isomorphisms $\mathcal{Q}_c^{(e,k)} \simeq S_{c,j}^{(\ell+3)}$, $c = 2\ell+5$, resp. $\mathcal{Q}_c^{(e,k)} \simeq G_{c,j}^{(\ell+2)}$, $c = 2\ell+4$, for $4 \leq c \leq 20$, that is, $0 \leq \ell \leq u = 8$. The algorithm is initialized by the starting group $R = M_3^{(2)}$ of coclass 2. The loop navigates through the mainline vertices $M_c^{(2)}$, $c \geq 3$, of the coclass tree $\mathcal{T}^{(2)}\left(M_3^{(2)}\right)$. The subroutine `IsAdmissible()` tests the transfer kernel type of all descendants and selects either the unique capable descendant with type c.18, resp. c.21, for the flag `0`, or the unique descendant with type E.6, resp. E.8, for the flag `1`, or the first or second descendant with type E.9, for the flag `2`. The selected non-mainline vertex is always checked for isomorphism to the metabelianization of the appropriate quotient $\mathcal{Q}_c^{(e,k)}$. See also [2, § 21.2, pp. 189–193], [15, pp. 751–756], the proof of Theorem 4.1, and Figures 5, 6, 7. $\square$

Here again, a pure bottom up approach without top down constructions, instead of using Algorithm 3.2, is able to reach coclass $r = 32$, nilpotency class $c = 63$, and logarithmic order $r + c = 95$, without surpassing internal limits of MAGMA, and strongly supports Conjecture 3.3.

Conjecture 3.3. Theorem 3.5 remains true for any upper bound $u > 8$.

FIGURE 3. Projections $\mathcal{Q}_c^{(e,k)} \to \mathcal{Q}_c^{(e,k)} / \left(\mathcal{Q}_c^{(e,k)}\right)''$ of the covers onto their metabelianizations

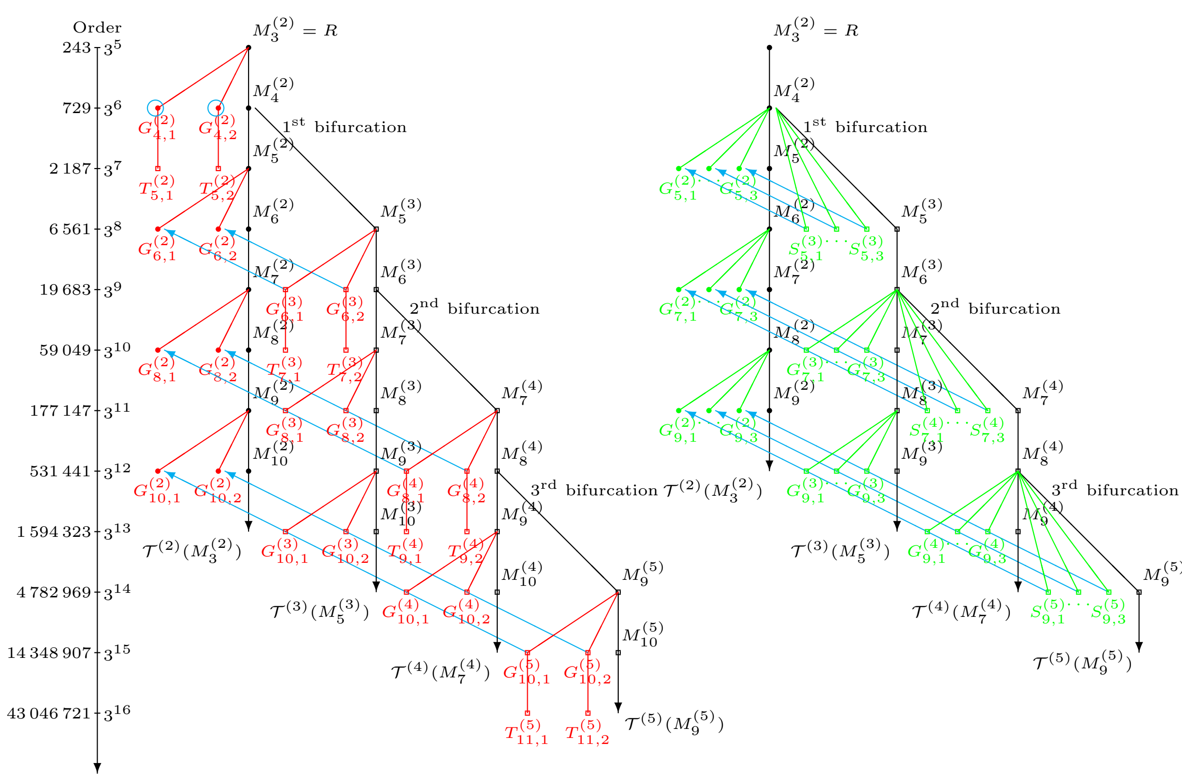


Figure 3 shows exactly the same situation as Figure 1, supplemented by blue arrows indicating the projections of the quotients $\mathcal{Q}_c^{(e,k)}$ onto their metabelianizations, that is, $S_{c,j}^{(\ell+3)} \to G_{c,j}^{(2)}$, for odd class $c = 2\ell+5$, in the right diagram with green branches, and $G_{c,j}^{(\ell+2)} \to G_{c,j}^{(2)}$, for even class $c = 2\ell+4$, in the left diagram with red branches. For $c = 4$, a degeneration occurs, since $\mathcal{Q}_4^{(e,k)}$ is metabelian already, indicated by surrounding blue circles.

