ISOCLINIC PROPAGATION OF ALGEBRAIC INVARIANTS BETWEEN FINITE p-GROUPS OF TYPE (p^m, p)

DANIEL C. MAYER

Abstract. Let p be a prime number, $m \geq 1$ a positive integer, and denote by $\mathcal{T}(m)$ the tree consisting of all finite two-generated p-groups G with $G/G' \simeq C(p^m) \times C(p)$ as vertices and ordered (child,parent)-pairs $(G, \pi(G))$ as directed edges. The propagation of algebraic invariants from the stem $\Phi_s(0) \subset \mathcal{T}(m)$ to branches $\Phi_s(i) \subset \mathcal{T}(m+i)$ with $i \geq 1$ of isoclinism families Φ_s of finite p-groups is described by simple transformation laws. These invariants include order, class, coclass, derived length, relation rank, nuclear rank, automorphism group, transfer kernels and abelian quotient invariants of maximal subgroups. Groups with fixed commutator quotient of rank two and type (p^m, p) are collected and visualized in descendant trees $\mathcal{T}(m)$ for $p = 2$ and $p = 3$. A shock wave of p-groups $G \in \mathcal{T}(m)$ with nilpotency class $\mathrm{cl}(G) = m$ reveals singular behavior with multifurcation and propagates through the collection of trees $(\mathcal{T}(m))_{m \geq 1}$ leaving behind a bounded domain (increasing with m) of capable vertices which give rise to infinite isogenerative α_1-chains and neogenerative α_2-chains. These phenomena explain why the p-group generation algorithm must be applied carefully for the recursive construction of the trees $\mathcal{T}(m)$ with $m \geq 2$ whose vertices possess non-elementary abelianizations.

1. Introduction

We start with theoretical foundations of power structure, commutator structure, quotient relations, descendant trees and the recently discovered *shock wave* in sections §§ 2 - 4. Then we thoroughly investigate 3-groups in section § 5, and continue with 2-groups in section § 6.

2. Power structure and commutator structure

Let p be a prime number. The key for investigating a finite p-group G with non-elementary commutator quotient G/G' is the *power structure* of G which is illuminated by a finite descending sequence of characteristic subgroups,

$$G = \mho_0(G) > \mho_1(G) > \mho_2(G) > \ldots > \mho_{\ell-1}(G) > \mho_\ell(G) = 1. \tag{2.1}$$

The length of this sequence is $\ell := \min\{n \geq 0 \mid (\forall\, g \in G)\, g^{p^n} = 1\}$, and p^ℓ is the *exponent* of G.

Definition 2.1. For each integer $k \geq 0$, let the *kth agemo subgroup* $\mho_k(G)$ of G be generated by the p^kth powers of elements $g \in G$,

$$\mho_k(G) := \langle\, g^{p^k} \mid g \in G \,\rangle. \tag{2.2}$$

Remark 2.1. We point out that the length m of the agemo series $(\mho_k(G/G'))_{0 \leq k \leq m}$ of the commutator quotient $G/G' \simeq C(p^m) \times C(p)$ will be of decisive importance for our investigations, rather than the length ℓ of the agemo series $(\mho_k(G))_{0 \leq k \leq \ell}$ of the entire group G itself.

The *commutator structure* of G is described by another finite descending sequence of characteristic subgroups,

$$G = \gamma_1(G) > \gamma_2(G) > \gamma_3(G) > \ldots > \gamma_c(G) > \gamma_{c+1}(G) = 1, \tag{2.3}$$

Date: April 24, 2018.
2000 *Mathematics Subject Classification.* 20D15, 20E18, 20E22, 20F05, 20F12, 20F14, 20–04.
Key words and phrases. Finite p-groups, extensions, descendant trees, pro-p groups, coclass forests, generator rank, relation rank, nuclear rank, parametrized polycyclic pc-presentations, automorphism groups, two-step centralizers, central series, agemo series, commutator calculus, transfer kernels, abelian quotient invariants, shock wave, p-group generation algorithm.
Research supported by the Austrian Science Fund (FWF): P 26008-N25.

whose length c is called the *nilpotency class* $\mathrm{cl}(G)$ of G.

Definition 2.2. For each integer $k \ge 1$, let the kth term $\gamma_k(G)$ of the *lower central series* of G be defined recursively by

$$\gamma_1(G) := G, \quad \text{and} \quad (\forall\, k \ge 2)\ \gamma_k(G) := [\gamma_{k-1}(G), G]. \tag{2.4}$$

In particular, $\gamma_2(G) = [G,G] = G'$ is the *commutator subgroup* of G.

3. Quotient relations and descendant trees

The p-group generation algorithm by Newman [12] and O'Brien [13], which is implemented in the computational algebra system MAGMA [5], is based on a combination of the power and commutator structure of a finite p-group G.

Definition 3.1. For each integer $k \ge 0$, let the kth term $P_k(G)$ of the *lower exponent-p central series* of G be defined recursively by

$$P_0(G) := G, \quad \text{and} \quad (\forall\, k \ge 1)\ P_k(G) := [P_{k-1}(G), G] \cdot P_{k-1}(G)^p. \tag{3.1}$$

In particular, $P_1(G) = [G,G] \cdot G^p = \Phi(G)$ is the *Frattini subgroup* of G.

As the following lemma shows, the combined power and commutator structure of a finite p-group G is described by the finite descending sequence of characteristic subgroups

$$G = P_0(G) > P_1(G) > P_2(G) > \ldots > P_{c_p-1}(G) > P_{c_p}(G) = 1, \tag{3.2}$$

whose length c_p is called the *p-class* $\mathrm{cl}_p(G)$ of G.

Lemma 3.1. *Each term of the lower exponent-p central series contains both, a term of the lower central series and a term of the agemo series of G:*

$$(\forall\, k \ge 0)\ P_k(G) \ge \gamma_{k+1}(G) \text{ and } P_k(G) \ge \mho_k(G). \tag{3.3}$$

Consequently, the p-class is not smaller than the class of G: $c_p = \mathrm{cl}_p(G) \ge c = \mathrm{cl}(G)$.

Proof. The proof is conducted by means of induction. For $k = 0$, we have $P_0(G) = \gamma_1(G) = \mho_0(G) = G$. Let $k \ge 1$ and suppose that $P_{k-1}(G) \ge \gamma_k(G)$ and $P_{k-1}(G) \ge \mho_{k-1}(G)$, then $P_k(G) = [P_{k-1}(G), G] \cdot P_{k-1}(G)^p \ge [\gamma_k(G), G] \cdot \mho_{k-1}(G)^p = \gamma_{k+1}(G) \cdot \mho_k(G)$, whence we obtain $P_k(G) \ge \gamma_{k+1}(G)$ and $P_k(G) \ge \mho_k(G)$, as required. Thus, $P_{c-1}(G) \ge \gamma_c(G) > 1$ and $c_p \ge c$. □

Several *quotient relations* between finite p-groups can be used to visualize finite sets of such groups in tree diagrams [8]. We shall be concerned with the *child-parent* relation, which is useful in the theory of coclass trees [2], and with the *successor-ancestor* relation, which adequately describes the p-group generation algorithm [12, 13].

Definition 3.2. Let p be a prime number and G be a finite non-abelian p-group.

- The *parent* $\pi(G)$ of G is the quotient $G/\gamma_c(G)$, where $(\gamma_i(G))_{i\ge1}$ denotes the lower central series of G and $c := \mathrm{cl}(G) \ge 2$ is the nilpotency class of G such that $\gamma_c(G) > \gamma_{c+1}(G) = 1$. Conversely, G is called a *child* of $\pi(G)$. By abuse of notation, the natural surjection $\pi : G \to G/\gamma_c(G)$ is also denoted by π and is called the *π-projection*.
- The *ancestor* $\alpha(G)$ of G is the quotient $G/P_{c_p-1}(G)$, where $(P_i(G))_{i\ge0}$ denotes the lower exponent-p central series of G and $c_p := \mathrm{cl}_p(G) \ge 2$ is the p-class of G such that $P_{c_p-1}(G) > P_{c_p}(G) = 1$. Conversely, G is called a *successor* of $\alpha(G)$. By abuse of notation, the canonical epimorphism $\alpha : G \to G/P_{c_p-1}(G)$ is also denoted by α and is called the *α-projection*.
- G is a *descendant*, respectively *p-descendant*, of a finite p-group H, if there exists a non-negative integer $i \ge 0$ such that $H = \pi^i(G)$, respectively $H = \alpha^i(G)$, is the i-th iterated parent, respectively ancestor, of G.

By means of these quotient relations, we can define digraphs which turn out to be rooted directed in-trees [10].

Definition 3.3. Let p be a prime number and R be a finite p-group.

- The *abelian quotient invariants* (AQI) of R are the abelian type invariants of its commutator quotient R/R'.
- The *descendant tree* $\mathcal{T}(R)$, respectively *p-descendant tree* $\mathcal{A}(R)$, of R is a rooted in-tree with root vertex R and with directed edges. It consists of all (isomorphism classes of) descendants, respectively p-descendants, G of R as vertices, and of all ordered pairs $(G, \pi(G))$, respectively $(G, \alpha(G))$, of vertices as directed edges.
- R is called AQI-*settled* if it shares a common abelianization R/R' with all of its p-descendants. (Note that R always shares a common abelianization with all of its descendants.)

Descendant trees are also called π-trees, and p-descendant trees are called α-trees, according to the respective projections which give rise to the directed edges of the trees. Now we are in the position to define the principal objects of investigation in the present article. For each integer $m \geq 1$, let R_m be the abelian p-group with type invariants (p^m, p), respectively $(m, 1)$ in logarithmic form, that is $R_m \simeq C(p^m) \times C(p)$. Denote by $\mathcal{T}(m)$ the descendant tree $\mathcal{T}(R_m)$ with abelian root R_m.

4. Propagation of the shock wave through the π-trees $\mathcal{T}(m)$

For each $m \geq 1$, the π-tree $\mathcal{T}(m)$ consists precisely of all finite p-groups G sharing the common abelian quotient invariants $(m, 1)$ in logarithmic form. Each of them is connected with the abelian tree root R_m by a unique finite path of iterated parents

$$R_m = \pi^c(G) \leftarrow \pi^{c-1}(G) \leftarrow \cdots \leftarrow \pi^2(G) \leftarrow \pi^1(G) \leftarrow \pi^0(G) = G, \tag{4.1}$$

which is called the *root path* of G, and where $c = \mathrm{cl}(G)$ denotes the nilpotency class of G.

Only for $G \in \mathcal{T}(1)$, the abelianization G/G' is *elementary* abelian of type $(1, 1)$. For all $m \geq 2$, however, the abelianization G/G' of $G \in \mathcal{T}(m)$ is *non-elementary* abelian of type $(m, 1)$. The key for understanding (two-generated) p-groups with non-elementary commutator quotient is a partition of each tree $\mathcal{T}(m)$, $m \geq 2$, into three domains in dependence on the nilpotency class.

