GROUP THEORETIC APPROACH TO CYCLIC CUBIC FIELDS

SIHAM AOUISSI AND DANIEL C. MAYER

Abstract. Let $(k_\mu)_{\mu=1}^4$ be a quartet of cyclic cubic number fields sharing a common conductor $c = pqr$ divisible by exactly three prime(power)s p, q, r. For those components of the quartet whose 3-class group $\mathrm{Cl}_3(k_\mu) \simeq (\mathbb{Z}/3\mathbb{Z})^2$ is elementary bicyclic, the automorphism group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$ of the maximal metabelian unramified 3-extension of k_μ is determined by conditions for cubic residue symbols between p, q, r and for ambiguous principal ideals in subfields of the common absolute 3-genus field k^* of all k_μ. With the aid of the relation rank $d_2(\mathfrak{M})$, it is decided whether $\mathfrak{M}$ coincides with the Galois group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu)$ of the maximal unramified pro-3-extension of k_μ.

Contents

Date: Wednesday, 27 December 2023.

2010 *Mathematics Subject Classification.* 11R16, 11R20, 11R27, 11R29, 11R37, 20D15.

Key words and phrases. Cyclic cubic number fields, conductor, combined cubic residue symbol, principal factors, absolute genus field, bicyclic bicubic fields, unramified cyclic cubic relative extensions, capitulation, maximal unramified pro-3-extension, finite 3-groups, elementary bicyclic commutator quotient, maximal subgroups, kernels of Artin transfers, abelian quotient invariants, relation rank.

Research of second author supported by the Austrian Science Fund (FWF): projects J0497-PHY, P26008-N25, and by the Research Executive Agency of the European Union (EUREA).

1. Introduction

Let k be a cyclic cubic number field, that is, an abelian extension of the rational number field $\mathbb{Q}$ with degree $[k:\mathbb{Q}]=3$ and some positive integer conductor $c>1$ (see § 2.1). In 1973, Georges Gras [11] determined the rank $\varrho=\varrho(k)$ of the 3-class group $\mathrm{Cl}_3(k)$ in dependence on the number t of prime(power) divisors $q_1,\ldots,q_t$ of c and on the cubic residue symbols $\left(\frac{q_i}{q_j}\right)_3$ for $i\neq j$. For mutual cubic residues, $\left(\frac{q_i}{q_j}\right)_3=\left(\frac{q_j}{q_i}\right)_3=1$, we write $q_i\leftrightarrow q_j$, otherwise $q_i\not\leftrightarrow q_j$.

It turned out that $\varrho=0$ for $t=1$, and $\varrho=1$ if $t=2$ and $q_1\not\leftrightarrow q_2$. So in the former case, the maximal unramified pro-3-extension $\mathrm{F}_3^\infty(k)$ of k is the base field k itself, and in the latter case, it is the Hilbert 3-class field $\mathrm{F}_3^1(k)$ of k, in fact, $[\mathrm{F}_3^1(k):k]=3$, since $\varrho=1$ iff $\mathrm{Cl}_3(k)\simeq\mathbb{Z}/3\mathbb{Z}$ is elementary cyclic. If $t=2$ and $q_1\leftrightarrow q_2$, then $\varrho=2$, $\mathrm{Cl}_3(k)$ is bicyclic, but may be non-elementary (singular).

In 1995, Ayadi [2] proved that there are only two possibilities for the Galois group $\mathfrak{G}=\mathrm{Gal}(\mathrm{F}_3^\infty(k)/k)$ of the 3-class field tower of k with length $\ell_3(k)$, when $t=2$, $q_1\leftrightarrow q_2$, and $\mathrm{Cl}_3(k)\simeq(\mathbb{Z}/3\mathbb{Z})^2$ is elementary bicyclic (regular), namely, in the notation of [5], either $\mathfrak{G}\simeq\mathrm{SmallGroup}(9,2)\simeq(\mathbb{Z}/3\mathbb{Z})^2$ is abelian or $\mathfrak{G}\simeq\mathrm{SmallGroup}(27,4)$ is the extra special 3-group with exponent 9.

The impact of t and $q_1,\ldots,q_t$ on the tower group $\mathfrak{G}$ and its metabelianization $\mathfrak{M}=\mathfrak{G}/\mathfrak{G}''$, i.e., the group $\mathfrak{M}=\mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ of the second Hilbert 3-class field $\mathrm{F}_3^2(k)$ of k, is shown in Table 1.

Table 1. Known and unknown impact of t and $q_1,\ldots,q_t$ on $\varrho(k)$ and $\mathfrak{M}$, $\mathfrak{G}$

t	conditions	$\varrho(k)$	$\mathrm{F}_3^\infty(k)$	$\mathfrak{M}$	$\mathfrak{G}$	$\ell_3(k)$
$t=1$		$\varrho=0$	$=k$	$=1$	$=1$	$=0$
$t=2$	$q_1\not\leftrightarrow q_2$	$\varrho=1$	$=\mathrm{F}_3^1(k)$	$=\mathbb{Z}/3\mathbb{Z}$	$=\mathfrak{M}$	$=1$
$t=2$	$q_1\leftrightarrow q_2$, $\mathrm{Cl}_3(k)$ elem.	$\varrho=2$	$=\mathrm{F}_3^1(k)$	$=\mathrm{SmallGroup}(9,2)$	$=\mathfrak{M}$	$=1$
		or	$=\mathrm{F}_3^2(k)$	$=\mathrm{SmallGroup}(27,4)$	$=\mathfrak{M}$	$=2$
$t=2$	$q_1\leftrightarrow q_2$, $\mathrm{Cl}_3(k)$ non-elem.	$\varrho=2$	$\geq\mathrm{F}_3^2(k)$	?	?	≥ 2
$t=3$		$2\leq\varrho\leq 4$	$\geq\mathrm{F}_3^1(k)$	?	?	≥ 1

However, according to Gras [11], $\varrho=2$ is also possible for $t=3$, and, according to Ayadi [2], $\varrho=2$ iff $\mathrm{Cl}_3(k)\simeq(\mathbb{Z}/3\mathbb{Z})^2$ is elementary bicyclic, when $t=3$.

For this situation $t=3$, $c=pqr$, $\varrho=2$, $\mathrm{Cl}_3(k)\simeq(\mathbb{Z}/3\mathbb{Z})^2$, the present article identifies the Galois group $\mathfrak{M}=\mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ in dependence on the cubic residue symbols between p,q,r. The crucial techniques are based on the lucky coincidence that the four unramified cyclic extensions of degree $[E_i:k]=3$, $1\leq i\leq 4$, can always be found among the 13 bicyclic bicubic subfields $B_1,\ldots,B_{13}$ of the absolute 3-genus-field k^* of k, for which Parry [22] has established a useful class number relation and a structure theory of the unit group. With the aid of the relation rank $d_2(\mathfrak{M})\leq 4$ or $d_2(\mathfrak{M})\geq 5$, it is decided whether $\mathfrak{M}$ coincides with the tower group $\mathfrak{G}$ or not.