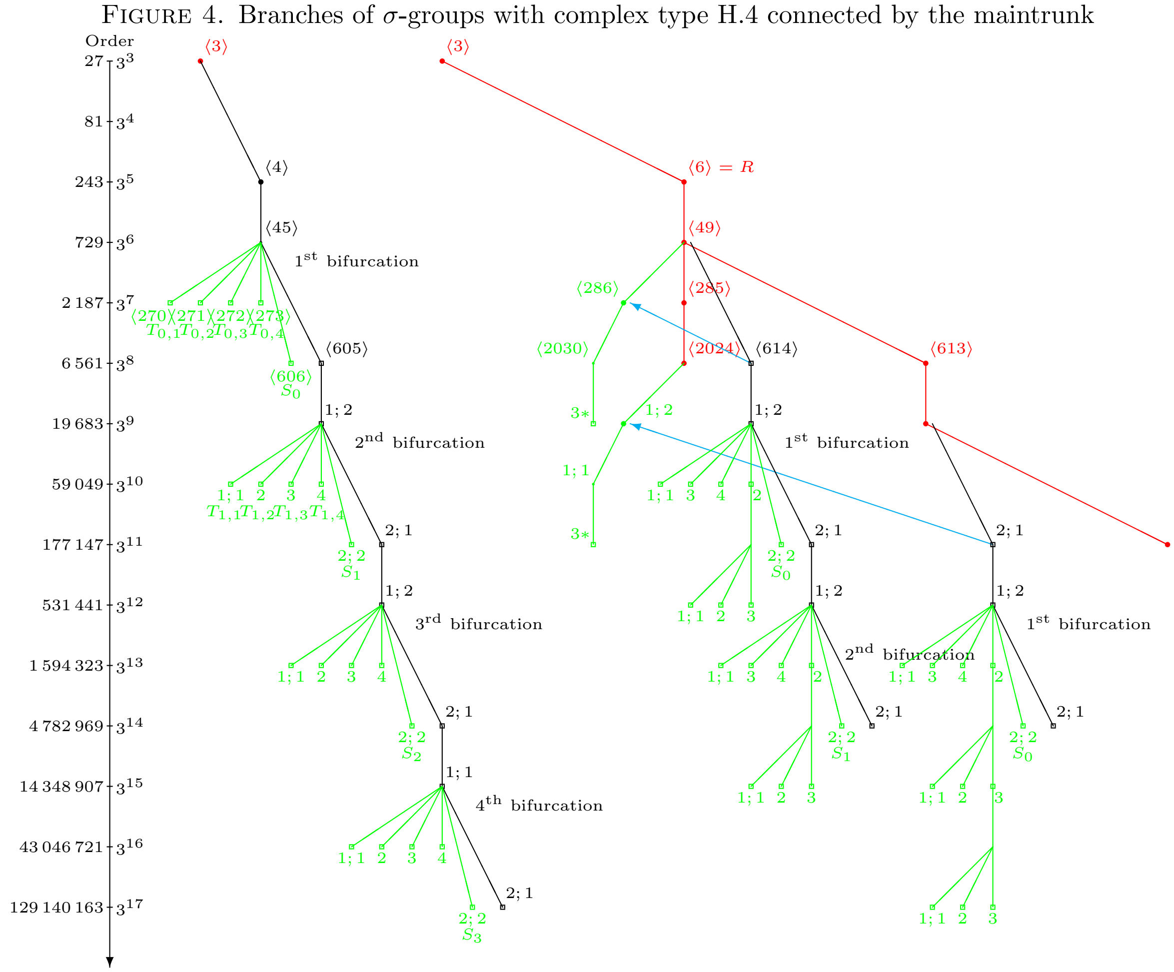
Strictly speaking, the caption of Figure 3 in its full generality is valid for $e = 1$, $M_3^{(2)} = \langle 243, 8\rangle$ only. For $e = 0$, $M_3^{(2)} = \langle 243, 6\rangle$, all blue arrows have the same meaning as before but the interpretation of the covers as quotients $\mathcal{Q}_c^{(e,k)}$ is slightly restricted. Whereas we have the following supplement to Formula (3.8):

$$\text{type E.14}: \quad \mathcal{Q}_c^{(0,-1)} \simeq \begin{cases} S_{c,3}^{(\ell+3)} & \text{for odd class } c = 2\ell+5,\ 0 \le \ell \le u, \\ G_{c,2}^{(\ell+2)} & \text{for even class } c = 2\ell+4,\ 0 \le \ell \le u, \end{cases} \tag{3.10}$$

the quotients $\mathcal{Q}_c^{(0,+1)}$ lead into a completely different realm, namely the complicated brushwood of the complex transfer kernel type H.4.

Figure 4 shows three pruned descendant trees $\mathcal{T}_*(\mathfrak{R})$ with roots $\mathfrak{R} = \langle 243, 4\rangle$, $\mathfrak{R} = \langle 6561, 615\rangle$, and $\mathfrak{R} = \langle 6561, 613\rangle-\#1;1-\#2;1$, all of whose vertices are of type H.4 exclusively. We restrict the trees to σ-groups indicated by green color. The top vertex $\langle 27, 3\rangle$ is intentionally drawn twice to avoid an overlap of the dense trees and to admit a uniform representation of periodic bifurcations.

FIGURE 4. Branches of σ-groups with complex type H.4 connected by the maintrunk



The tree with root $\langle 243,4\rangle$ is not concerned by the quotients $\mathcal{Q}_c^{(0,+1)}$. It is sporadic and consists of periodically repeating finite saplings of depth 2 and increasing coclass $2,3,\ldots$. Connected by the maintrunk with vertices of type c.18 (red color) in the descendant tree $\mathcal{T}(\langle 243,6\rangle)$, the trees with roots $\langle 6561,615\rangle$ and $\langle 6561,613\rangle-\#1;1-\#2;1$ form the beginning of an infinite sequence of similar trees, which are, however, not isomorphic as graphs, since the depth of the constituting saplings increases in steps of 2. The projections of the quotients $\mathcal{Q}_c^{(0,+1)}$ with odd class $c\in\{5,7\}$ onto their metabelianizations are indicated by blue arrows.