Definition 4.1. Let p be a prime and $m \geq 1$ be an integer. Suppose that $G \in \mathcal{T}(m)$ is a finite p-group with nilpotency class $c = \mathrm{cl}(G) \geq 1$. We say G is located *behind*, respectively *on*, respectively *before*, the *shock wave*, if $c < m$, respectively $c = m$, respectively $c > m$.

4.1. Usual behavior before the shock wave.

The *infinite* region *before* the shock wave has the nice property that α- and π-projections are the same, which we call the *usual* behavior.

Theorem 4.1. *Let $m \geq 1$ be fixed. Each vertex $G \in \mathcal{T}(m)$ before the shock wave, that is, with nilpotency class $c = \mathrm{cl}(G) > m$, is* AQI-*settled and satisfies the following coincidences:*

$$\begin{aligned} c_p = \mathrm{cl}_p(G) = \mathrm{cl}(G) = c \ &\textit{(p-class and class)}, \\ P_{c_p-1}(G) = \gamma_c(G) \ &\textit{(lower exponent-p central and lower central)}, \\ \alpha(G) = G/P_{c_p-1}(G) = G/\gamma_c(G) = \pi(G) \ &\textit{(ancestor and parent)}. \end{aligned} \tag{4.2}$$

Proof. For a finite p-group G with commutator quotient $G/G' \simeq C(p^m) \times C(p)$, all powers in $\mho_m(G) = \langle g^{p^m} \mid g \in G \rangle$ are contained in the commutator subgroup $\gamma_2(G) = G'$, which can also be expressed by $\mho_m(G/G') = 1$. We consider the terms of the lower p-central series, $P_1(G) = [G, G] \cdot G^p = \gamma_2(G) \cdot \mho_1(G)$, $P_2(G) = [P_1(G), G] \cdot P_1(G)^p = [\gamma_2(G), G] \cdot [\mho_1(G), G] \cdot \gamma_2(G)^p \cdot \mho_1(G)^p = \gamma_3(G) \cdot \mho_2(G)$, and generally $P_i(G) = [P_{i-1}(G), G] \cdot P_{i-1}(G)^p = [\gamma_i(G), G] \cdot [\mho_{i-1}(G), G] \cdot \gamma_i(G)^p \cdot \mho_{i-1}(G)^p = \gamma_{i+1}(G) \cdot \mho_i(G)$. Before the shock wave, the nilpotency class $c = \mathrm{cl}(G) > m$ is so large that the powers in $\mho_i(G)$ do not contribute to the lower p-centrals and $P_i(G) = \gamma_{i+1}(G)$, for $i \geq m$. In particular, we have $c - 1 \geq m$ by assumption, and thus $P_{c-1}(G) = \gamma_c(G) > P_c(G) = \gamma_{c+1}(G) = 1$, which proves the coincidence $c_p = c$. It follows that $\alpha(G) = G/P_{c_p-1}(G) = G/P_{c-1}(G) = G/\gamma_c(G) = \pi(G)$. Consequently, G is AQI-settled. □

Example 4.1. For $m = 1$, all non-abelian finite p-groups $G \in \mathcal{T}(1)$ reveal the usual behavior. An exception is only the abelian root $R_1 \simeq C(p) \times C(p)$ of $\mathcal{T}(1)$.

4.2. **Anomalous behavior behind the shock wave.** In this section, we describe several striking phenomena which we have discovered most recently. The anomalous behavior behind the shock wave is illustrated with the aid of finite 3-groups G, characterized by their identifiers $\langle o, i\rangle$ consisting of the order $o = \mathrm{ord}(G)$ and an integer counter $i \ge 1$ in the SmallGroups database [1].

The *shock wave* of p-groups $G \in \mathcal{T}(m)$ with nilpotency class $\mathrm{cl}(G) = m$ reveals singular behavior with multifurcation and propagates through the collection of trees $(\mathcal{T}(m))_{m\ge 1}$ leaving behind a bounded domain, whose cardinality increases with m. The bounded domain contains capable vertices G with nuclear rank $\nu(G) \ge 1$ which are not AQI-settled and give rise to infinite isogenerative α_1-chains and neogenerative α_2-chains. We intentionally use the designation *chain* instead of path, because the terms of a chain are totally scattered among different trees $\mathcal{T}(m)$ whereas a path is entirely contained in a single π-tree.

Definition 4.2. Let $s \ge 1$ be a step size in the p-group generation algorithm. An finite sequence $(G_i)_{0\le i\le \ell}$ ($\ell \ge 1$) of finite p-groups is called a finite α_s-*chain* if it consists of iterated ancestors located behind or on the shock wave, that is

$$(\forall_{i=1}^{\ell})\, G_{i-1} = \alpha(G_i) \text{ and } (\exists\, m \ge \mathrm{cl}(G_0))\, (\forall_{i=0}^{\ell})\, G_i \in \mathcal{T}(m+i) \tag{4.3}$$

where $\alpha(G_i) = G_i/P_{c_p-1}(G_i)$ with $c_p = \mathrm{cl}_p(G_i)$ and $\#(P_{c_p-1}(G_i)) = p^s$. Explicitly:

$$G_0 = \alpha^{\ell}(G_\ell) \leftarrow G_1 = \alpha^{\ell-1}(G_\ell) \leftarrow \ldots \leftarrow G_{\ell-2} = \alpha^2(G_\ell) \leftarrow G_{\ell-1} = \alpha^1(G_\ell) \leftarrow G_\ell = \alpha^0(G_\ell), \tag{4.4}$$

where $\mathrm{cl}(G_i) \le m + i$ for all $0 \le i \le \ell$, and G_0, respectively G_ℓ, is called the *initial*, respectively *final*, term of the chain. (Note that no term except the final term can be AQI-settled.)

An infinite sequence $(G_i)_{i\ge 0}$ of finite p-groups is called a *sustainable chain* or, more precisely, an infinite α_s-*chain* if each finite subsequence $(G_i)_{0\le i\le \ell}$ with $\ell \ge 1$ is a finite α_s-chain.

An α_s-chain and its terms are called *isogenerative* if $s = 1$ and all terms share a common nilpotency class. (Then the nuclear rank $\nu(G_i) = 1$ for $i \ge 1$.)

An α_s-chain and its terms are called *neogenerative* if $s \ge 2$ and two distinct terms never share a common nilpotency class. (Then the nuclear rank $\nu(G_i) \ge 2$ for $i \ge 0$.)

The smallest possible initial term of an α_s-chain $(G_i)_{i\ge 0}$, in the sense that there does not exist a finite p-group G such that $G = \alpha(G_0)$, where $\alpha(G_0) = G_0/P_{c_p-1}(G_0)$ with $c_p = \mathrm{cl}_p(G_0)$ and $\#(P_{c_p-1}(G_0)) = p^s$, is called a *regenerative center*.

Theorem 4.2. *Suppose for an isogenerative chain $(G_i)_{i\ge 0}$ there exists another isogenerative chain $(\pi(G_i))_{i\ge 0}$ consisting of parents. Then, for each term G_i, possibly except the initial term G_0, the commutator relation $\alpha \circ \pi = \pi \circ \alpha$ for α- and π-projections holds, more precisely, for $i \ge 1$,*

$$\alpha(\pi(G_i)) = \Big(G_i/\gamma_c(G_i)\Big)/P_{\tilde{c}_p-1}(\pi(G_i)) = \Big(G_i/P_{c_p-1}(G_i)\Big)/\gamma_{\tilde{c}}(\alpha(G_i)) = \pi(\alpha(G_i)), \tag{4.5}$$

where $c = \mathrm{cl}(G_i)$, $\tilde{c}_p = \mathrm{cl}_p(\pi(G_i))$, $c_p = \mathrm{cl}_p(G_i)$ and $\tilde{c} = \mathrm{cl}(\alpha(G_i))$. The term G_0 must be excluded if and only if it is a regenerative center, which must necessarily be located on the shock wave.

Proof. If G_0 is the smallest possible initial term of the isogenerative α_1-chain, i.e. a regenerative center, then $\#(P_{c_p-1}(G_0)) = p^s$ necessarily with step size $s \ge 2$. Consequently, both, $\alpha(G_0)$ and G_0 are located on the shock wave. □

Corollary 4.1. *Let $m_0 \ge 2$ be a fixed integer. For each $m \ge m_0$, there exists a finite subtree $\mathcal{S}(m) \subset \mathcal{T}(m)$, whose cardinality depends on m_0, such that $\mathcal{S}(m) \simeq \mathcal{S}(m-1)$ are isomorphic as directed graphs, for $m > m_0$.*

Proof. Let $\mathcal{S}(m) := \{G \in \mathcal{T}(m) \mid \mathrm{cl}(G) < m \text{ and } \nu(G) = 1\}$. By abuse of notation, we consider the local natural projection $\alpha : G \to \alpha(G) = G/P_{c_p-1}(G)$ (onto the quotient by the last non-trivial term of the lower p-central series) as a global mapping $\alpha : \mathcal{S}(m) \to \mathcal{S}(m-1)$, $G \mapsto \alpha(G)$. The validity of the commutator relation 4.5 in Theorem 4.2 is precisely the necessary and sufficient condition that α is an isomorphism of directed trees. □

Example 4.2. An infinite family of finite 3-groups G with varying nilpotency class $c = \mathrm{cl}(G) \ge 2$ and abelian quotient invariants $(m,1)$, $m \ge c$, that is $G/G' \simeq C(p^m) \times C(p)$, is given by the parametrized power commutator presentation

$$
\begin{aligned}
G_m^c(x,y) :=\langle\ & x, y, s_1, \dots, s_c, \tau_0, \dots, \tau_m \ \mid \\
& s_1 = y,\ (\forall_{i=2}^{c+1})\, s_i = [s_{i-1}, x],\ s_{c+1} = 1, \\
& \tau_0 = x,\ (\forall_{j=1}^{m})\, \tau_j = \tau_{j-1}^3,\ \tau_m = 1, \\
& (\forall_{i=1}^{c})\, s_i^3 = s_{i+1},\ (\forall_{j+i<c,\ j>0})\, [\tau_j, s_i] = s_{j+i+1}\ \rangle,
\end{aligned}
\tag{4.6}
$$

where the second row defines the commutator structure, the third row defines the power structure, and the fourth row gives powers of commutators and commutators of powers.

All members share the common punctured transfer kernel type (pTKT) A.20, $\varkappa = (4444)$. In Table 1, resp. 2, resp. 3, the first few terms of isogenerative chains $(G_m^c)_{m \ge c}$ with constant nilpotency class $c = 2$, resp. $c = 3$, resp. $c = 4$, are shown together with their SmallGroup identifiers, type of abelianization G/G', and descendant numbers (N_1, C_1) for step size $s = 1$. In each case, only the regenerative center G_c^c has descendant numbers (N_2, C_2) for step size $s = 2$.