The examination of cyclic cubic fields k with $\varrho=3$ and elementary tricyclic $\mathrm{Cl}_3(k)\simeq(\mathbb{Z}/3\mathbb{Z})^3$ is reserved for a future paper, since among the 13 unramified cyclic extensions of degree $[E_i:k]=3$, $1\leq i\leq 13$, only four are bicyclic bicubic, and the remaining nine E_i arise in three triplets of pairwise isomorphic non-Galois nonic fields. Similarly, $\mathrm{Cl}_3(k)$ non-elementary for $t=2$ and $p\leftrightarrow q$.

The present work illuminates Ayadi's doctoral thesis [2] from the perspective of group theory, and completely clarifies the question mark "?" for the group $\mathfrak{M}$ in the last row of Table 1, partially also the "?" for the group $\mathfrak{G}$, provided that $\mathrm{Cl}_3(k)\simeq(\mathbb{Z}/3\mathbb{Z})^2$ is elementary bicyclic.

2. Construction of cyclic fields of odd prime degree

2.1. **Multiplicity of conductors and discriminants.** For a fixed odd prime number $\ell\geq 3$, let k be a *cyclic number field* of degree ℓ, that is, $k/\mathbb{Q}$ is a Galois extension of degree $[k:\mathbb{Q}]=\ell$ with absolute automorphism group $\mathrm{Gal}(k/\mathbb{Q})=\langle\sigma\mid\sigma^\ell=1\rangle$. According to the **Theorem of Kronecker, Weber and Hilbert** on abelian extensions of the rational number field $\mathbb{Q}$, the *conductor* c of k is the smallest positive integer such that $k = k_c$ is contained in the cyclotomic field $K = \mathbb{Q}(\zeta_c)$, where $\zeta_c = \exp(2\pi\sqrt{-1}/c)$ denotes a primitive c-th root of unity, more precisely, in the *ℓ-ray class field modulo c* of $\mathbb{Q}$, denoted by $\mathrm{F}_{\ell,c}(\mathbb{Q})$, which lies in the maximal real subfield $K^+ = \mathbb{Q}(\zeta_c + \zeta_c^{-1})$ of $K = \mathbb{Q}(\zeta_c)$.

Theorem 1. *The* ***conductor*** *of a cyclic field of odd prime degree ℓ has the shape $c = \ell^e \cdot q_1 \cdots q_\tau$, where $e \in \{0, 2\}$ and the q_i are pairwise distinct prime numbers $q_i \equiv +1 \pmod{\ell}$, for $1 \le i \le \tau$. The* ***discriminant*** *of $k = k_c$ is the perfect $(\ell-1)$-th power $d_k = c^{\ell-1}$, and the number of rational primes which are (totally) ramified in k is given by*

$$t := \begin{cases} \tau & \text{if } e = 0 \ (\ell \text{ is unramified in } k), \\ \tau + 1 & \text{if } e = 2 \ (\ell \text{ is ramified in } k). \end{cases} \tag{2.1}$$

In the last case, we formally put $q_{\tau+1} := \ell^2$. The number of non-isomorphic cyclic number fields $k_{c,1}, \ldots, k_{c,m}$ of degree ℓ, sharing the common conductor c, is given by the ***multiplicity formula***

$$m = m(c) = (\ell - 1)^{t-1}. \tag{2.2}$$

Proof. See [14, p. 831]. □

2.2. **Construction as ray class fields.** For the construction of all cyclic number fields $k = k_c$ of degree ℓ with ascending conductors $b \le c \le B$ between an assigned lower bound b and upper bound B by means of the computational algebra system Magma [12], the class field theoretic routines by Fieker [9] can be used without the need of preparing a list of suitable generating polynomials of ℓ-th degree. The big advantage of this technique is that the cyclic number fields of degree ℓ are produced as a *multiplet* $(k_{c,1}, \ldots, k_{c,m})$ of pairwise non-isomorphic fields sharing the common conductor c with *multiplicity* $m \in \{1, \ell - 1, (\ell - 1)^2, (\ell - 1)^3, \ldots\}$ in dependence on the number $t \in \{1, 2, 3, 4, \ldots\}$ of primes dividing the conductor c, according to Formula (2.2). Our algorithms for the construction, and statistics of ℓ-class groups, have been presented in [20, Alg. 1–3, pp. 4–7, Tbl. 1.1–1.6, pp. 7–11]. From now on, let $\ell = 3$, for the remainder of this article.

3. Arithmetic of cyclic cubic fields

Generally, t denotes the number of prime divisors of the conductor c of a cyclic cubic number field k, and $\varrho(k) = \varrho_3(k)$ denotes the rank $\dim_{\mathbb{F}_3}(\mathrm{Cl}_3(k)/\mathrm{Cl}_3(k)^3)$ of the 3-class group $\mathrm{Cl}_3(k) = \mathrm{Syl}_3\mathrm{Cl}(k)$. In formulas concerning principal factors (§ 3.2), the prime power conductor 3^2 must be replaced by 3.

3.1. **Rank of 3-class groups.** Since the rank $\varrho_3(k)$ of the 3-class group $\mathrm{Cl}_3(k)$ of a cyclic cubic field k depends on the mutual cubic residue conditions between the prime(power) divisors $q_1, \ldots, q_t$ of the conductor c, Gras [11, pp. 21–22] has introduced directed graphs with t vertices $q_1, \ldots, q_t$ whose directed edges $q_i \to q_j$ describe values of cubic residue symbols. We use a simplified notation of these graphs, fitting in a single line, but occasionally requiring the repetition of a vertex.

Definition 1. Let ζ_3 be a fixed primitive third root of unity. For each pair (q_i, q_j) with $1 \le i \ne j \le t$, the value of the *cubic residue symbol* $\left(\frac{q_i}{q_j}\right)_3 = \zeta_3^{a_{ij}}$ is determined uniquely by the integer $a_{ij} \in \{-1, 0, 1\}$. Let a *directed edge* $q_i \to q_j$ be defined if and only if $\left(\frac{q_i}{q_j}\right)_3 = 1$, that is, q_i is a cubic residue modulo q_j (and thus $a_{ij} = 0$). The **combined cubic residue symbol** $[q_1, \ldots, q_t]_3 :=$

$$\left\{ q_i \to q_j \;\middle|\; i \ne j, \left(\frac{q_i}{q_j}\right)_3 = 1 \right\} \bigcup \left\{ q_i \;\middle|\; (\forall j \ne i) \ \left(\frac{q_i}{q_j}\right)_3 \ne 1, \left(\frac{q_j}{q_i}\right)_3 \ne 1 \right\} \tag{3.1}$$

where the subscripts i and j run from 1 to t, is defined as the union of the set of all *directed edges* which occur in the graph associated with $q_1, \ldots, q_t$ in the sense of Gras, and the set of all *isolated vertices*. For $t = 3$, we additionally need the invariant $\delta := a_{12}a_{23}a_{31} - a_{13}a_{32}a_{21}$ in order to distinguish two subcases of the case with three isolated vertices.