3.6. **Topologies in descendant trees.** Tree topologies describe the mutual location of distinct higher p-class groups $\mathrm{G}_p^{(m)}F$ and $\mathrm{G}_p^{(n)}F$, with $n>m\ge 1$, of an algebraic number field F. The case $(m,n)=(3,4)$ will be crucial for finding the first examples of *four-stage towers* of p-class fields with length $\ell_pF:=\mathrm{dl}(\mathrm{G}_p^{(\infty)}F)=4$, which are unknown up to now, for any prime $p\ge 2$. Fork topologies with $(m,n)=(2,3)$ have proved to be essential for discovering p-class towers with length $\ell_pF=3$, for odd primes $p\ge 3$. In [13, Prp. 5.3, p. 89], we have pointed out that the *qualitative* topology problem for $(m,n)=(1,2)$ is trivial, since the fork of $\mathrm{G}_p^{(1)}F$ and $\mathrm{G}_p^{(2)}F$ is simply the abelian root $\mathrm{G}_p^{(1)}F\simeq\mathrm{Cl}_pF$ of the entire relevant descendant tree. However, the *quantitative* structure of the root path between $\mathrm{G}_p^{(2)}F$ and $\mathrm{G}_p^{(1)}F$ is not at all trivial and can be given in a general theorem for $\mathrm{Cl}_pF\simeq(p,p)$ and $p\in\{2,3\}$ only. In the following Theorem 3.6, we establish a purely group theoretic version of this result by replacing $\mathrm{G}_p^{(2)}F$ with an arbitrary *metabelian* 3-group $\mathfrak{M}$ having abelianization $\mathfrak{M}/\mathfrak{M}'$ of type $(3,3)$. Any attempt to determine the group $G:=\mathrm{Gal}(F_p^{(\infty)}/F)$ of the p-class tower $F_p^{(\infty)}$ of an algebraic number field F begins with a search for the metabelianization $\mathfrak{M}:=G/G''$, i.e., the second derived quotient, of the p-tower group G. $\mathfrak{M}$ is also called the *second p-class group* $\mathrm{Gal}(F_p^{(2)}/F)$ of F, and $F_p^{(2)}$ can be viewed as a metabelian approximation of the p-class tower $F_p^{(\infty)}$. In the case of the smallest odd prime $p=3$ and a number field F with 3-class group Cl_3F of type $(3,3)$, the structure of the root path from $\mathfrak{M}$ to the root $\langle 9,2\rangle$ is known explicitly. For its description, it suffices to use the set of possible transfer kernel types

$$\mathrm{X}\in\{\mathrm{A},\mathrm{D},\mathrm{E},\mathrm{F},\mathrm{G},\mathrm{H},\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\}$$

of the ancestors $\pi^j\mathfrak{M}$, $0\le j\le\ell$, and the symbol $\stackrel{s}{\to}$ for a weighted edge of step size $s\ge 1$ with formal exponents denoting iteration. A capable vertex is indicated by an asterisk X^*.

Theorem 3.6. *(Periodic root paths.)*
There exist basically three kinds of root paths $P:=\left(\pi^j\mathfrak{M}\right)_{0\le j\le\ell}$ of metabelian 3-groups $\mathfrak{M}$ with abelianization $\mathfrak{M}/\mathfrak{M}'$ of type $(3,3)$, which are located on coclass trees. Let c denote the nilpotency class $\mathrm{cl}(\mathfrak{M})$ and r the coclass $\mathrm{cc}(\mathfrak{M})$ of $\mathfrak{M}$.

(1) *If $r=1$ and $c\ge 1$, then $P=\mathrm{X}\left\{\stackrel{1}{\to}\mathrm{a}^*\right\}^{c-1}$, where $\mathrm{X}\in\{\mathrm{A},\mathrm{a},\mathrm{a}^*\}$.*

(2) *If $r=2$ and $c\ge 3$, then either $P=\mathrm{X}\left\{\stackrel{1}{\to}\mathrm{b}^*\right\}^{c-3}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{d},\mathrm{b},\mathrm{b}^*\}$, or $P=\mathrm{X}\left\{\stackrel{1}{\to}\mathrm{c}^*\right\}^{c-3}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{E},\mathrm{G}^*,\mathrm{H}^*,\mathrm{c}^*\}$. An additional variant arises for $r=2$, $c\ge 5$, with $P=\mathrm{X}\stackrel{1}{\to}\mathrm{X}^*\left\{\stackrel{1}{\to}\mathrm{c}^*\right\}^{c-4}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{G},\mathrm{H}\}$.*

(3) *If $r\ge 3$ and $c\ge r+1$, then either $P=\mathrm{X}\left\{\stackrel{1}{\to}\mathrm{b}^*\right\}^{c-(r+1)}\left\{\stackrel{2}{\to}\mathrm{b}^*\right\}^{r-2}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{d},\mathrm{b},\mathrm{b}^*\}$, or $P=\mathrm{X}\left\{\stackrel{1}{\to}\mathrm{d}^*\right\}^{c-(r+1)}\left\{\stackrel{2}{\to}\mathrm{b}^*\right\}^{r-2}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{F},\mathrm{G}^*,\mathrm{H}^*,\mathrm{d}^*\}$. An additional variant arises for $r\ge 3$, $c\ge r+3$, with $P=\mathrm{X}\stackrel{1}{\to}\mathrm{X}^*\left\{\stackrel{1}{\to}\mathrm{d}^*\right\}^{c-(r+2)}\left\{\stackrel{2}{\to}\mathrm{b}^*\right\}^{r-2}\stackrel{2}{\to}\mathrm{a}^*\stackrel{1}{\to}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{G},\mathrm{H}\}$.*

In particular, the maximal possible step size is $s=2$, and the $r-1$ edges with step size $s=2$ arise successively without gaps at the end of the path, except the trailing edge of step size $s=1$.

Proof. X always denotes the type of the starting vertex $\mathfrak{M}$. The remaining vertices of the root path form the *scaffold*, which connects the starting vertex with the ending vertex (the root $R=\langle 9,2\rangle$).