Table 1. Beginning of the isogenerative chain $(G_m^2)_{m \ge 2}$ with $\varkappa = (4444)$

(m,c)	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
G	$\langle 81,4\rangle$	$\langle 243,21\rangle$	$\langle 729,91\rangle$	$\langle 2187,382\rangle$	$\langle 6561,2217\rangle$
G/G'	$(9,3)$	$(27,3)$	$(81,3)$	$(243,3)$	$(729,3)$
(N_1,C_1)	$(2,1)$	$(2,1)$	$(2,1)$	$(2,1)$	$(2,1)$
(N_2,C_2)	$(1,1)$				

Table 2. Beginning of the isogenerative chain $(G_m^3)_{m \ge 3}$ with $\varkappa = (4444)$

(m,c)	$(3,3)$	$(4,3)$	$(5,3)$
G	$\langle 729,22\rangle$	$\langle 2187,193\rangle$	$\langle 6561,1785\rangle$
G/G'	$(27,3)$	$(81,3)$	$(243,3)$
(N_1,C_1)	$(2,1)$	$(2,1)$	$(2,1)$
(N_2,C_2)	$(1,1)$		

Table 3. Beginning of the isogenerative chain $(G_m^4)_{m \ge 4}$ with $\varkappa = (4444)$

(m,c)	$(4,4)$
G	$\langle 6561,199\rangle$
G/G'	$(81,3)$
(N_1,C_1)	$(2,1)$
(N_2,C_2)	$(1,1)$

4.3. **Singular behavior on the shock wave.** The subsets

$$\mathcal{C}(m) := \{G \in \mathcal{T}(m) \mid \mathrm{cl}(G) = m \text{ and } \nu(G) \geq 2\} \tag{4.7}$$

of regenerative centers in the trees $\mathcal{T}(m)$ form a *shock wave* in the sense that there exist tight lower and upper bounds $\ell(m) \leq u(m)$ for the logarithmic orders $\mathrm{lo} := \log_p \circ \mathrm{ord}$ of their elements:

$$(\forall\, G \in \mathcal{C}(m))\ \ell(m) \leq \mathrm{lo}(G) \leq u(m). \tag{4.8}$$

Theorem 4.3. *For each regenerative center $G \in \mathcal{C}(m)$ with $m \geq 2$, that is, with exception of the unique regenerative center $R_1 \simeq C(p) \times C(p) \in \mathcal{C}(1)$, the annihilator relation $\alpha \circ \pi = \alpha$ holds for α- and π-projections, more precisely,*

$$\alpha(\pi(G)) = \Big(G/\gamma_c(G)\Big)/P_{\tilde{c}_p-1}(\pi(G)) = G/P_{c_p-1}(G) = \alpha(G), \tag{4.9}$$

where $c = \mathrm{cl}(G) = m$, $\tilde{c}_p = \mathrm{cl}_p(\pi(G))$ and $c_p = \mathrm{cl}_p(G)$.

Proof. □

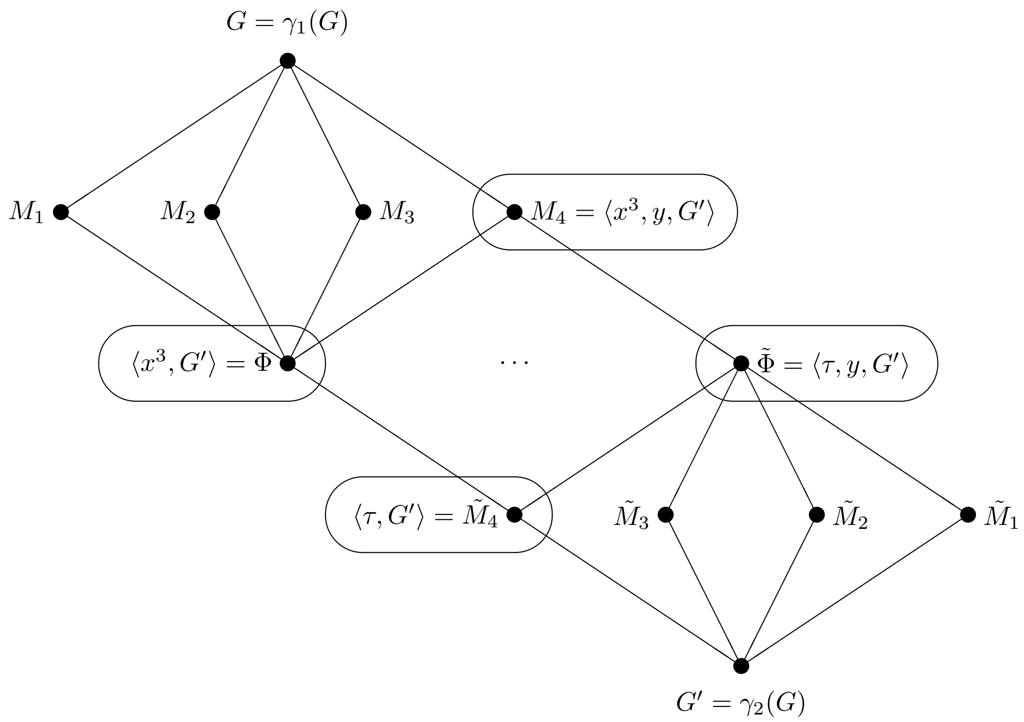
5. Isoclinism of finite 3-groups

For an integer exponent $m \geq 2$, we consider metabelian 3-groups $G = \langle x, y\rangle$ with two generators satisfying $x^{3^m} \in G'$ and $y^3 \in G'$, and with non-elementary bicyclic commutator quotient G/G' having abelian type invariants $(m, 1) \hat{=} (3^m, 3)$. Generally, such a group possesses

- four maximal normal subgroups of index 3,
$$M_1 = \langle x, G'\rangle,\ M_2 = \langle xy, G'\rangle,\ M_3 = \langle xy^2, G'\rangle,\ M_4 = \langle x^3, y, G'\rangle,$$
- Frattini subgroup $\Phi = \Phi(G) = \cap_{i=1}^4 M_i = G^3G' = \langle x^3, G'\rangle$ of index 9,
- four normal subgroups of index 3^m,
$$\tilde{M}_1 = \langle y, G'\rangle,\ \tilde{M}_2 = \langle \tau y, G'\rangle,\ \tilde{M}_3 = \langle \tau y^2, G'\rangle,\ \tilde{M}_4 = \langle \tau, G'\rangle,$$
where $\tau := x^{3^{m-1}}$, and
- a distinguished subgroup $\tilde{\Phi} = \prod_{i=1}^4 \tilde{M}_i = \langle \tau, y, G'\rangle$ of index 3^{m-1}.

We use the subscript 4 to indicate that the quotient $M_4/G' = \langle x^3, y\rangle$ is bicyclic of type $(3^{m-1}, 3)$, whereas M_i/G' is cyclic of order 3^m, for $1 \leq i \leq 3$, and that $\tilde{M}_4 \leq \Phi(G)$ is contained in the Frattini subgroup of G, whereas $\tilde{M}_i$ is only contained in M_4, for $1 \leq i \leq 3$.

Figure 1. Commutator quotient G/G' of type $(3^m, 3)$



5.1. S_3**-double orbits of punctured transfer kernel types.** The *transfer* V_i (Verlagerung) from G to its maximal subgroup M_i is given by

$$\mathrm{V}_i = \mathrm{V}_{G,M_i} : G/G' \to M_i/M_i',\ g \mapsto \begin{cases} g^3, & \text{if } g \in G \setminus M_i\,, \\ g^{\mathrm{S}_3(h)}, & \text{if } g \in M_i\,, \end{cases} \tag{5.1}$$

where $\mathrm{S}_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$, with an arbitrary element $h \in G \setminus M_i$, denotes the third *trace element* (Spur) in the group ring, acting as a symbolic exponent.

There are five possibilities for the kernel of V_i, for each $1 \le i \le 4$. Either $\ker(\mathrm{V}_i) = \tilde{M}_j/G'$, for some $1 \le j \le 4$, and we denote the *one-dimensional* transfer kernel by the singulet $\varkappa(i) = j$, or $\ker(\mathrm{V}_i) = \tilde{\Phi}/G'$, and we denote the *two-dimensional* transfer kernel by $\varkappa(i) = 0$. Due to the distinguished role of the subscript 4, we combine the singulets to form a multiplet (quartet)

$$\varkappa = (\ \varkappa(1), \varkappa(2), \varkappa(3);\ \varkappa(4)\ ) \in [0,4]^3 \times [0,4]$$

which we call the *punctured transfer kernel type* (TKT) of the group G with respect to our selection of generators x, y.

To be independent of the choice of generators and the order of M_1, M_2, M_3 and $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$, we define the *double orbit*

$$\varkappa^{S_3 \times S_3} = \{\tilde{\sigma} \circ \varkappa \circ \hat{\tau} \mid \sigma, \tau \in S_3\}$$

of $\varkappa$ under the operation of $S_3 \times S_3$ as an isomorphism invariant $\varkappa(G)$ of G. Here, $\tilde{\sigma}$ denotes the extension of σ from $[1,3]$ to $[0,4]$ which fixes 0 and 4 and $\hat{\tau}$ denotes the extension of τ from $[1,3]$ to $[1,4]$ which fixes 4.

Two further *isomorphism invariants* of G are $\mu = \mu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 4\}$ and the number of two-dimensional transfer kernels $\nu = \nu(G) = \#\{1 \le i \le 4 \mid \varkappa(i) = 0\}$.

5.2. **Combinatorially possible punctured transfer kernel types.** In this section, we arrange all combinatorially possible S_3-double orbits of the $5^4 = 625$ punctured quartets $\varkappa \in [0,4]^3 \times [0,4]$ by increasing invariant $0 \le \mu \le 4$ and cardinality of the image. Table 4 shows the punctured quartets with invariant $\nu = 0$ and Table 5 shows the punctured quartets with invariant $1 \le \nu \le 4$ as possible punctured transfer kernel types of 3-groups G with G/G' of type $(3^m, 3)$, respectively *punctured* 3*-principalization types* of number fields K with 3-class group $\mathrm{Cl}_3(K)$ of type $(3^m, 3)$, according to Artin's reciprocity law [6]. The double orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\varkappa) = (|\varkappa^{-1}\{i\}|)_{0 \le i \le 4}$ the family of occupation numbers of the selected double orbit representative $\varkappa$ (in particular, $o(\varkappa)_0 = \nu$ and $o(\varkappa)_4 = \mu$) and by κ the quartet of Taussky's conditions [14] associated with $\varkappa$.

If a double orbit $\varkappa^{S_3 \times S_3}$ can be realized as a punctured transfer kernel type $\varkappa(G)$, then a suitable 3-group G with G/G' of type $(9,3)$ is given in the notations of James [4], using Hall's isoclinism families [3], and of the SmallGroups database [1].