Theorem 2. *(Rank Distribution, **G. Gras, 1973**, [11, Prop. VI.5, pp. 21–22].)*
*Let k be a cyclic cubic field of conductor $c = q_1 \cdots q_t$ with $1 \le t \le 3$. We indicate **mutual cubic residues** simply by writing $q_1 \leftrightarrow q_2$ instead of $q_1 \to q_2 \to q_1$.*

- *If $t = 1$, then $m = 1$, k forms a singlet, $[q_1]_3 = \{q_1\}$, and $\varrho(k) = 0$.*
- *If $t = 2$, then $m = 2$, k is member of a doublet (k_1, k_2), and there arise two possibilities.*
  (1) *$(\varrho(k_1), \varrho(k_2)) = (1, 1)$, if*

$$[q_1, q_2]_3 = \begin{cases} \{q_1, q_2\} & \textit{, Graph } \mathbf{1}, \text{ or} \\ \{q_i \to q_j\} & \textit{, Graph } \mathbf{2}, \text{ with } i \ne j. \end{cases} \tag{3.2}$$

  (2) *$(\varrho(k_1), \varrho(k_2)) = (2, 2)$, if*

$$[q_1, q_2]_3 = \{q_1 \leftrightarrow q_2\}\textit{, Graph } \mathbf{3}. \tag{3.3}$$

- *If $t = 3$, then $m = 4$, k is member of a quartet $(k_1, \ldots, k_4)$, and there arise five cases.*
  (1) *$(\varrho(k_1), \ldots, \varrho(k_4)) = (2, 2, 2, 2)$, called **Category III**, if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta \not\equiv 0 \,(\mathrm{mod}\, 3)\} & \textit{, Graph } \mathbf{1}, \text{ or} \\ \{q_i \to q_j; q_l\} & \textit{, Graph } \mathbf{2}, \text{ or} \\ \{q_i \to q_j \to q_l\} & \textit{, Graph } \mathbf{3}, \text{ or} \\ \{q_i \to q_j \to q_l \to q_i\} & \textit{, Graph } \mathbf{4}, \text{ or} \\ \{q_i \leftrightarrow q_j; q_l\} & \textit{, Graph } \mathbf{5}, \text{ or} \\ \{q_i \leftrightarrow q_j \to q_l\} & \textit{, Graph } \mathbf{6}, \text{ or} \\ \{q_i \leftrightarrow q_j \leftarrow q_l\} & \textit{, Graph } \mathbf{7}, \text{ or} \\ \{q_l \to q_i \leftrightarrow q_j \leftarrow q_l\} & \textit{, Graph } \mathbf{8}, \text{ or} \\ \{q_l \to q_i \leftrightarrow q_j \to q_l\} & \textit{, Graph } \mathbf{9} \end{cases} \tag{3.4}$$

  with i, j, l pairwise distinct.
  (2) *$(\varrho(k_1), \ldots, \varrho(k_4)) = (3, 2, 2, 2)$, called **Category I**, if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta \equiv 0 \,(\mathrm{mod}\, 3)\} & \textit{, Graph } \mathbf{1}, \text{ or} \\ \{q_i \leftarrow q_j \to q_l\} & \textit{, Graph } \mathbf{2} \end{cases} \tag{3.5}$$

  with i, j, l pairwise distinct.
  (3) *$(\varrho(k_1), \ldots, \varrho(k_4)) = (3, 3, 2, 2)$, called **Category II**, if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \to q_j \leftarrow q_l\} & \textit{, Graph } \mathbf{1}, \text{ or} \\ \{q_i \to q_j \leftarrow q_l \to q_i\} & \textit{, Graph } \mathbf{2} \end{cases} \tag{3.6}$$

  with i, j, l pairwise distinct.
  (4) *$(\varrho(k_1), \ldots, \varrho(k_4)) = (3, 3, 3, 3)$, called **Category IV**, if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \leftarrow q_j \leftrightarrow q_l \to q_i\} & \textit{, Graph } \mathbf{1}, \text{ or} \\ \{q_i \leftrightarrow q_j \leftrightarrow q_l\} & \textit{, Graph } \mathbf{2}, \text{ or} \\ \{q_i \leftrightarrow q_j \leftrightarrow q_l \to q_i\} & \textit{, Graph } \mathbf{3} \end{cases} \tag{3.7}$$

  with i, j, l pairwise distinct.
  (5) *$(\varrho(k_1), \ldots, \varrho(k_4)) = (4, 4, 4, 4)$, called **Category V**, if*

$$[q_1, q_2, q_3]_3 = \{q_1 \leftrightarrow q_2 \leftrightarrow q_3 \leftrightarrow q_1\}. \tag{3.8}$$

Proof. See [11, Prp. VI.5, pp. 21–22]. Multiplicities $m \in \{1, 2, 4\}$ are taken from Theorem 1. □

Remark 1. Ayadi introduced categories in [2, pp. 45–47]. He investigated the cases $t = 2$, Formula (3.3), and $t = 3$, Formulae (3.4), (3.5), (3.6), in Theorem 2. For $t = 3$, he denoted the nine subcases of Formula (3.4) by Graph 1,2,3,4,5,6,7,8,9 of Category III, the two subcases of Formula (3.5) by Graph 1,2 of Category I, and the two subcases of Formula (3.6) by Graph 1,2 of Category II. For the Categories I and II, Ayadi did **not** study the fields with 3-class rank

$\varrho_3(k_\mu) = 3$, $1 \le \mu \le 4$. Our algorithms for the classification by categories and graphs, and their statistics, have been presented in [20, Alg. 4–5, Tbl. 2.1, pp. 15–19].

For $t = 3$, we also write briefly $p = q_1$, $q = q_2$ and $r = q_3$ for the prime(power)s dividing the conductor $c = pqr$. Graph 1 of Category I with symbol $[p, q, r]_3 = \{p, q, r; \delta \equiv 0 \,(\mathrm{mod}\, 3)\}$ and Graph 1 of Category III with symbol $[p, q, r]_3 = \{p, q, r; \delta \not\equiv 0 \,(\mathrm{mod}\, 3)\}$ are the only two situations without any trivial cubic residue conditions between p, q, r. We show the impact of the δ-invariant.