The unique coclass tree $\mathcal{T}^{(1)}\langle 9,2\rangle$ with $r=1$ has a mainline of type a^*. Two of the coclass trees $\mathcal{T}^{(2)}\langle 243,n\rangle$ with $r=2$, those with $n\in\{6,8\}$, have mainlines of type c^* and an additional scaffold of type a^*. For $n=3$, the mainline is of type b^*. The coclass trees $\mathcal{T}^{(r)}$ with $r\ge 3$ behave uniformly with mainlines of type b^* or d^* and scaffold types b^*, a^*. For details, see Nebelung [11, Satz 3.3.7, p. 70, Lemma 5.2.6, p. 183, Satz 6.9, p. 202, Satz 6.14, p. 208]. □

Remark 3.3. The final statement of Theorem 3.6 is a graph theoretic reformulation of the quotient structure of the lower central series $(\gamma_j\mathfrak{M})_{j\ge 1}$ of a metabelian 3-group $\mathfrak{M}$, observing that the root $R=\langle 9,2\rangle$ corresponds to the bicyclic quotient $\gamma_1/\gamma_2\simeq(3,3)$ and the conspicuous trailing edge $\xrightarrow{1}\mathrm{a}^*$ corresponds to the cyclic bottleneck $\gamma_2/\gamma_3\simeq(3)$. The structure is drawn ostensively in Formula (2.12) of [16, § 2.2], using the CF-invariant $e=r+1$ instead of the coclass r.

Theorem 3.6 concerns periodic vertices on coclass trees. Sporadic vertices outside of coclass trees must be treated separately in Corollary 3.1.

Corollary 3.1. *(Sporadic root paths.)*
As before, let $\mathfrak{M}$ be a metabelian 3-group with abelianization $\mathfrak{M}/\mathfrak{M}'\simeq(3,3)$, nilpotency class $c:=\mathrm{cl}(\mathfrak{M})$ and coclass $r:=\mathrm{cc}(\mathfrak{M})$. Assume that $\mathfrak{M}$ is located outside of coclass trees.

(1) *If $r=2$ and $c=3$, then $P=\mathrm{X}\xrightarrow{2}\mathrm{a}^*\xrightarrow{1}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{D},\mathrm{G}^*,\mathrm{H}^*\}$.*
(2) *If $r=2$ and $c=4$, then $P=\mathrm{X}\xrightarrow{1}\mathrm{X}^*\xrightarrow{2}\mathrm{a}^*\xrightarrow{1}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{G},\mathrm{H}\}$.*
(3) *If $r\ge 3$ and $c=r+1$, then $P=\mathrm{X}\left\{\xrightarrow{2}\mathrm{b}^*\right\}^{r-2}\xrightarrow{2}\mathrm{a}^*\xrightarrow{1}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{F},\mathrm{G}^*,\mathrm{H}^*\}$.*
(4) *If $r\ge 3$ and $c=r+2$, then $P=\mathrm{X}\xrightarrow{1}\mathrm{X}^*\left\{\xrightarrow{2}\mathrm{b}^*\right\}^{r-2}\xrightarrow{2}\mathrm{a}^*\xrightarrow{1}\mathrm{a}^*$, where $\mathrm{X}\in\{\mathrm{G},\mathrm{H}\}$.*

Proof. As in the proof of Theorem 3.6, see the dissertation of Nebelung [11]. □

3.7. **Computing Artin patterns of p-groups.** In both Algorithms 3.1 and 3.2, we made use of a subroutine `IsAdmissible()` which filters p-groups G with abelianization $G/G'\simeq(p,p)$ having a prescribed transfer kernel type (TKT). Since an algorithm of this kind is not implemented in MAGMA, we briefly communicate a succinct form of the code for this subroutine.

Algorithm 3.3. (Transfer kernel type.)
Input: a prime number p and a finite p-group `G`.
Code:

```
if ([p,p] eq AbelianQuotientInvariants(G)) then
    x := G.1; y := G.2; // main generators
    A := []; B := []; // generators and transversal
    Append(~A,y);
    Append(~B,x);
    for e in [0..p-1] do
        Append(~A,x*y^e);
        Append(~B,y);
    end for;
    DG := DerivedSubgroup(G);
    nTotal := 0; nFixed := 0;
    TKT := [];
    for i in [1..p+1] do
        M := sub<G|A[i],DG>;
        DM := DerivedSubgroup(M);
        AQM,pr := M/DM;
        ImA := (A[i]*B[i]^-1)^p*B[i]^p; // inner transfer
        ImB := B[i]^p; // outer transfer
        T := hom<G->AQM|<A[i],(ImA)@pr>,<B[i],(ImB)@pr>>;
        KT := sub<G|DG,Kernel(T)>;
        if KT eq G then // total kernel
```

```
            Append(~TKT,0);
            nTotal := nTotal+1;
        else
            for j in [1..p+1] do
                if A[j] in KT then
                    Append(~TKT,j);
                    if (i eq j) then // fixed point
                        nFixed := nFixed+1;
                    end if;
                end if;
            end for;
        end if;
    end for;
    image := [];
    for i in [1..p+1] do
        if not (TKT[i] in image) then
            Append(~image,TKT[i]);
        end if;
    end for;
    occupation := #image;
    repetitions := 0; // maximal occupation number
    intersection := 0; // meet of repetitions and fixed points
    doublet := 0;
    for digit in [1..p+1] do
        counter := 0;
        for j in [1..#TKT] do
            if (digit eq TKT[j]) then
                counter := counter + 1;
            end if;
        end for;
        if (counter ge 2) then
            doublet := digit;
        end if;
        if (counter gt repetitions) then
            repetitions := counter;
        end if;
    end for;
    if (doublet ge 1) then
        if (doublet eq TKT[doublet]) then
            intersection := 1;
        end if;
    end if;
end if;
```

Output: transfer kernel type `TKT`, number `nTotal` of total kernels, number `nFixed` of fixed points, and further invariants `occupation,repetitions,intersection` describing the orbit of the TKT.