Theorem 5.1. *For $1 \le i \le 4$, the singulet of Taussky's conditions is given by*

$$\kappa(i) = \begin{cases} A & \text{if } i < 4,\ \varkappa(i) \in \{0,4\} \text{ or } i = 4,\ \varkappa(4) > 0, \\ A_2 & \text{if } i = 4,\ \varkappa(4) = 0, \\ B & \text{if } i < 4,\ \varkappa(i) \in \{1,2,3\}. \end{cases} \tag{5.2}$$

Proof. For $1 \le i \le 3$, the intersection of the maximal subgroup M_i with the transfer kernel $\ker(V_i)$ may be trivial,

$$M_i \cap \ker(V_i) = \begin{cases} 1 & \text{if } \varkappa(i) \in \{1,2,3\}, \\ \tilde{M}_4/G' & \text{if } \varkappa(i) \in \{0,4\}, \end{cases} \tag{5.3}$$

but for the distinguished case $i = 4$, we always have a non-trivial intersection,

$$M_4 \cap \ker(V_4) = \begin{cases} \tilde{M}_j/G' & \text{if } \varkappa(4) = j \in \{1,2,3,4\}, \\ \tilde{\Phi}/G' & \text{if } \varkappa(4) = 0. \quad \square \end{cases} \tag{5.4}$$

Table 4 gives a coarse classification into sections by uppercase letters A to E, an identification by ordinal numbers 1 to 20, and a set theoretical characterization.

TABLE 4. The 20 S_3-double orbits of $\varkappa \in [1,4]^4$ with $\nu = 0$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
A	1	(1111)	(04000)	$(BBBA)$	constant	3	$\Phi_2(31)$
B	2	(1112)	(03100)	$(BBBA)$	nearly	6	???
B	3	(1121)	(03100)	$(BBBA)$	constant	18	
C	4	(1122)	(02200)	$(BBBA)$		18	???
D	5	(1123)	(02110)	$(BBBA)$		18	
D	6	(1231)	(02110)	$(BBBA)$		18	???
B	7	(1114)	(03001)	$(BBBA)$	nearly	3	$\Phi_6(321)_{b_{1,1}}, \Phi_6(321)_{b_{1,2}}$
B	8	(1141)	(03001)	$(BBAA)$	constant	9	
D	9	(1124)	(02101)	$(BBBA)$		18	???
D	10	(1142)	(02101)	$(BBAA)$		18	???
D	11	(1241)	(02101)	$(BBAA)$		36	$\Phi_6(321)_{a_1}, \Phi_6(321)_{a_2}$
E	12	(1234)	(01111)	$(BBBA)$	per-	6	$\Phi_6(321)_{b_{2,1}}, \Phi_6(321)_{b_{2,2}}$
E	13	(1243)	(01111)	$(BBAA)$	mutation	18	
C	14	(1144)	(02002)	$(BBAA)$		9	
C	15	(1441)	(02002)	$(BAAA)$		9	
D	16	(1244)	(01102)	$(BBAA)$		18	???
D	17	(1442)	(01102)	$(BAAA)$		18	???
B	18	(1444)	(01003)	$(BAAA)$	nearly	9	???
B	19	(4441)	(01003)	$(AAAA)$	constant	3	???
A	20	(4444)	(00004)	$(AAAA)$	constant	1	$\Phi_6(2^2 1^2)_g, \Phi_2(2^2), \Phi_8(32)$
					Total number:	256	

Table 5 gives a coarse classification into sections by lowercase letters a to e, an identification by ordinal numbers 1 to 32, and a set theoretical characterisation.

TABLE 5. The 32 S_3-double orbits of $\varkappa \in [0,4]^4 \setminus [1,4]^4$ with $1 \le \nu \le 4$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_3\times S_3}\rvert$	realising 3-group G
a	1	(0000)	(40000)	$(AAAA_2)$	constant	1	$\Phi_2(21^2)_c, \Phi_3(21^3)_d, \Phi_3(21^3)_e$
b	2	(0001)	(31000)	$(AAAA)$	nearly	3	$\Phi_3(31^2)_a$
b	3	(0010)	(31000)	$(AABA_2)$	constant	9	$\Phi_3(31^2)_{b_1}, \Phi_3(31^2)_{b_2}$
c	4	(0011)	(22000)	$(AABA)$		9	
c	5	(0110)	(22000)	$(ABBA_2)$		9	
d	6	(0012)	(21100)	$(AABA)$		18	
d	7	(0120)	(21100)	$(ABBA_2)$		18	
b	8	(0111)	(13000)	$(ABBA)$	nearly	9	
b	9	(1110)	(13000)	$(BBBA_2)$	constant	3	
d	10	(0112)	(12100)	$(ABBA)$		18	$\Phi_6(31^3)_a$
d	11	(0121)	(12100)	$(ABBA)$		36	
d	12	(1120)	(12100)	$(BBBA_2)$		18	
e	13	(0123)	(11110)	$(ABBA)$	per-	18	
e	14	(1230)	(11110)	$(BBBA_2)$	mutation	6	$\Phi_6(31^3)_{b_1}, \Phi_6(31^3)_{b_2}$
b	15	(0004)	(30001)	$(AAAA)$	nearly	1	$\Phi_3(2^21)_{b_1}, \Phi_3(2^21)_{b_2}, \Phi_6(21^4)_d$
b	16	(0040)	(30001)	$(AAAA_2)$	constant	3	$\Phi_3(2^21)_a$
d	17	(0014)	(21001)	$(AABA)$		9	
d	18	(0041)	(21001)	$(AAAA)$		9	
d	19	(0140)	(21001)	$(ABAA_2)$		18	
d	20	(0114)	(12001)	$(ABBA)$		9	
d	21	(0141)	(12001)	$(ABAA)$		18	
d	22	(1140)	(12001)	$(BBAA_2)$		9	
e	23	(0124)	(11101)	$(ABBA)$	per-	18	
e	24	(0142)	(11101)	$(ABAA)$	muta-	36	
e	25	(1240)	(11101)	$(BBAA_2)$	tion	18	
c	26	(0044)	(20002)	$(AAAA)$		3	
c	27	(0440)	(20002)	$(AAAA_2)$		3	$\Phi_6(2^21^2)_{h_1}$
d	28	(0144)	(11002)	$(ABAA)$		18	
d	29	(0441)	(11002)	$(AAAA)$		9	
d	30	(1440)	(11002)	$(BAAA_2)$		9	
b	31	(0444)	(10003)	$(AAAA)$	nearly	3	$\Phi_6(2^21^2)_{h_2}$
b	32	(4440)	(10003)	$(AAAA_2)$	constant	1	
				Total number:	$625 - 256 =$	369	