Lemma 1. *Consider three cubic residue symbols for products of two primes,* $\left(\frac{qr}{p}\right)_3$, $\left(\frac{pr}{q}\right)_3$, $\left(\frac{pq}{r}\right)_3$ *with respect to triviality, i.e., being equal to* 1.

If $\delta \equiv 0$*, then zero or two of the symbols are trivial.*

If $\delta \not\equiv 0$*, then one or three of the symbols are trivial.*

Proof. For each of the two triplets (a_{12}, a_{23}, a_{31}) and (a_{32}, a_{13}, a_{21}) of exponents in Definition 1, there are $2^3 = 8$ combinatorial possibilities. The product of the components is $+1$ if zero or two components are negative, and it is -1 if one or three components are negative.

For Graph I.1 with $\delta \equiv 0$, triplets with equal product must be combined. Consequently, for each choice of a fixed first triplet, one of the four admissible second triplets (namely $(a_{32}, a_{13}, a_{21}) = (a_{12}, a_{23}, a_{31})$) produces no trivial symbol, and three of the second triplets produce two trivial symbols each.

For Graph III.1 with $\delta \not\equiv 0$, triplets with distinct product must be combined. Consequently, for each choice of a fixed first triplet, one of the four admissible second triplets (namely $(a_{32}, a_{13}, a_{21}) = -(a_{12}, a_{23}, a_{31})$) produces three trivial symbols, and three of the second triplets produce a single trivial symbol each. □

3.2. **Ambiguous principal ideals.** The number of *primitive ambiguous ideals* of a cyclic cubic field k, which are invariant under $\mathrm{Gal}(k/\mathbb{Q}) = \langle\sigma\rangle$, increases with the number t of prime factors of the conductor c. According to Hilbert's Theorem 93, the number is given by

$$\#\left(\mathcal{I}_k^{\langle\sigma\rangle}/\mathcal{I}_{\mathbb{Q}}\right) = 3^t. \tag{3.9}$$

However, the number of *primitive ambiguous* **principal** *ideals* of k is a fixed invariant of all cyclic cubic fields, regardless of the number t.

Theorem 3. *The number of ambiguous principal ideals of any cyclic cubic field* k *is given by*

$$\#\left(\mathcal{P}_k^{\langle\sigma\rangle}/\mathcal{P}_{\mathbb{Q}}\right) = 3. \tag{3.10}$$

Proof. The well-known theorem on the **Herbrand quotient** of the unit group U_k of k as a Galois module over the group $\mathrm{Gal}(k/\mathbb{Q}) = \langle\sigma\rangle$, which can be expressed by abstract cohomology groups $\#\mathrm{H}^{-1}(\langle\sigma\rangle, U_k)/\#\hat{\mathrm{H}}^0(\langle\sigma\rangle, U_k) = [k : \mathbb{Q}]$, can also be stated more ostensively as $\#\left(\mathcal{P}_k^{\langle\sigma\rangle}/\mathcal{P}_{\mathbb{Q}}\right) = \#\left(E_{k/\mathbb{Q}}/U_k^{1-\sigma}\right) = [k : \mathbb{Q}] \cdot \#\left(U_{\mathbb{Q}}/N_{k/\mathbb{Q}}(U_k)\right) = 3$, since the unit norm index is given by $\left(U_{\mathbb{Q}} : N_{k/\mathbb{Q}}(U_k)\right) = 1$. Here, $E_{k/\mathbb{Q}} = \{\varepsilon \in U_k \mid N_{k/\mathbb{Q}}(\varepsilon) = 1\}$ are the relative units. □

Consequently, if we speak about a *non-trivial primitive ambiguous principal ideal* of k, then we either mean $(\alpha) = \alpha\mathcal{O}_k$ or $(\alpha^2/b) = (\alpha^2/b)\mathcal{O}_k$, where $\mathcal{P}_k^{\langle\sigma\rangle}/\mathcal{P}_{\mathbb{Q}} = \{1, (\alpha), (\alpha^2/b)\}$. The norms of these two elements are divisors of the square $c^2 = q_1^2 \cdots q_t^2$ of the conductor c of k, where q_t must be replaced by 3 if $q_t = 9$. When $N_{k/\mathbb{Q}}(\alpha) = a \cdot b^2$ with square free coprime integers a, b, then $N_{k/\mathbb{Q}}(\alpha^2/b) = a^2 \cdot b^4/b^3 = a^2 \cdot b$. It follows that both norms are cube free integers.

Definition 2. The minimum of the two norms of non-trivial primitive *ambiguous principal ideals* $(\alpha), (\alpha^2/b)$ of a cyclic cubic field k is called the **principal factor** (of the discriminant $d_k = c^2$) of the field k, denoted by $A(k) := \min\{a \cdot b^2, a^2 \cdot b\}$, that is

$$A(k) = \begin{cases} a \cdot b^2 & \text{if } b < a, \\ a^2 \cdot b & \text{if } a < b. \end{cases} \tag{3.11}$$

Ayadi [2, Rem. 2.6, p. 18], [3] speaks about the *Parry constant* or *Parry invariant* $A(k)$ of k, and Derhem [8] calls $A(k) = N_{k/\mathbb{Q}}(R)$ with $R = 1 + \varepsilon + \varepsilon^{1+\sigma}$, $\varepsilon = R^{1-\sigma}$, the *Kummer resolvent* of k, when $U_k = \langle -1, \varepsilon, \varepsilon^\sigma \rangle$ as a $\langle\sigma\rangle$-module is generated by -1 and the fundamental unit ε. However, the concept of *principal factors* has been coined much earlier by Barrucand and Cohn [4]. Our algorithm for the determination of principal factors has been presented in [20, Alg. 6, pp. 20–21].