The output of Algorithm 3.3 is used for the subroutine `IsAdmissible(G,p,t)` in dependence on the parameter flag `t`. When the root $R = \langle 243, 8\rangle$ is selected for the tree $\mathcal{T}(R)$ the return value is determined in the following manner:

```
if (0 eq t) then
    return ((1 eq nTotal) and (2 eq nFixed)); // type c.21
elif (1 eq t) then
    return ((0 eq nTotal) and (3 eq nFixed)); // type E.8
elif (2 eq t) then
```

```
    return ((0 eq nTotal) and (2 eq nFixed) and (3 eq occupation)); // type E.9
end if;
```

For the root $R = \langle 243, 6\rangle$, we have

```
if (0 eq t) then
    return ((1 eq nTotal) and (0 eq nFixed)); // type c.18
elif (1 eq t) then
    return ((0 eq nTotal) and (1 eq nFixed)); // type E.6
elif (2 eq t) then
    return ((0 eq nTotal) and (0 eq nFixed) and (3 eq occupation)); // type E.14
end if;
```

3.8. **Benefits and drawbacks of bottom up and top down techniques.** In this chapter, we have presented several convenient ways of expressing information about *infinite sequences* of finite p-groups. Each of them has its benefits and drawbacks.

The *bottom up strategy* of constructing finite p-groups as successive extensions of a (metabelian or even abelian) starting group R, called the *root*, by recursive applications of the p-group algorithm by Newman [7] and O'Brien [8] has the benefit of visualizing the graph theoretic *root path* in the descendant tree $\mathcal{T}(R)$. Its implementation in MAGMA is incredibly stable and robust without surpassing any internal limits up to logarithmic orders of 95 and even more. Only the consumption of CPU time becomes considerable in such extreme regions.

The *top down strategy* of expressing finite p-groups as quotients of an infinite pro-p group with given pro-p presentation has the benefit of including non-metabelian groups with arbitrary coclass $r \geq 3$, periodic mainline vertices in Algorithm 3.1 and sporadic Schur σ-leaves in Algorithm 3.2. The drawback is that the evaluation of the pro-p presentation in MAGMA exceeds the maximal permitted word length for nilpotency class $c \geq 36$.

Up to this point, we have not yet touched upon *parametrized polycyclic power-commutator presentations* [17]. For the root $R = \langle 243, 6\rangle$, the metabelian vertices G of the coclass tree $\mathcal{T}^{(2)}(R)$ with class $c = \mathrm{cl}(G) \geq 5$, down to depth $\mathrm{dp}(G) \leq 1$, can be presented in the form

$$
\begin{aligned}
G_c(\alpha,\beta) = \langle\ & x, y, s_2, t_3, s_3, \ldots, s_c \ \mid \\
& s_2 = [y,x],\ t_3 = [s_2,y],\ s_i = [s_{i-1},x] \text{ for } 3 \leq i \leq c, \\
& x^3 = s_c^{\alpha},\ y^3 s_3^{-2} s_4^{-1} = s_c^{\beta},\ s_i^3 = s_{i+2}^2 s_{i+3} \text{ for } 2 \leq i \leq c-3,\ s_{c-2}^3 = s_c^2\ \rangle,
\end{aligned}
\tag{3.11}
$$

where the parameters α and β depend on the transfer kernel type $\varkappa(G)$,

$$
(\alpha,\beta) = \begin{cases} (0,0) & \text{for } \varkappa(G) \sim (0122),\ \text{c.18}, \\ (1,0) & \text{for } \varkappa(G) \sim (1122),\ \text{E.6}, \\ (0,1) \text{ or } (0,2) & \text{for } \varkappa(G) \sim (2122),\ \text{H.4}, \\ (1,1) \text{ or } (1,2) & \text{for } \varkappa(G) \sim (3122) \sim (4122),\ \text{E.14}. \end{cases}
\tag{3.12}
$$

This presentation has the benefit of including six periodic sequences with distinct transfer kernel types, and the drawback of being restricted to the fixed coclass 2.

4. The first 3-class towers of length 3

In our long desired disproof of the claim by Scholz and Taussky [18, p. 41] concerning the 3-class tower of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-9\,748})$, we presented the first p-class towers with exactly three stages, for an odd prime p, in cooperation with Bush [19, Cor. 4.1.1, p. 775]. The underlying fields F were of type E.9 in its ground state, which admits two possibilities for the second 3-class group, $\mathfrak{M} \simeq \langle 2187, 302\rangle$ or $\langle 2187, 306\rangle$. Now we want to illustrate the way which led to the *fork topologies* in Theorem 3.5 by using the more convenient type E.8, where the group $\mathfrak{M}$ is unique for every state, in particular, $\mathfrak{M} \simeq \langle 2187, 304\rangle$ for the ground state.

Remark 4.1. Concerning the notation, we are going to use *logarithmic type invariants* of abelian 3-groups, for instance $(21) \widehat{=} (9,3)$, $(32) \widehat{=} (27,9)$, $(43) \widehat{=} (81,27)$, and $(54) \widehat{=} (243,81)$.

Let $F=\mathbb{Q}(\sqrt{d})$ be an *imaginary* quadratic number field with 3-class group $\mathrm{Cl}_3F\simeq(3,3)$, and let $E_1,\ldots,E_4$ be the unramified cyclic cubic extensions of F.

Theorem 4.1. *(First towers of type* E.8.*) Let the capitulation of* 3*-classes of* F *in* $E_1,\ldots,E_4$ *be of type* $\varkappa_1F\sim(\mathbf{1},\mathbf{2},\mathbf{3},1)$*, which is called type* E.8*. Assume further that the* 3*-class groups of* $E_1,\ldots,E_4$ *possess the abelian type invariants* $\tau_1F\sim[T_1,21,21,21]$*, where* $T_1\in\{32,43,54\}$*.*