FIGURE 2. Finite 3-groups G with commutator quotient $G/G' \simeq (3,3)$

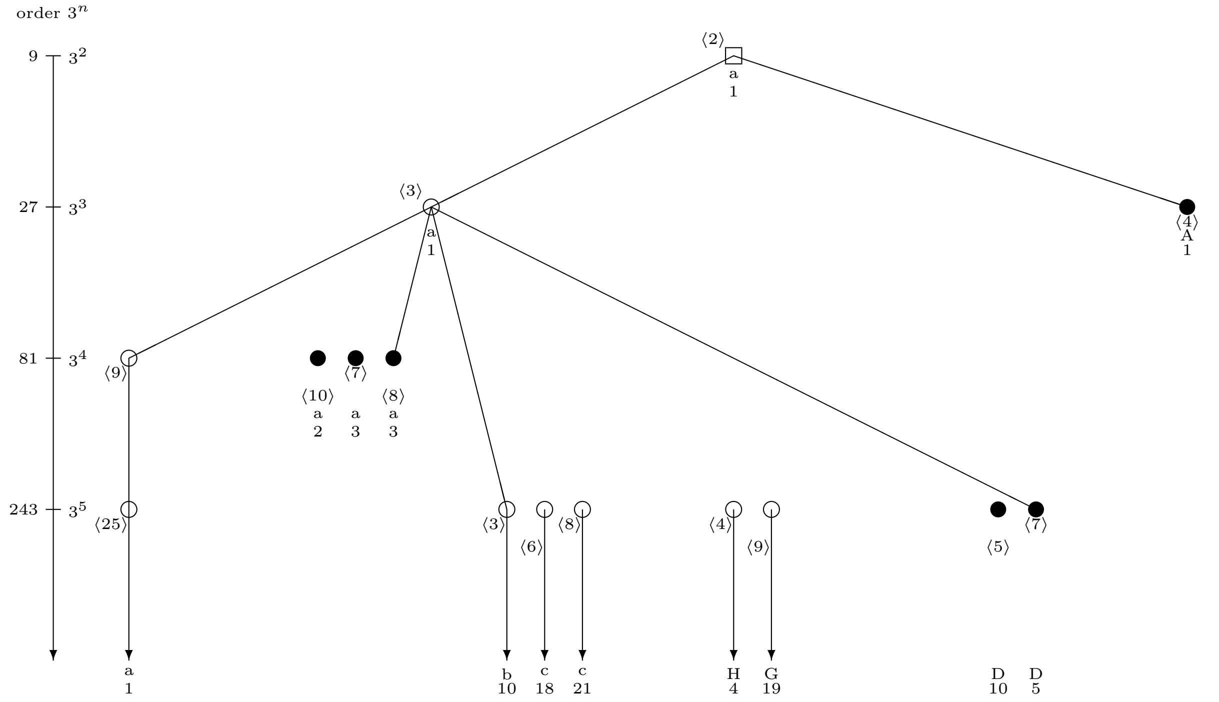


FIGURE 3. Finite 3-groups G with commutator quotient $G/G' \simeq (9,3)$

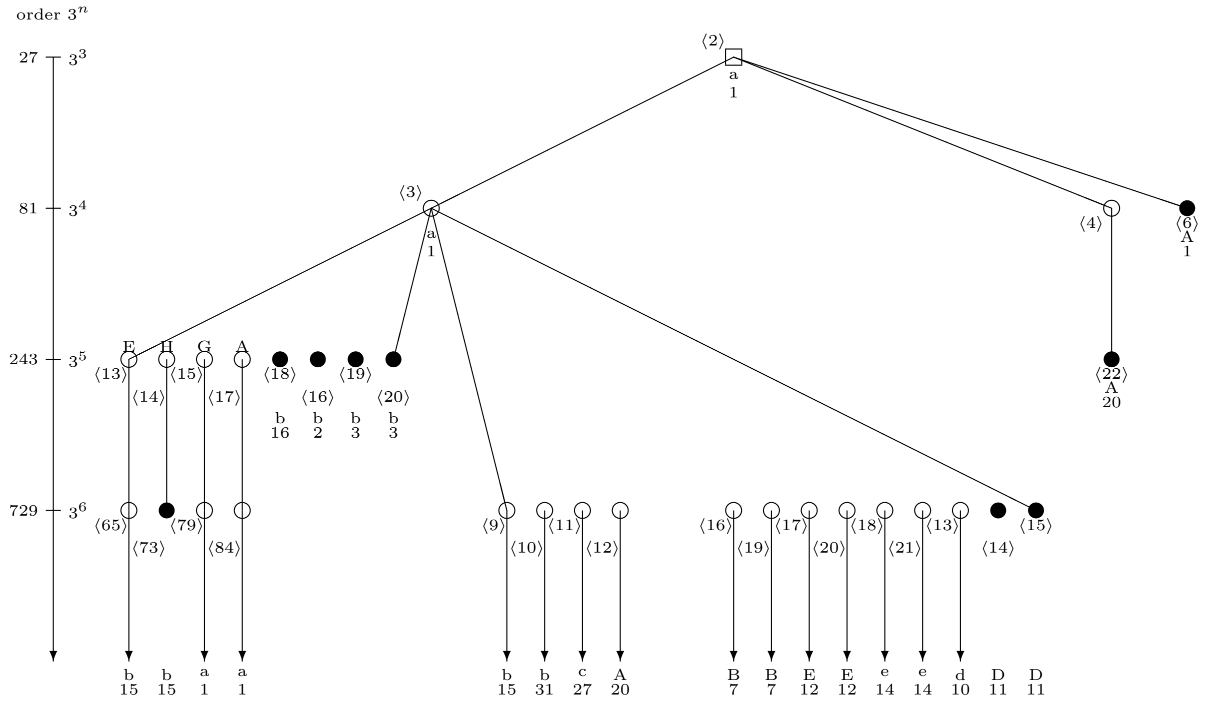


FIGURE 4. Finite 3-groups G with commutator quotient $G/G' \simeq (27,3)$

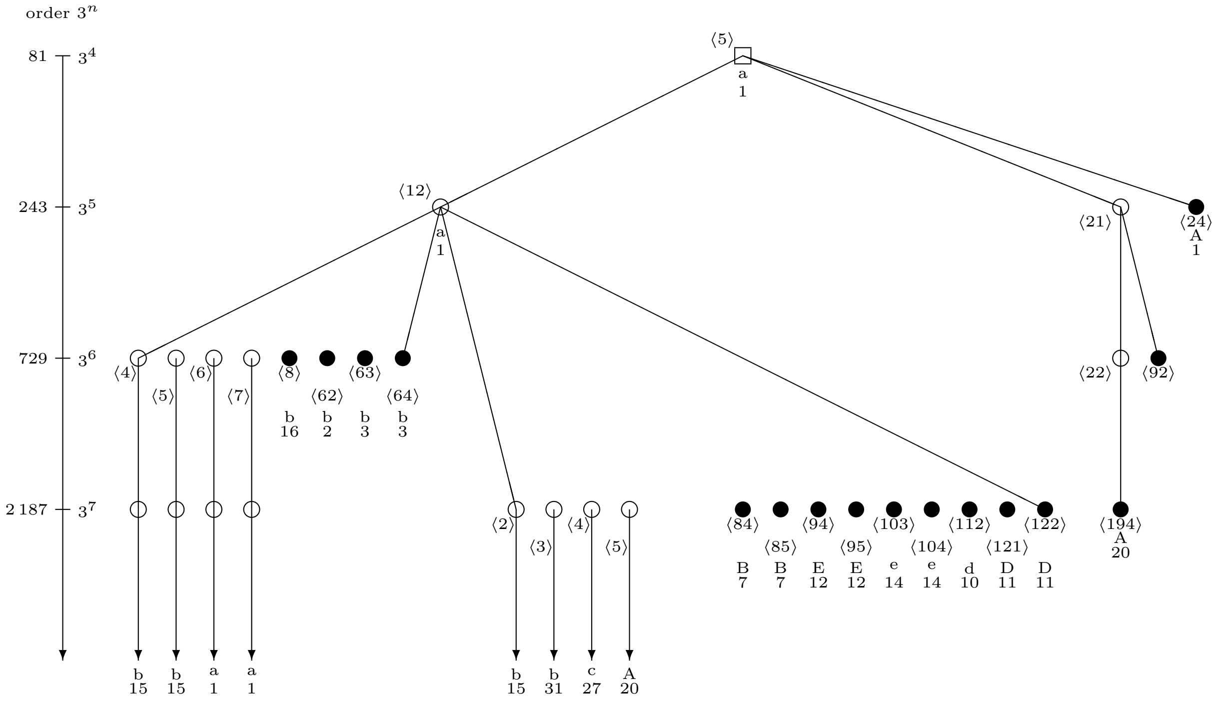


FIGURE 5. Finite 3-groups G with commutator quotient $G/G' \simeq (81,3)$

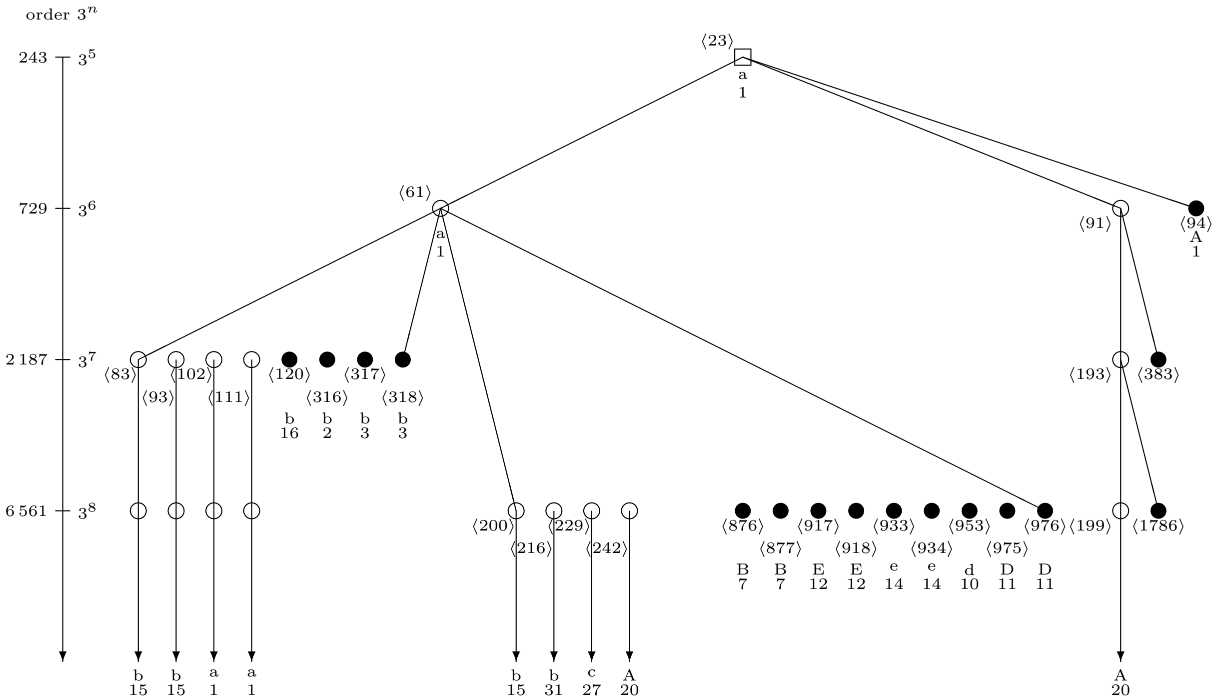


FIGURE 6. Finite 3-groups G with commutator quotient $G/G' \simeq (243, 3)$

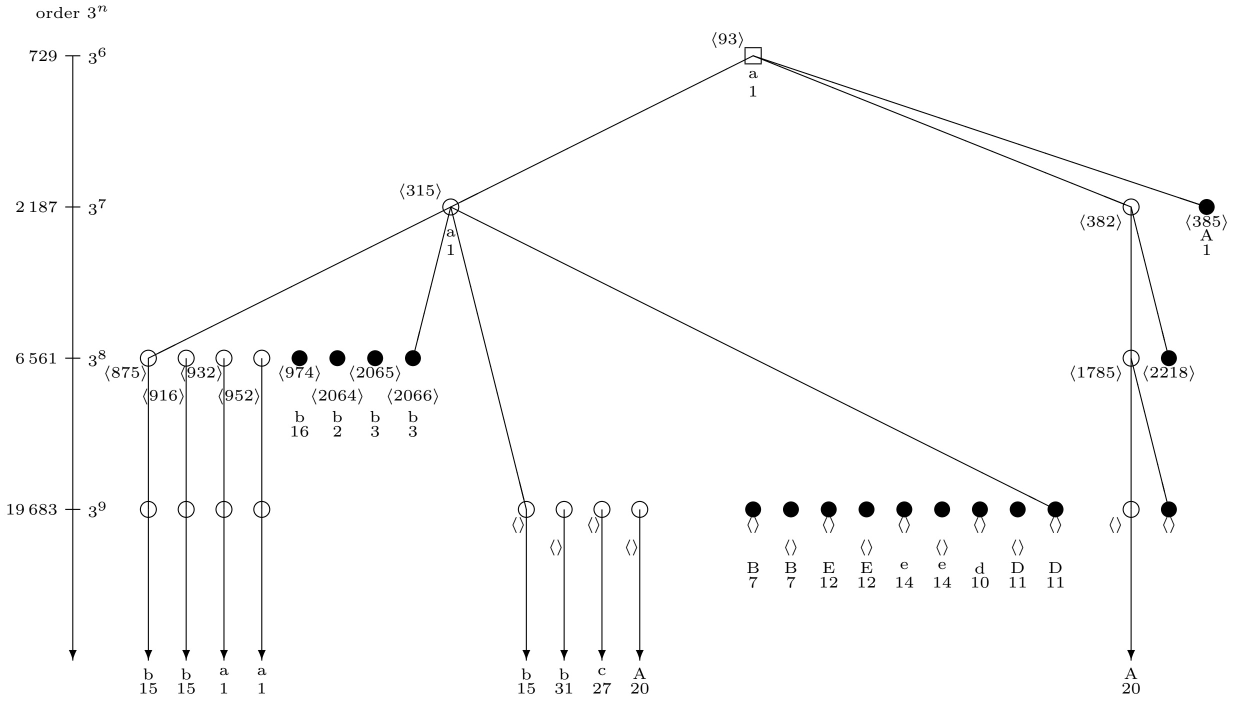
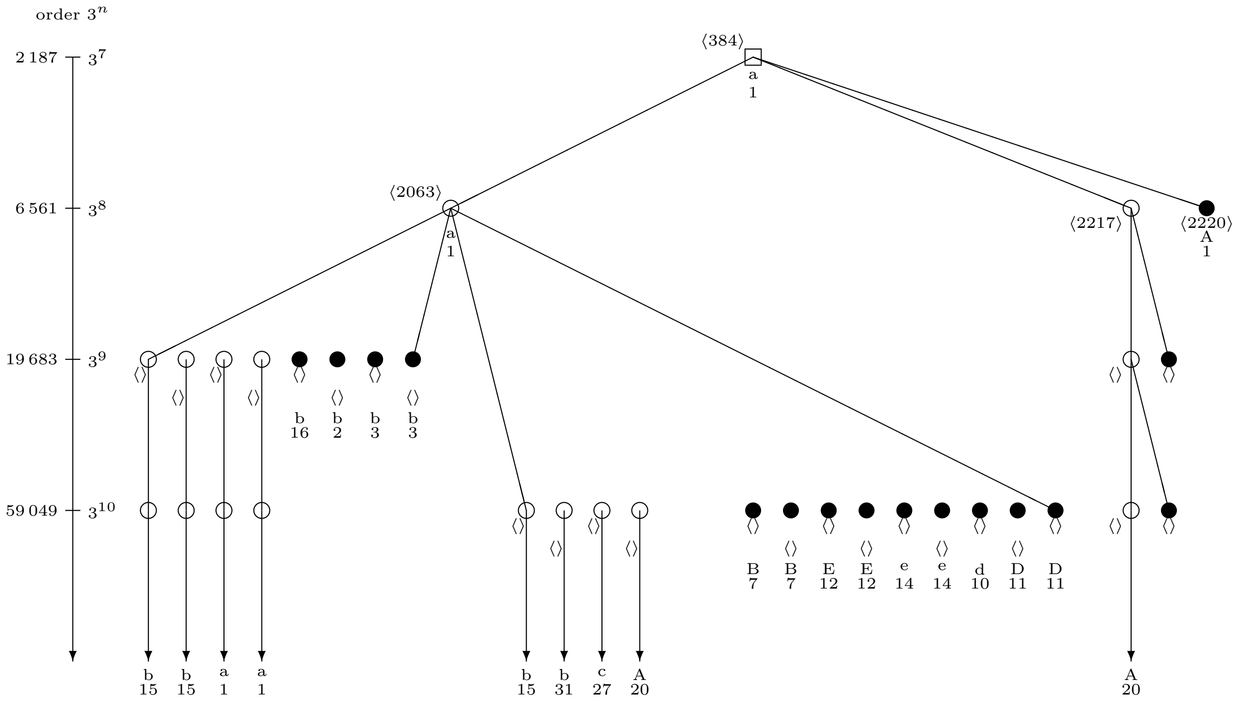