Theorem 4. *(Principal factor criterion,* ***Ayadi, 1995****,* [2, Thm. 3.3, p. 37]*.)*
Let c be a conductor divisible by two primes, $t = 2$, such that $\mathrm{Cl}_3(k_{c,\mu}) \simeq (3,3)$ for both cyclic cubic fields $k_{c,\mu}$, $1 \le \mu \le 2$, with conductor c. Denote by $\mathcal{P}$ the number of prime divisors of the norm $A(k) = \mathrm{N}_{k/\mathbb{Q}}(\alpha)$ of a non-trivial primitive ambiguous principal ideal (α), i.e. a ***principal factor****, of any of the two fields $k = k_{c,\mu}$. Then $\mathcal{P} \in \{1,2\}$,*
and the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ of both fields $k = k_{c,\mu}$ is given by

$$\mathfrak{M} \simeq \begin{cases} \langle 9,2\rangle \text{ with capitulation type a.1, } \varkappa(k) = (0000), & \text{if } \mathcal{P} = 2, \\ \langle 27,4\rangle \text{ with capitulation type A.1, } \varkappa(k) = (1111), & \text{if } \mathcal{P} = 1. \end{cases} \tag{3.12}$$

The length of the Hilbert 3-class field tower is $\ell_3(k) = 1$ with $\mathrm{F}_3^\infty(k) = \mathrm{F}_3^1(k)$ if $\mathcal{P} = 2$, and $\ell_3(k) = 2$ with $\mathrm{F}_3^\infty(k) = \mathrm{F}_3^2(k)$ if $\mathcal{P} = 1$. In both cases, $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k)/k) = \mathfrak{M}$.

Proof. See [2, Prp. 3.6, p. 32, Thm. 3.1, p. 34, Thm. 3.3, p. 37] and [20, pp. 31–33]. □

The first example $c = 19 \cdot 1129 = 21\,451$ for $\mathfrak{M} \simeq \langle 27,4\rangle$ is due to Scholz and Taussky [25, pp. 209–210]. It was misprinted as $19 \cdot 1429 = 27\,151$ in [26, p. 383]. Systematic tables have been presented at `http://www.algebra.at/ResearchFrontier2013ThreeByThree.htm` in §§ 1.1–1.2.

Concerning the 3-capitulation types a.1 and A.1, viewed as transfer kernel types (TKT), and the related concept of transfer target types (TTT), i.e., abelian type invariants (ATI), see [15].

4. Unramified extensions of cyclic cubic fields

In this crucial section, we first introduce the absolute 3-genus field k^* (§ 4.1) of a cyclic cubic number field k. Then we show that the bicyclic bicubic subfields $B < k^*$ constitute unramified cyclic cubic relative extensions B/k of a cyclic cubic number field k with $t = 3$. Finally, using the unramified cyclic cubic relative extensions E/k as capitulation targets (§ 4.3), we define the capitulation kernels (§ 4.2) of a cyclic cubic number field k with non-trivial 3-class group $\mathrm{Cl}_3(k)$.

4.1. **The absolute 3-genus field.** The *absolute* 3*-genus field* $k^* = (k/\mathbb{Q})^*$ of a cyclic cubic field k is the maximal unramified 3-extension k^*/k with abelian absolute Galois group $\mathrm{Gal}(k^*/\mathbb{Q})$. If the conductor $c = q_1 \cdots q_t$ of $k = k_c$ has t prime divisors, then k^* is the compositum of the multiplet $(k_{c,1}, \ldots, k_{c,m})$ of all cyclic cubic fields sharing the common conductor c, where $m = m(c) = 2^{t-1}$, according to the multiplicity formula (2.2). The absolute Galois group $\mathrm{Gal}(k^*/\mathbb{Q})$ is the elementary abelian 3-group $(\mathbb{Z}/3\mathbb{Z})^t$. In particular, if $t = 1$, $c = q_1$, then $k^* = k$ is the cyclic cubic field itself, and if $t = 2$, $c = q_1 q_2$, then $k^* = k_{c,1} \cdot k_{c,2}$ is a bicyclic bicubic field with conductor c and discriminant

$$d(k^*) = d(k_{q_1}) \cdot d(k_{q_2}) \cdot d(k_{c,1}) \cdot d(k_{c,2}) = q_1^2 \cdot q_2^2 \cdot (q_1 q_2)^2 \cdot (q_1 q_2)^2 = c^6. \tag{4.1}$$

In 1990, Parry [22] investigated the arithmetic of a general *bicyclic bicubic field* $B/\mathbb{Q}$ with conductor $c = q_1 \cdots q_t$, $t \ge 2$, and four cyclic cubic subfields $k_1, \ldots, k_4$. In particular, he determined the *class number relation* in terms of the *index I of subfield units* of B.

Theorem 5. *Let $M := (e_{i,j})$ be the $(4 \times t)$-**matrix of** integer exponents in the following representation of the* ***principal factors*** *$A(k_i) = \prod_{j=1}^t q_j^{e_{i,j}}$, for $1 \le i \le 4$. Then:*

1. *The Galois group $\mathrm{Gal}(B/\mathbb{Q}) \simeq (3,3)$ is elementary bicyclic.*
2. *The index $I := (U : V)$ of the subgroup $V := \langle U_1, \ldots, U_4 \rangle$ generated by the unit groups $U_i := U_{k_i}$, $1 \le i \le 4$, in the unit group $U := U_B$ is bounded by $I = 3^e$, $0 \le e \le 3$.*
3. *The* ***class number*** *of B satisfies the following* ***relation****:*

$$h(B) = \frac{I}{3^5} \cdot \prod_{i=1}^{4} h(k_i) = \frac{(U:V)}{243} \cdot h(k_1) \cdot h(k_2) \cdot h(k_3) \cdot h(k_4), \tag{4.2}$$

*where I denotes the abovementioned **index of subfield units** of B.*

(4) *$3\nmid h(B)$ if and only if $c=pq$, i.e. $t=2$, and $p\not\leftrightarrow q$ are **not** mutual cubic residues, i.e., the graph of p,q is either Graph 1 or Graph 2. If $3\nmid h(B)$, then $I=27$.*

(5) *In dependence on the rank $2\le r_M:=\mathrm{rank}(M)\le 4$ of the matrix M, the **index** I takes the following values:*

$$I=(U:V)=\begin{cases}1 & \text{if } r_M=4,\\ 3 & \text{if } r_M=3,\\ 9 \text{ or } 27 & \text{if } r_M=2.\end{cases} \tag{4.3}$$

Proof. For the *class number relation*, see Parry [22, Prp. 7, p. 496, Thm. 9, p. 497]. Generally, the *index of subfield units*, I, is a divisor of $27=3^3$ [22, Lem. 11, p. 500, Thm. 13, p. 501]. See also Ayadi [2, Prop. 2.7.(2) and Prop. 2.8, p. 20]. Note that $p\not\leftrightarrow q$ implies $\prod_{i=1}^4 h(k_i)=9$. □

Corollary 1. *Let $t=3$ and B be a bicyclic bicubic field with conductor $c=pqr$ such that there are **no mutual cubic residues** among p,q,r. Then:*

(1) *For all $1\le j\le 4$, $h_3(B_j)=\frac{(U_j:V_j)}{3^2}h_3(k_j)$.*

(2) *For all $5\le j\le 10$, $h_3(B_j)=\frac{(U_j:V_j)}{3^4}h_3(k_i)h_3(k_\ell)$, where $1\le i,\ell\le 4$, $i\ne\ell$, and k_i, k_ℓ are the two components of the quartet which are contained in B_j.*

Proof. By (4.8), the first statement is valid, since $h_3(k)=3$ for the six subfields k with $t=2$. By (4.9), the second statement holds, since $h_3(k)=1$ for the three subfields k with $t=1$. □

For a cyclic cubic field k with $t=2$, $c=pq$, the 3-class numbers of the 3-genus field k^*, which is bicyclic bicubic, and of its four cyclic cubic subfields can be summarized as follows.