Then the length of the 3*-class tower of* F *is precisely* $\ell_3F=3$*.*

Proof. We employ the p-group generation algorithm [7, 8] for searching the Artin pattern $\mathrm{AP}(F)=(\tau_1F,\varkappa_1F)$ among the descendants of the root $R:=C_3\times C_3=\langle 9,2\rangle$ in the tree $\mathcal{T}(R)$. After two steps, $\langle 9,2\rangle\leftarrow\langle 27,3\rangle\leftarrow\langle 243,8\rangle$, we find the next root $U_5:=\langle 243,8\rangle$ of the unique relevant coclass tree $\mathcal{T}^{(2)}(U_5)$, using the assigned simple TKT E.8, $\varkappa_3=(1231)$, and its associated scaffold TKT c.21, $\varkappa_0=(0231)$. Finally, the first component $T_1=\tau_1(1)\in\{32,43,54\}$ of the TTT provides the break-off condition, according to [13, Thm. 1.21, p. 79], resp. Theorem M in [20, p. 14], and we get $\mathfrak{M}\simeq\langle 2187,304\rangle=\langle 729,54\rangle-\#1;4$ for the ground state $T_1=(32)$, $\mathfrak{M}\simeq\langle 6561,2050\rangle-\#1;2$ for the 1st excited state $T_1=(43)$, and $\mathfrak{M}\simeq\langle 6561,2050\rangle(-\#1;1)^2-\#1;2$ for the 2nd excited state $T_1=(54)$, where $\langle 2187,303\rangle=\langle 729,54\rangle-\#1;3$ and $\langle 6561,2050\rangle=\langle 2187,303\rangle-\#1;1$. The situation is visualized by Figure 2, where the three metabelianizations $\mathfrak{M}\simeq G/G''$ of the 3-tower group G, for the ground state and two excited states, are emphasized with red color. Figure 2, showing the second 3-class groups $\mathfrak{M}$, was essentially known to Ascione in 1979 [21, 22], and to Nebelung in 1989 [11]. Compare the historical remarks [2, § 3, p. 163].

In the next three Figures 5, 6, and 7, which were unknown until 2012, we present the decisive break-through establishing the first rigorous proof for three-stage towers of 3-class fields. The key ingredient is the discovery of periodic bifurcations [2, § 3, p. 163] in the complete descendant tree $\mathcal{T}(U_5)$ which is of considerably higher complexity than the coclass tree $\mathcal{T}^{(2)}(U_5)$.

For the ground state $T_1=(32)$, the first bifurcation yields the cover

$$\mathrm{cov}(\mathfrak{M})=\{\mathfrak{M},\langle 6561,622\rangle\}$$

of $\mathfrak{M}\simeq\langle 2187,304\rangle$. The relation rank $d_2\mathfrak{M}=3$ eliminates $\mathfrak{M}$ as a candidate for the 3-tower group G, according to the Corollary [20, p. 7] of the Shafarevich Theorem [13, Thm. 1.3, pp. 75–76], and we end up getting $G\simeq\langle 6561,622\rangle=\langle 729,54\rangle-\#2;4$ with a siblings topology

$$\mathrm{E}\overset{1}{\to}\ \mathrm{c}\ \overset{2}{\leftarrow}\mathrm{E}$$

which describes the relative location of $\mathfrak{M}$ and G.

For the first excited state $T_1=(43)$, the second bifurcation yields the cover

$$\mathrm{cov}(\mathfrak{M})=\{\mathfrak{M},\langle 6561,621\rangle-\#1;1-\#1;2,\langle 6561,621\rangle-\#1;1-\#2;2\}$$

of $\mathfrak{M}\simeq\langle 6561,2050\rangle-\#1;2$, where $\langle 6561,621\rangle=\langle 729,54\rangle-\#2;3$. The relation rank $d_2=3$ eliminates $\mathfrak{M}$ and $\langle 6561,621\rangle-\#1;1-\#1;2$ as candidates for the 3-tower group G, according to Shafarevich, and we get the unique $G\simeq\langle 6561,621\rangle-\#1;1-\#2;2$ with fork topology

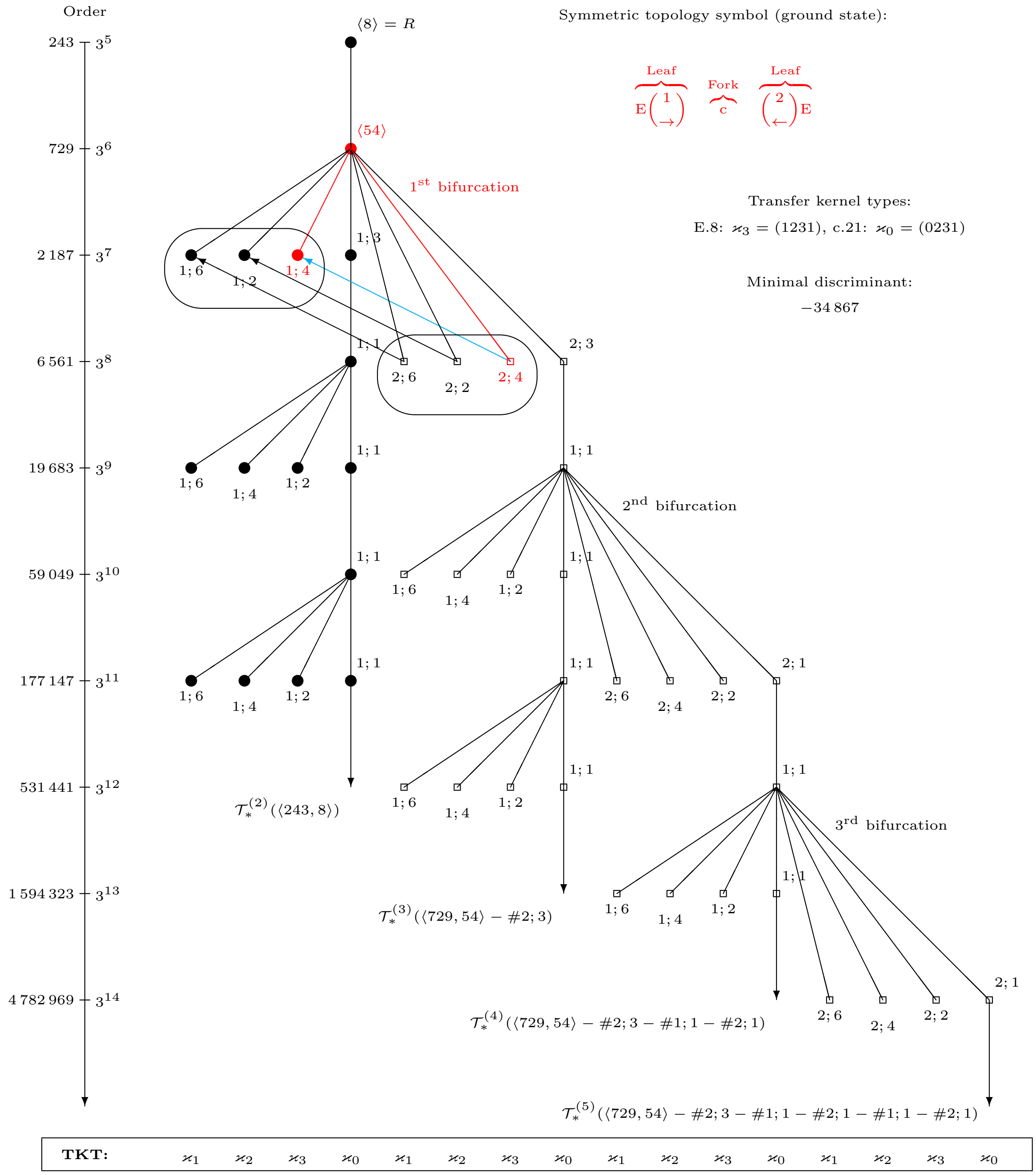
$$\mathrm{E}\overset{1}{\to}\ \left\{\mathrm{c}\overset{1}{\to}\right\}^2\ \mathrm{c}\ \left\{\overset{2}{\leftarrow}\mathrm{c}\overset{1}{\leftarrow}\mathrm{c}\right\}\ \overset{2}{\leftarrow}\mathrm{E}.$$