FIGURE 7. Finite 3-groups G with commutator quotient $G/G' \simeq (729, 3)$



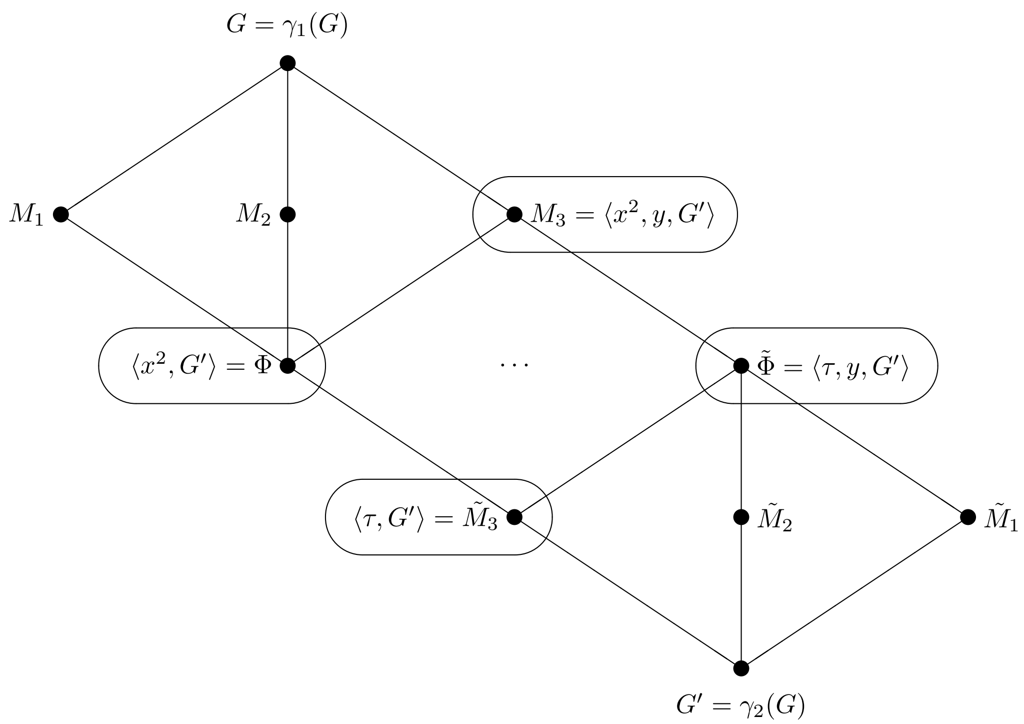
6. Isoclinism of finite 2-groups

For an integer exponent $m \geq 2$, we consider metabelian 2-groups $G = \langle x, y \rangle$ with two generators satisfying $x^{2^m} \in G'$ and $y^2 \in G'$, and with non-elementary bicyclic commutator quotient G/G' having abelian type invariants $(2^m, 2)$, in logarithmic form $(m, 1)$. Generally, such a group possesses

- three maximal normal subgroups of index 2,
$$M_1 = \langle x, G' \rangle, \ M_2 = \langle xy, G' \rangle, \ M_3 = \langle x^2, y, G' \rangle,$$
- Frattini subgroup $\Phi = \Phi(G) = \cap_{i=1}^{3} M_i = G^2 G' = \langle x^2, G' \rangle$ of index 4,
- four normal subgroups of index 2^m,
$$\tilde{M}_1 = \langle y, G' \rangle, \ \tilde{M}_2 = \langle \tau y, G' \rangle, \ \tilde{M}_3 = \langle \tau, G' \rangle,$$
where $\tau := x^{2^{m-1}}$, and
- a distinguished subgroup $\tilde{\Phi} = \prod_{i=1}^{3} \tilde{M}_i = \langle \tau, y, G' \rangle$ of index 2^{m-1}.

We use the subscript 3 to indicate that the quotient $M_3/G' = \langle x^2, y \rangle$ is bicyclic of type $(2^{m-1}, 2)$, whereas M_i/G' is cyclic of order 2^m, for $1 \leq i \leq 2$, and that $\tilde{M}_3 \leq \Phi(G)$ is contained in the Frattini subgroup of G, whereas $\tilde{M}_i$ is only contained in M_3, for $1 \leq i \leq 2$.

Figure 8. Commutator quotient G/G' of type $(2^m, 2)$



6.1. S_2**-double orbits of punctured transfer kernel types.** The *transfer* V_i (Verlagerung) from G to its maximal subgroup M_i is given by

$$\mathrm{V}_i = \mathrm{V}_{G,M_i} : G/G' \to M_i/M_i',\ g \mapsto \begin{cases} g^2, & \text{if } g \in G \setminus M_i\,, \\ g^{\mathrm{S}_2(h)}, & \text{if } g \in M_i\,, \end{cases} \tag{6.1}$$

where $\mathrm{S}_2(h) = 1 + h \in \mathbb{Z}[G]$, with an arbitrary element $h \in G \setminus M_i$, denotes the second *trace element* (Spur) in the group ring, acting as a symbolic exponent.

There are four possibilities for the kernel of V_i, for each $1 \le i \le 3$. Either $\ker(\mathrm{V}_i) = \tilde{M}_j/G'$, for some $1 \le j \le 3$, and we denote the *one-dimensional* transfer kernel by the singulet $\varkappa(i) = j$, or $\ker(\mathrm{V}_i) = \tilde{\Phi}/G'$, and we denote the *two-dimensional* transfer kernel by $\varkappa(i) = 0$. Due to the distinguished role of the subscript 3, we combine the singulets to form a multiplet (triplet)

$$\varkappa = (\,\varkappa(1), \varkappa(2);\ \varkappa(3)\,) \in [0,3]^2 \times [0,3]$$

which we call the *punctured transfer kernel type* (TKT) of the group G with respect to our selection of generators x, y.

To be independent of the choice of generators and the order of M_1, M_2 and $\tilde{M}_1, \tilde{M}_2$, we define the *double orbit*

$$\varkappa^{S_2 \times S_2} = \{\tilde{\sigma} \circ \varkappa \circ \hat{\tau} \mid \sigma, \tau \in S_2\}$$

of $\varkappa$ under the operation of $S_2 \times S_2$ as an isomorphism invariant $\varkappa(G)$ of G. Here, $\tilde{\sigma}$ denotes the extension of σ from $[1,2]$ to $[0,3]$ which fixes 0 and 3 and $\hat{\tau}$ denotes the extension of τ from $[1,2]$ to $[1,3]$ which fixes 3.

Two further *isomorphism invariants* of G are $\mu = \mu(G) = \#\{1 \le i \le 3 \mid \varkappa(i) = 3\}$ and the number of two-dimensional transfer kernels $\nu = \nu(G) = \#\{1 \le i \le 3 \mid \varkappa(i) = 0\}$.

6.2. **Combinatorially possible punctured** 2**-transfer kernel types.** In this section, we arrange all combinatorially possible S_2-double orbits of the $4^3 = 64$ punctured quartets $\varkappa \in [0,3]^2 \times [0,3]$ by increasing invariant $0 \le \mu \le 3$ and cardinality of the image. Table 6 shows the punctured triplets with invariant $\nu = 0$ and Table 7 shows the punctured triplets with invariant $1 \le \nu \le 3$ as possible punctured transfer kernel types of 2-groups G with G/G' of type $(2^m, 2)$, respectively *punctured* 2*-principalization types* of number fields K with 2-class group $\mathrm{Cl}_2(K)$ of type $(2^m, 2)$, according to Artin's reciprocity law [6]. The double orbits are divided into sections, denoted by letters, and identified by ordinal numbers.

We denote by $o(\varkappa) = (|\varkappa^{-1}\{i\}|)_{0 \le i \le 3}$ the family of occupation numbers of the selected double orbit representative $\varkappa$ (in particular, $o(\varkappa)_0 = \nu$ and $o(\varkappa)_3 = \mu$) and by κ the triplet of Taussky's conditions [14] associated with $\varkappa$.

If a double orbit $\varkappa^{S_2 \times S_2}$ can be realized as a punctured transfer kernel type $\varkappa(G)$, then a suitable 2-group G is given in the notation of James [4], using Hall's isoclinism families [3].

Table 6 gives a coarse classification into sections by uppercase letters A to C, an identification by ordinal numbers 1 to 10, and a set theoretical characterization.