Theorem 6. *Let $k^*=k_p\cdot k_q\cdot k_{c,1}\cdot k_{c,2}$ be the genus field of the two cyclic cubic fields $k_{c,1}$ and $k_{c,2}$ with conductor $c=pq$. Denote the 3-valuations of the class numbers h^*, h_1, h_2, h_3, h_4 of k^*, k_p, k_q, $k_{c,1}$, $k_{c,2}$, respectively, by v^*, v_1, v_2, v_3, v_4. Then $v_1=v_2=0$, and*

$$v^*\begin{cases}=0,\ v_3=v_4=1,\ I=27, & \text{if } p\not\leftrightarrow q,\\ =1, & \text{if } p\leftrightarrow q,\ v_3=v_4=2,\ I=9,\\ =2, & \text{if } p\leftrightarrow q,\ v_3=v_4=2,\ I=27,\\ \ge 3, & \text{if } p\leftrightarrow q,\ v_3\ge 3,\ v_4\ge 3,\ I\ge 9.\end{cases} \tag{4.4}$$

Proof. According to Theorem 2, we generally have $v_1=v_2=0$, $v_3\ge 1$, $v_4\ge 1$ if $p\not\leftrightarrow q$, and $v_3\ge 2$, $v_4\ge 2$ if $p\leftrightarrow q$. Now, the claim is a consequence of Formula (4.2), which yields

$$v^*=v_3(h^*)=v_3(I)-5+\sum_{i=1}^4 v_3(h_i)=v_3(I)-5+v_1+v_2+v_3+v_4=v_3(I)-5+v_3+v_4.$$

The combination of [22, Thm. 9, p. 497] and [22, Cor. 1, p. 498] shows that $v^*=0$ if and only if $p\not\leftrightarrow q$, and $v^*=0$ implies $v_3(I)=3$, whence necessarily $v_3=v_4=1$. However, if $p\leftrightarrow q$, then $v_3=2$ is equivalent with $v_4=2$, according to [3, Thm. 4.1, p. 472]. □

Remark 2. For $v_3=v_4=2$, we have $\mathrm{Cl}_3(k_{pq,\mu})\simeq(3,3)$. The smallest occurrences of $v_3=v_4=3$ are the conductors $7\cdot 673=4\,711$ ("Eau de Cologne", **singular** with $\mathrm{Cl}_3(k^*)\simeq(3,3,3)$) and $7\cdot 769=5\,383$ (**super-singular** with $\mathrm{Cl}_3(k^*)\simeq(9,3,3)$) both with $\mathrm{Cl}_3(k_{pq,\mu})\simeq(9,3)$, for $\mu\in\{1,2\}$.

For a cyclic cubic field k with $t=3$ and conductor $c=q_1q_2q_3$, the 3-genus field k^* contains 13 bicyclic bicubic subfields. Three of them are the *sub genus fields* $B_i:=(k_{f_{i-10}})^*$, $11\le i\le 13$, of the cyclic cubic fields with conductors $f_1=q_1q_2$, $f_2=q_1q_3$, $f_3=q_2q_3$, respectively. In the numerical tables of [20], we always start with the leading three sub genus fields B_i, $11\le i\le 13$, separated by a semicolon from the trailing ten remaining bicyclic bicubic subfields, when we give a family of invariants for these 13 subfields $B_1,\ldots,B_{13}$,

$$\text{in particular, } [\mathrm{Cl}_3 B_i]_{1\le i\le 13}:=[\mathrm{Cl}_3(B_{11}),\ldots,\mathrm{Cl}_3(B_{13});\mathrm{Cl}_3(B_1),\ldots,\mathrm{Cl}_3(B_{10})]. \tag{4.5}$$

4.2. **Capitulation kernels.** We recall the connection between the size of the capitulation kernel $\ker(T_{E/k})$ and the unit norm index $(U_k : \mathrm{N}_{E/k}(U_E))$ of an unramified cyclic cubic extension E/k of a cyclic cubic field k. Here, $T_{E/k} : \mathrm{Cl}_3(k) \to \mathrm{Cl}_3(E)$, $\mathfrak{a}\mathcal{P}_k \mapsto (\mathfrak{a}\mathcal{O}_E)\mathcal{P}_E$, denotes the extension homomorphism or **transfer** of 3-classes from k to E.

Theorem 7. *The order of the* 3*-capitulation kernel or* ***transfer kernel*** *of* E/k *is given by*

$$\#\ker(T_{E/k}) = \begin{cases} 3, \\ 9, \\ 27, \end{cases} \quad \text{if and only if} \quad (U_k : \mathrm{N}_{E/k}(U_E)) = \begin{cases} 1, \\ 3, \\ 9. \end{cases} \tag{4.6}$$

Proof. According to the **Herbrand Theorem** on the cohomology of the unit group U_E as a Galois module with respect to $G = \mathrm{Gal}(E/k)$, we have the relation $\#\ker(T_{E/k}) = [E : k] \cdot (U_k : \mathrm{N}_{E/k}(U_E))$, since $\ker(T_{E/k}) \simeq H^1(G, U_E)$ when E/k is unramified of odd prime degree $[E : k] = 3$ and $U_k/\mathrm{N}_{E/k}(U_E) \simeq \hat{H}^0(G, U_E)$. The cyclic cubic base field k has signature $(r_1, r_2) = (3, 0)$ and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 3 + 0 - 1 = 2$. Thus, there are three possibilities for the unit norm index $(U_k : \mathrm{N}_{E/k}(U_E)) \in \{1, 3, 9\}$. □

Remark 3. When k is a cyclic cubic field with 3-class group $O := \mathrm{Cl}_3(k)$ of elementary tricyclic type $(3, 3, 3)$, viewed as a vector space of dimension 3 over the finite field $\mathbb{F}_3$, then $\#\ker(T_{E/k}) = 3$ if and only if $\ker(T_{E/k}) = L_i$ is a **line** for some $1 \le i \le 13$, $\#\ker(T_{E/k}) = 9$ if and only if $\ker(T_{E/k}) = P_i$ is a **plane** for some $1 \le i \le 13$, and $\#\ker(T_{E/k}) = 27$ if and only if $\ker(T_{E/k}) = O$ is the **entire vector space** over $\mathbb{F}_3$. Details are reserved for a future paper. Our algorithms for the determination of the capitulation kernels for $\mathrm{Cl}_3(k)$ of type $(3, 3)$ and $(3, 3, 3)$ have been presented in [20, Alg. 8–9, pp. 26–30].