Similarly, the second excited state $T_1=(54)$ yields a more complex advanced fork topology

$$\mathrm{E}\overset{1}{\to}\ \left\{\mathrm{c}\overset{1}{\to}\right\}^4\ \mathrm{c}\ \left\{\overset{2}{\leftarrow}\mathrm{c}\overset{1}{\leftarrow}\mathrm{c}\right\}^2\ \overset{2}{\leftarrow}\mathrm{E}.\qquad\square$$

Figure 5 impressively shows that entering the unnoticed secret door, which is provided by the bifurcation at the vertex $\langle 729,54\rangle$, immediately leads to the long desired 3-tower group $G\simeq\langle 6561,622\rangle=\langle 729,54\rangle-\#2;4$ of the imaginary quadratic field $F=\mathbb{Q}(\sqrt{-34\,867})$. The siblings topology is emphasized with red color, and the projection $G\to\mathfrak{M}\simeq G/G''$ is drawn in blue color.

FIGURE 5. Tree topology of type E in the ground state



In Figure 6, we see that the path to the 3-tower group $G \simeq \langle 729, 54\rangle - \#2;3 - \#1;1 - \#2;2$ of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-370\,740})$ contains two bifurcations at $\langle 729, 54\rangle$ and $\langle 729, 54\rangle - \#2;3 - \#1;1$. As before, the fork topology is emphasized with red color, and the projection $G \to \mathfrak{M} \simeq G/G''$ is drawn in blue color. Two projection arrows of type E.9 are black.