Table 6. The 10 S_2-double orbits of $\varkappa \in [1,3]^3$ with $\nu = 0$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_2\times S_2}\rvert$	realising 2-group G
A	1	(111)	(0300)	(BBA)	constant	2	
B	2	(112)	(0210)	(BBA)	nearly	2	
B	3	(121)	(0210)	(BBA)	constant	4	
B	4	(113)	(0201)	(BBA)	nearly	2	
B	5	(131)	(0201)	(BBA)	constant	4	
C	6	(123)	(0111)	(BBA)	per-	2	
C	7	(132)	(0111)	(BAA)	mutation	4	
B	8	(133)	(0102)	(BAA)	nearly	4	
B	9	(331)	(0102)	(AAA)	constant	2	
A	10	(333)	(0003)	(AAA)	constant	1	
					Total number:	27	

Table 7 gives a coarse classification into sections by lowercase letters a to c, an identification by ordinal numbers 1 to 14, and a set theoretical characterisation.

Table 7. The 14 S_2-double orbits of $\varkappa \in [0,3]^3 \setminus [1,3]^3$ with $1 \le \nu \le 3$

Sec.	Nr.	repres. of dbl.orb. $\varkappa$	occupation numbers $o(\varkappa)$	Taussky cond. κ	charact. property	cardinality of dbl.orb. $\lvert\varkappa^{S_2\times S_2}\rvert$	realising 2-group G
a	1	(000)	(3000)	(AAA)	constant	1	
b	2	(001)	(2100)	(AAA)	nearly	2	
b	3	(010)	(2100)	(ABA)	constant	4	
b	4	(011)	(1200)	(ABA)	nearly	4	
b	5	(110)	(1200)	(BBA)	constant	2	
c	6	(012)	(1110)	(ABA)	per-	4	
c	7	(120)	(1110)	(BBA)	mutation	2	
b	8	(003)	(2001)	(AAA)	nearly	1	
b	9	(030)	(2001)	(AAA)	constant	2	
c	10	(013)	(1101)	(ABA)	per-	4	
c	11	(031)	(1101)	(AAA)	muta-	4	
c	12	(130)	(1101)	(BAA)	tion	4	
b	13	(033)	(1002)	(AAA)	nearly	2	
b	14	(330)	(1002)	(AAA)	constant	1	
				Total number:	$64 - 27 =$	37	