In our theorems on cyclic cubic fields with $t = 3$ belonging to the various graphs of each category, we shall frequently find particular statements which relate several similar capitulation types.

Definition 3. Let G be a 3-group with generator rank $d_1(G) = 2$ and elementary bicyclic commutator quotient $G/G' \simeq (3, 3)$. By $T_{G,H_i} : G/G' \to H_i/H_i'$ we denote the transfers from G to the four maximal normal subgroups H_i, $1 \le i \le 4$. Then the set of all *ordered transfer kernel types* $\varkappa = (\varkappa_i)_{1 \le i \le 4}$ with $\varkappa_i := \ker(T_{G,H_i})$ is endowed with a *partial order* relation $\varkappa \le \varkappa'$ by $(\forall\, 1 \le i \le 4)\ \varkappa_i \le \varkappa_i'$. The order is strict, $\varkappa < \varkappa'$, when $\varkappa \le \varkappa'$ and $(\exists\, 1 \le j \le 4)\ \varkappa_j < \varkappa_j'$.

The possibilities for a strict order are rather limited, since a transfer kernel is either cyclic of order 3 (*partial* — by Hilbert's Theorem 94, it cannot be trivial) or bicyclic of type $(3, 3)$ (*total*). As usual, we abbreviate $\varkappa_i = j$ if $(\exists\, 1 \le j \le 4)\ \ker(T_{G,H_i}) = H_j/G'$, and $\varkappa_i = 0$ if $\ker(T_{G,H_i}) = G/G'$, for fixed $1 \le i \le 4$. So, $\varkappa < \varkappa' \iff (\exists\, 1 \le j, i \le 4)\ \varkappa_j = H_i/G' < G/G' = \varkappa_j'$. The arithmetical application of this group theoretic Definition 3 is given in the following definition.

Definition 4. Let K be an algebraic number field with elementary bicyclic 3-class goup $\mathrm{Cl}_3(K) \simeq (3, 3)$. Then K has four unramified cyclic cubic relative extensions E_i/K, $1 \le i \le 4$, and corresponding class extension homomorphisms $T_{E_i/K} : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(E_i)$. Let $\mathfrak{M} := \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ be the Galois group of the second Hilbert 3-class field of K, that is, the maximal metabelian unramified 3-extension of K. Then $\varkappa(K) := \varkappa(\mathfrak{M})$ is called the **minimal transfer kernel type** (mTKT) of K, if $\varkappa(K) \le \varkappa'(K)$, for any other possible capitulation type $\varkappa'(K)$.

4.3. **Capitulation targets.** The precise constitution of the lattice of all subfields of the absolute 3-genus field k^* of a cyclic cubic field $k = k_{pqr}$ with $t = 3$ and conductor $c = pqr$ is as follows.

Theorem 8. *The genus field* k^* *of* k *contains* 13 *cyclic cubic fields,*

$$\begin{aligned} &k_{p,1}, k_{q,1}, k_{r,1}, k_{pq,1}, k_{pq,2}, k_{pr,1}, k_{pr,2}, k_{qr,1}, k_{qr,2}, k_{pqr,1}, k_{pqr,2}, k_{pqr,3}, k_{pqr,4}, \ \textit{briefly} \\ &k_p, k_q, k_r, k_{pq}, \tilde{k}_{pq}, k_{pr}, \tilde{k}_{pr}, k_{qr}, \tilde{k}_{qr}, k_1, k_2, k_3, k_4. \end{aligned} \tag{4.7}$$

The composita $L := k_{pq}k_{pr}k_{qr}$ *and* $\tilde{L} := \tilde{k}_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$ *satisfy the* ***skew balance of degrees*** $[L : \mathbb{Q}] \cdot [\tilde{L} : \mathbb{Q}] = 243$ *with* $[L : \mathbb{Q}] = 9$ *and* $[\tilde{L} : \mathbb{Q}] = 27$*, or vice versa.*

Alert: *Always in the sequel, the* ***normalization*** $[L:\mathbb{Q}]=9$ *is assumed.*
The genus field $k^\ast$ *of* k *contains* 13 *bicyclic bicubic fields,*

$$\text{(4.8)}\qquad 4\ \textbf{single capitulation targets}\quad \begin{aligned} B_1 &:= k_{pq}k_{pr} = k_1k_{pq}k_{pr}k_{qr},\\ B_2 &:= \tilde{k}_{pr}\tilde{k}_{qr} = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr},\\ B_3 &:= \tilde{k}_{pq}\tilde{k}_{pr} = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr},\\ B_4 &:= \tilde{k}_{pq}\tilde{k}_{qr} = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr},\end{aligned}$$

$$\text{(4.9)}\qquad 6\ \textbf{double capitulation targets}\quad \begin{aligned} B_5 &:= k_p\tilde{k}_{qr} = k_1k_3k_p\tilde{k}_{qr},\\ B_6 &:= k_q\tilde{k}_{pr} = k_1k_4k_q\tilde{k}_{pr},\\ B_7 &:= k_r\tilde{k}_{pq} = k_1k_2k_r\tilde{k}_{pq},\\ B_8 &:= k_pk_{qr} = k_2k_4k_pk_{qr},\\ B_9 &:= k_qk_{pr} = k_2k_3k_qk_{pr},\\ B_{10} &:= k_rk_{pq} = k_3k_4k_rk_{pq},\end{aligned}$$

$$\text{(4.10)}\qquad \text{and } 3\ \textbf{sub genus fields}\quad \begin{aligned} B_{11} &:= k_{pq}\tilde{k}_{pq} = k_pk_qk_{pq}\tilde{k}_{pq},\\ B_{12} &:= k_{pr}\tilde{k}_{pr} = k_pk_rk_{pr}\tilde{k}_{pr},\\ B_{13} &:= k_{qr}\tilde{k}_{qr} = k_qk_rk_{qr}\tilde{k}_{qr},\end{aligned}$$

provided that k_{pq}, k_{pr}, k_{qr} *are normalized. The conductor of* $B_1,\ldots,B_{10}$ *is* $c=pqr$*, the conductor of* B_{11} *is* $f_1=pq$*, the conductor of* B_{12} *is* $f_2=pr$*, and the conductor of* B_{13} *is* $f_3=qr$*.*

Proof. See [2, Prop. 4.1, p. 40, Lem. 4.1, p. 42]. The **short form** suffices for construction. □

The algorithm for the determination of bicyclic bicubic fields has been presented in [20, Alg. 7, pp. 24–26], but $B_5,\ldots,B_{10}$ should be defined as in Formula (4.9) (short form without $k_1,\ldots,k_4$).