FIGURE 6. Tree topology of type E in the first excited state

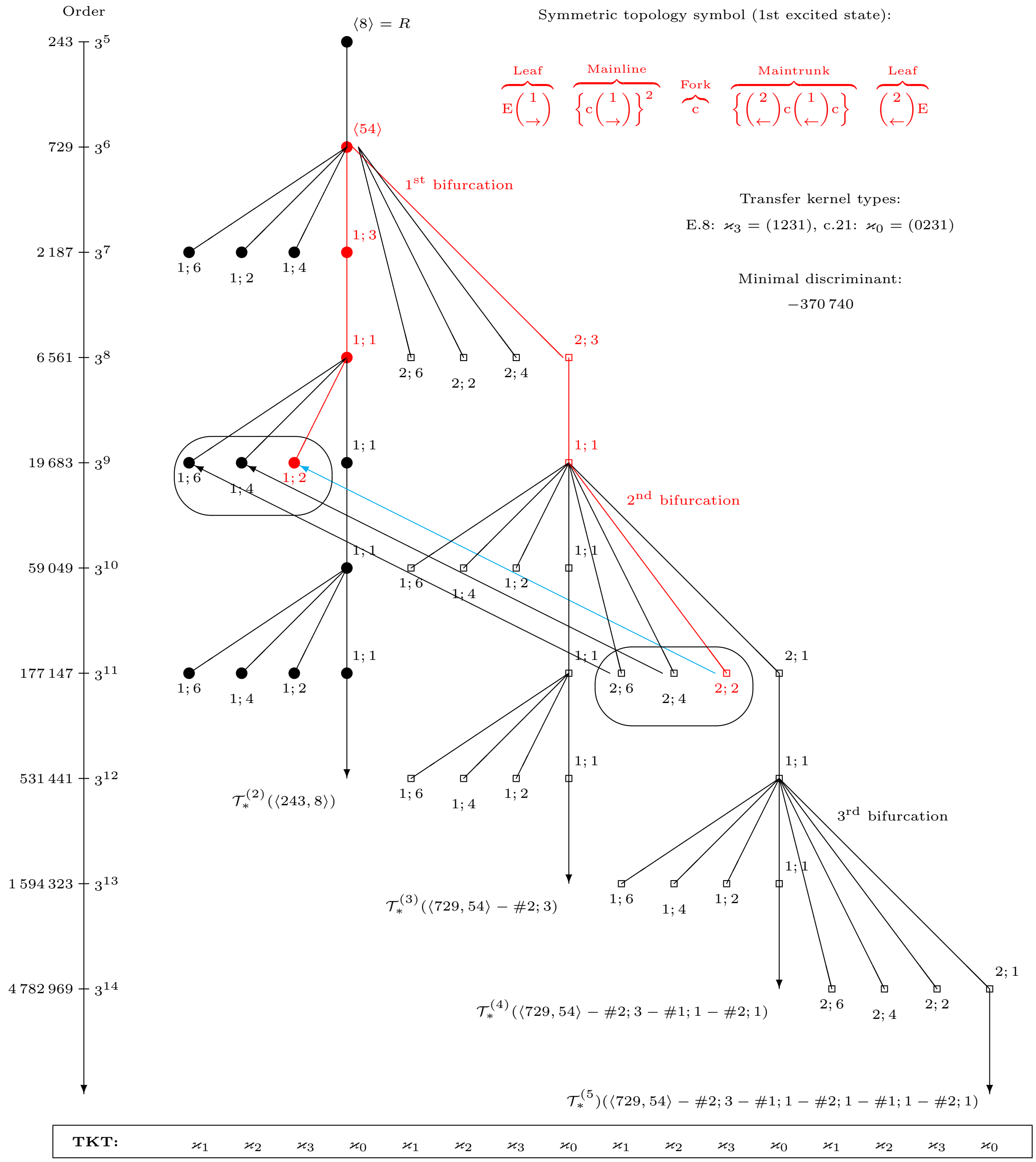
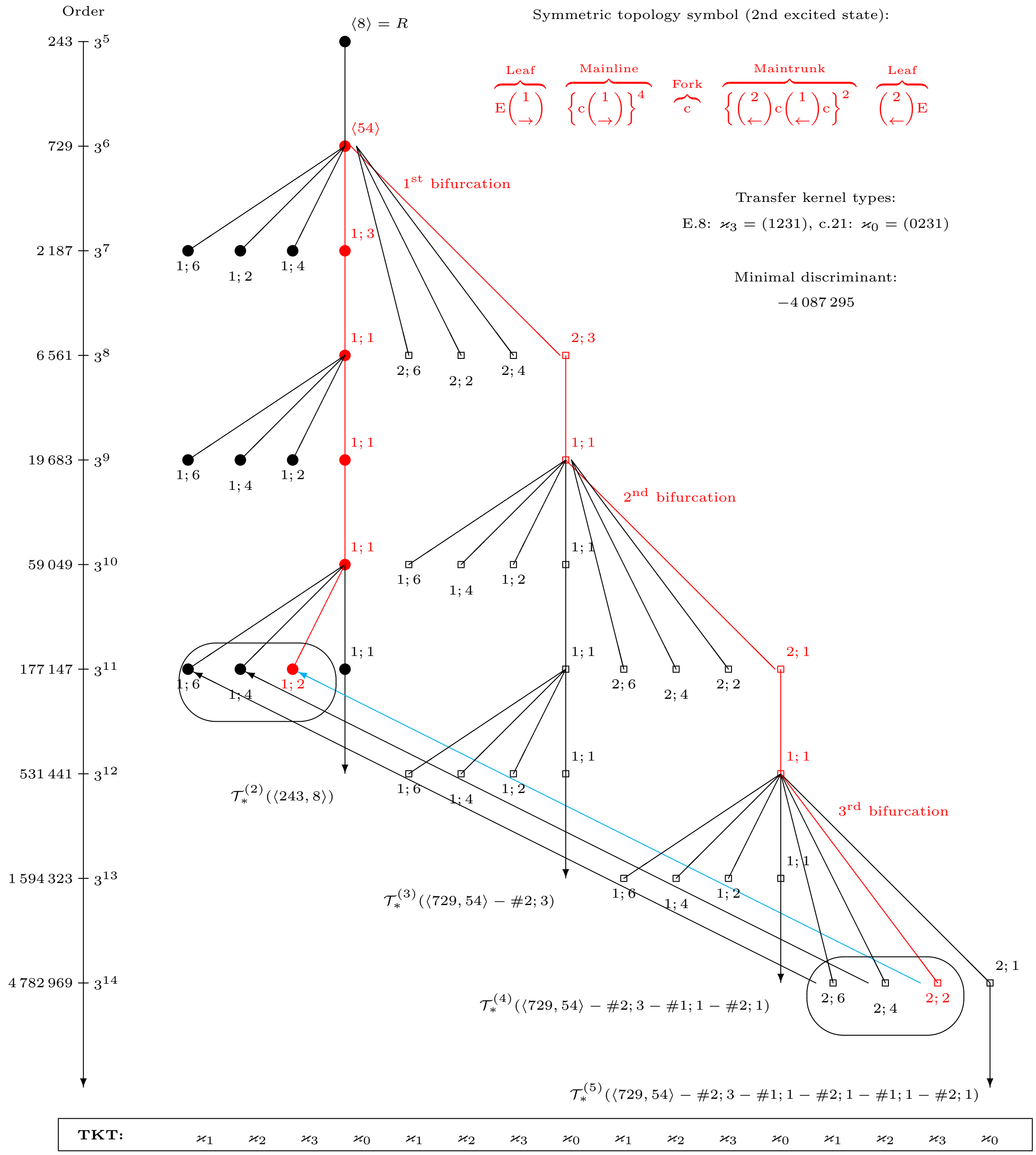


Figure 7 shows the path to the 3-tower group $G \simeq \langle 729, 54\rangle - \#2;3 - \#1;1 - \#2;1 - \#1;1 - \#2;2$ of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-4\,087\,295})$. It requires three bifurcations at $\langle 729, 54\rangle$, $\langle 729, 54\rangle - \#2;3 - \#1;1$ and $\langle 729, 54\rangle - \#2;3 - \#1;1 - \#2;1 - \#1;1$. Again, the fork topology is emphasized with red color, and the projection $G \to \mathfrak{M} \simeq G/G''$ is drawn in blue color.

FIGURE 7. Tree topology of type E in the second excited state



5. FUTURE DEVELOPMENTS

Fork topologies with significantly higher complexity and step sizes up to 3 and even 4 will be investigated in cooperation with M. F. Newman [23] for finite 3-groups with TKT F.

6. ACKNOWLEDGEMENTS

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