Figure 9. Finite 2-groups G with commutator quotient $G/G' \simeq (2, 2)$

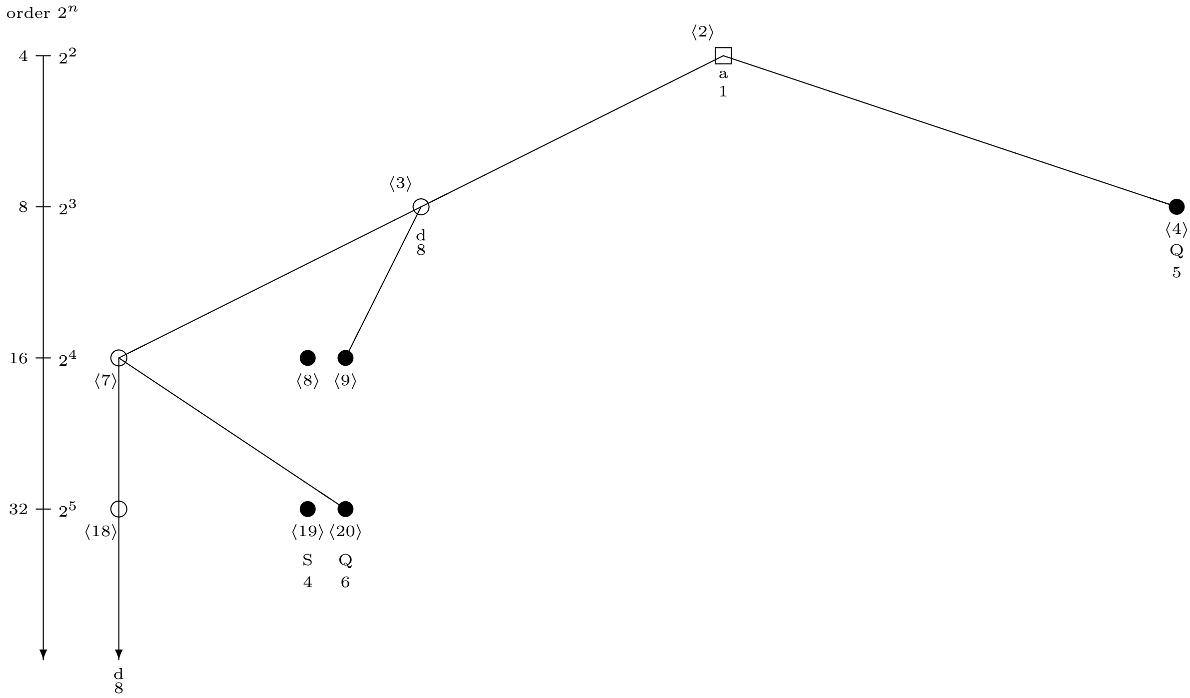


Figure 10. Finite 2-groups G with commutator quotient $G/G' \simeq (4, 2)$

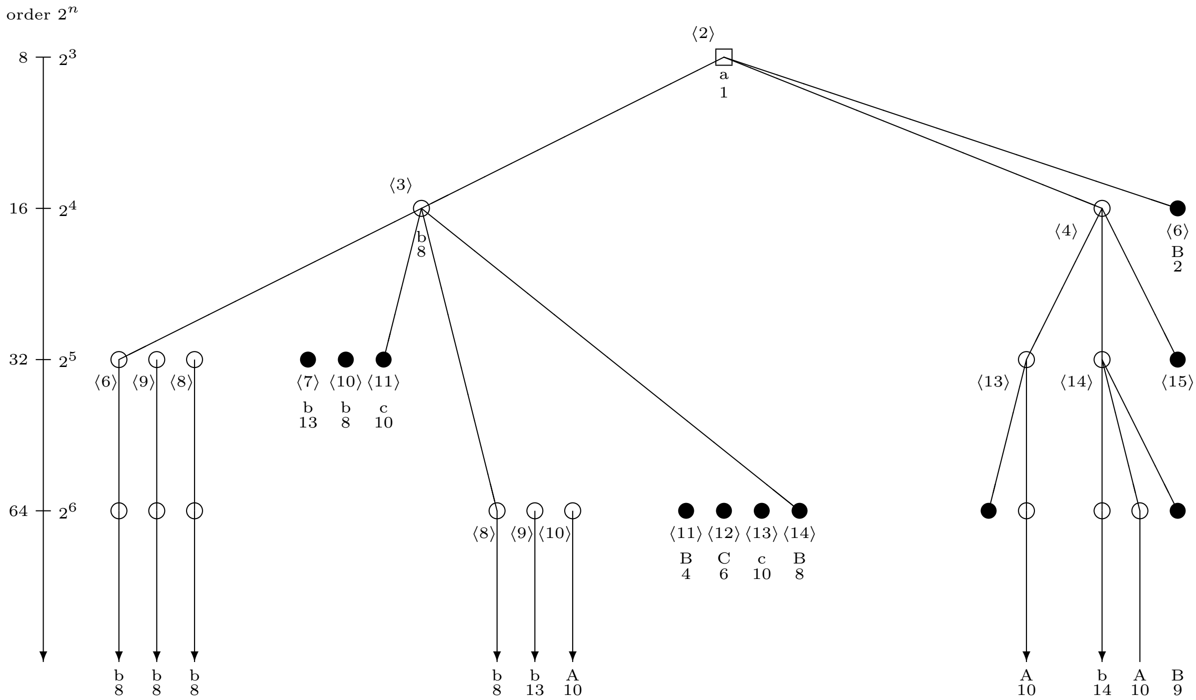


FIGURE 11. Finite 2-groups G with commutator quotient $G/G' \simeq (8,2)$

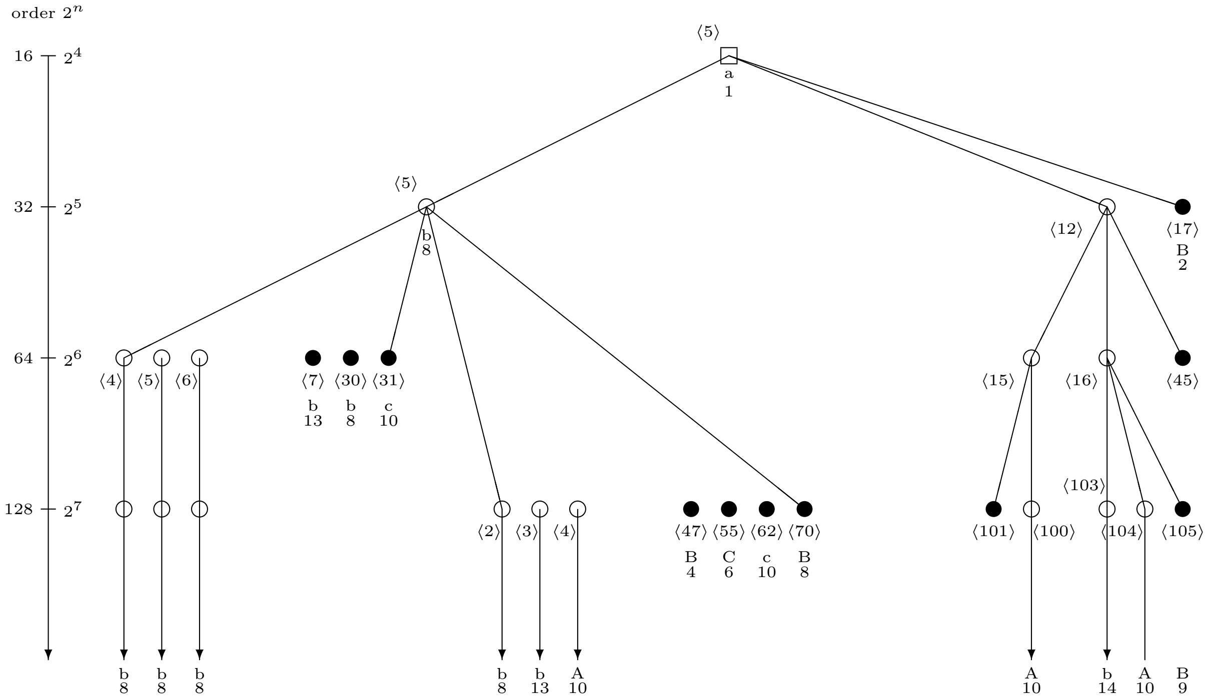


FIGURE 12. Finite 2-groups G with commutator quotient $G/G' \simeq (16,2)$

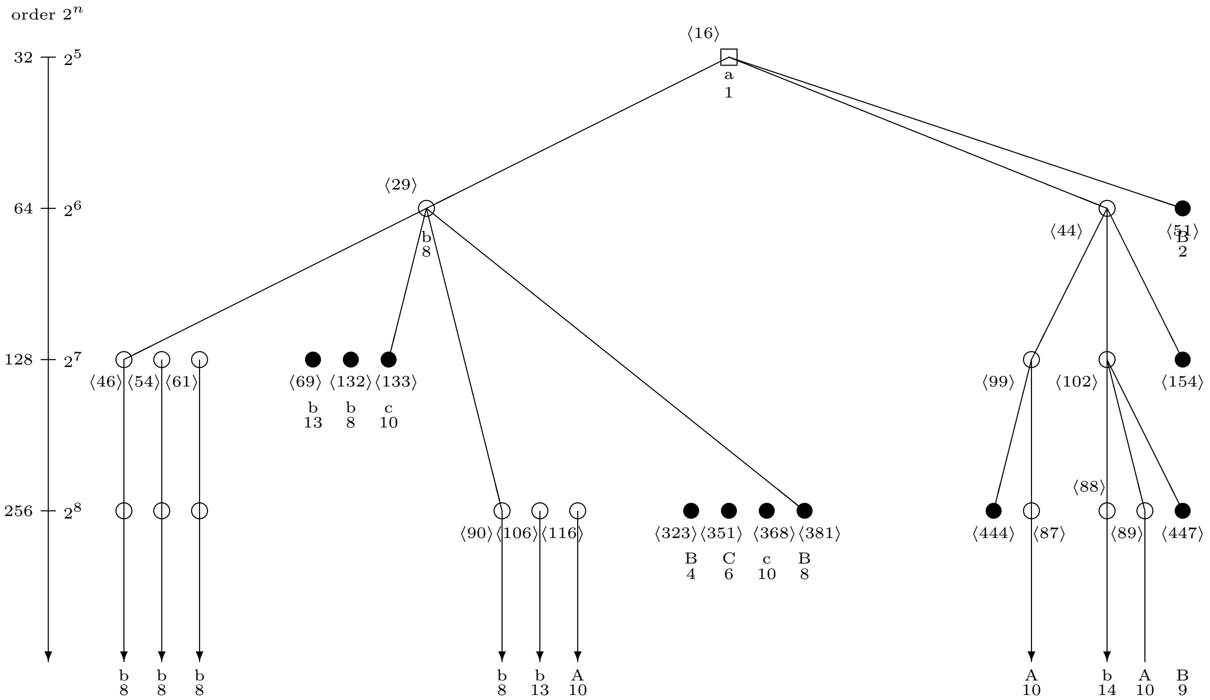


FIGURE 13. Finite 2-groups G with commutator quotient $G/G' \simeq (32, 2)$

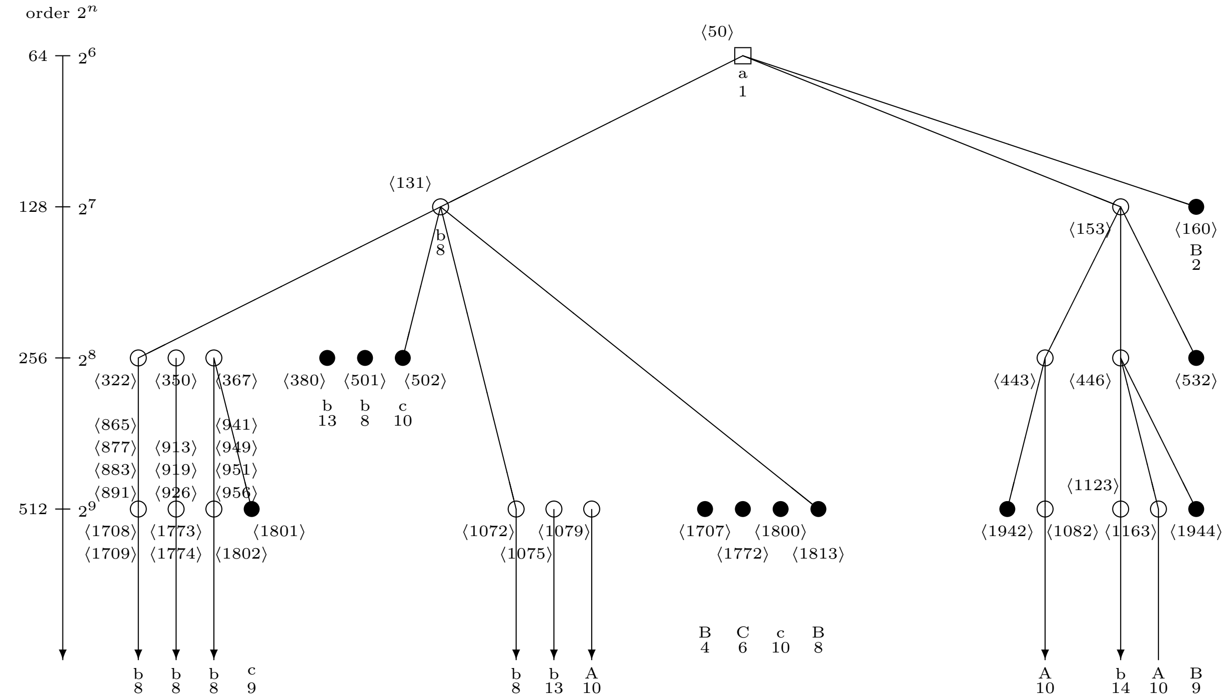
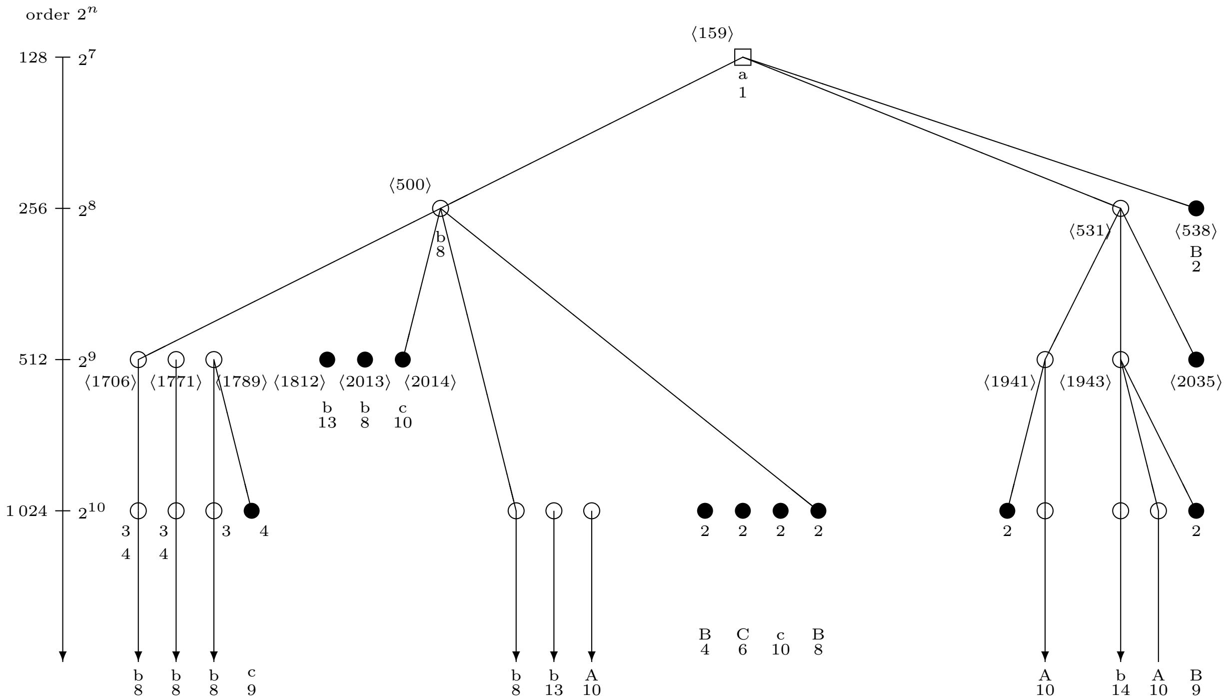


FIGURE 14. Finite 2-groups G with commutator quotient $G/G' \simeq (64, 2)$



7. Acknowledgement

We gratefully acknowledge that our research was supported by the Austrian Science Fund (FWF): Project P 26008-N25.

References

[1] H. U. Besche, B. Eick and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.
[2] B. Eick, *Metabelian p-groups and coclass theory*, J. Algebra **421** (2015), 102–118.
[3] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.
[4] R. James, *The groups of order* p^6 *(p an odd prime)*, Math. Comp. **34** (1980), nr. 150, 613–637.
[5] MAGMA Developer Group, MAGMA *Computational Algebra System*, Version 2.23-9, Sydney, 2018, (http://magma.maths.usyd.edu.au).
[6] D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x. (arXiv: 1403.3896v1 [math.GR] 16 Mar 2014.) [Green OA]
[7] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456, DOI 10.5802/jtnb.842. (arXiv: 1403.3833v1 [math.NT] 15 Mar 2014.) [Green OA]
[8] D. C. Mayer, *Periodic bifurcations in descendant trees of finite p-groups*, Adv. Pure Math. **5** (2015), no. 4, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015. (arXiv: 1502.03390v1 [math.GR] 11 Feb 2015.) [Gold OA]
[9] D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66–104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016. (arXiv: 1511.07819v1 [math.GR] 24 Nov 2015.) [Gold OA]
[10] D. C. Mayer, *Modeling rooted in-trees by finite p-groups*, Chapter 5, pp. 85–113, in the Open Access Book *Graph Theory — Advanced Algorithms and Applications*, Ed. B. Sirmacek, InTech d.o.o., Rijeka, January 2018, DOI 10.5772/intechopen.68703. (arXiv: 1701.08020v1 [math.GR] 27 Jan 2017.) [Gold OA]
[11] D. C. Mayer, *Co-periodicity isomorphisms between forests of finite p-groups*, Adv. Pure Math. **8** (2018), no. 1, 77–140, DOI 10.4236/apm.2018.81006, Special Issue on Group Theory Studies, January 2018. (arXiv: 1802.03344v1 [math.GR] 09 Feb 2018.) [Gold OA]
[12] M. F. Newman, *Determination of groups of prime-power order*, pp. 73–84, in: Group Theory, Canberra, 1975, Lecture Notes in Math., vol. **573**, Springer, Berlin, 1977.
[13] E. A. O'Brien, *The p-group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698, DOI 10.1016/S0747-7171(80)80082-X.
[14] O. Taussky, *A remark concerning Hilbert's Theorem* 94, J. Reine Angew. Math. **239/240** (1970), 435–438.

Naglergasse 53, 8010 Graz, Austria
E-mail address: algebraic.number.theory@algebra.at
URL: http://www.algebra.at