Corollary 2. *The* ***capitulation targets****, i.e. the unramified cyclic cubic relative extensions of* k_1*, respectively* k_2*, respectively* k_3*, respectively* k_4*, among the absolutely bicyclic bicubic subfields of the* 3*-genus field* $k^\ast=k_pk_qk_r$ *are* B_1,B_5,B_6,B_7*, respectively* B_2,B_7,B_8,B_9*, respectively* B_3,B_5,B_9,B_{10}*, respectively* B_4,B_6,B_8,B_{10}*. In particular,* B_7 *is common to both,* k_1 *and* k_2*,* B_5 *is common to* k_1 *and* k_3*,* B_6 *is common to* k_1 *and* k_4*,* B_9 *is common to* k_2 *and* k_3*,* B_8 *is common to* k_2 *and* k_4*, and* B_{10} *is common to* k_3 *and* k_4*.*

Proof. This follows immediately from Theorem 8, Equations (4.8) and (4.9). □

Proposition 1. *If there exists* $1\le j\le 10$ *such that* $h_3(B_j)=3$*, then* $h_3(B_\ell)=3$*, for all* $1\le \ell\le 10$*, and* $h_3(k_i)=9$*, for all* $1\le i\le 4$*.*

The 3*-class number of* B_j*,* $1\le j\le 10$*, satisfies the* ***tame*** *condition* $h_3(B_j)=(U_j:V_j)$ *if and only if for each cyclic cubic subfield* k *of* B_j *the Hilbert* 3*-class field* $\mathrm{F}_3^1(k)$ *of* k *coincides with the genus field* $k^\ast$ *of* k*. Otherwise the* ***wild*** *condition* $h_3(B_j)>(U_j:V_j)$ *holds.*

If there exists $1\le j\le 10$ *such that* $h_3(B_j)>(U_j:V_j)$*, then* $9\mid h_3(B_\ell)$*, for all* $1\le\ell\le 10$*.*

Proof. The condition is trivial for the subfields k with $t=1$, since $h_3(k)=[\mathrm{F}_3^1(k):k]=[k^\ast:k]=1$ is satisfied anyway. However, the subfields k with $t=2$ must have the 3-class number $h_3(k)=[\mathrm{F}_3^1(k):k]=[k^\ast:k]=3$, in particular, the prime divisors of the conductor are not mutual cubic residues, and the subfields k with $t=3$ must have 3-class number $h_3(k)=[\mathrm{F}_3^1(k):k]=[k^\ast:k]=9$, that is, they cannot have 3-class rank $\varrho(k)\ge 3$. For details see [2, pp. 47–48, i.p. Prop. 4.5]. □

Let $t=3$ and k_μ, $1\le\mu\le 4$, be one of the four cyclic cubic number fields sharing the common conductor $c=pqr$, and suppose B_j, $1\le j\le 10$, is one of the ten bicyclic bicubic subfields of the absolute 3-genus field $k^\ast$ of k_μ such that B_j/k_μ is an unramified cyclic extension of degree 3. We denote by U_j the unit group of B_j, by V_j the subgroup generated by all subfield units, by r_j the rank of the principal factor matrix M_j of B_j, and by $A=(a_{\iota\lambda})$ the right upper triangular (8×8)-matrix such that $(\gamma_1^3,\ldots,\gamma_8^3)=(\varepsilon_1,\ldots,\varepsilon_8)\cdot A$ (in the sense of exponentiation), for a suitable torsion free basis $(\gamma_1,\ldots,\gamma_8)$ of U_j and a canonical basis $(\varepsilon_1,\ldots,\varepsilon_8)$ of V_j, according to [22, pp. 497–503] and [2, pp. 19–22].

For several times, Ayadi [2] alludes to the following fact: the *minimal subfield unit index* $(U_j : V_j) = 3$ for the matrix rank $r_j = 3$ of B_j corresponds to the *maximal unit norm index* $(U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3$, associated with a *total transfer kernel* $\#\ker(T_{B_j/k_\mu}) = 9$ of B_j/k_μ. Since he does not give a prove, we summarize all related issues in a lemma.

Lemma 2. *The following statements are equivalent, row by row:*

$$\begin{aligned}(U_j : V_j) = 3 &\iff a_{77} = 3,\ a_{88} = 3,\ a_{66} = 1 \implies (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3,\\ (U_j : V_j) = 9 &\iff a_{77} = 3,\ a_{88} = 1,\ a_{66} = 1 \implies (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3,\\ (U_j : V_j) = 27 &\iff a_{77} = 1,\ a_{88} = 1,\ a_{66} = 1 \iff (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 1.\end{aligned} \tag{4.11}$$

Proof. According to Theorem 5, $r_j = 3 \iff (U_j : V_j) = 3$, and $r_j = 2 \iff (U_j : V_j) \in \{9, 27\}$. Now, $a_{77} = 1$ implies $\gamma_7^3 = (\prod_{\iota=1}^{6} \varepsilon_\iota^{a_{\iota 7}}) \cdot \varepsilon_7$, $N_{B_j/k_\mu}(\gamma_7) = \pm\varepsilon_7$, $\quad (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 1$, but $a_{77} = 3$ implies $\gamma_7^3 = (\prod_{\iota=1}^{6} \varepsilon_\iota^{a_{\iota 7}}) \cdot \varepsilon_7^3$, $N_{B_j/k_\mu}(\gamma_7) = \pm\varepsilon_7^3$, $\quad (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3$.

Finally, Theorem 7 on the **Herbrand quotient** of U_j shows the cardinality of the transfer kernel, $\#\ker T_{B_j/k_\mu} = [k_\mu : \mathbb{Q}] \cdot (U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3 \cdot (U(k_\mu) : N_{B_j/k_\mu}(U_j))$. □

Proposition 2. *Let ℓ be an odd prime, and suppose that $B = K \cdot L$ is a bicyclic field of degree ℓ^2, compositum of two cyclic fields K and L of degree ℓ. If p is a prime number which ramifies in both, K and L, i.e., $p\mathcal{O}_K = \mathfrak{p}_1^\ell$ and $p\mathcal{O}_L = \mathfrak{p}_2^\ell$, then the extension ideals $\mathfrak{p}_1\mathcal{O}_B = \mathfrak{p}_2\mathcal{O}_B$ coincide.*

Proof. If the decomposition invariants of p in B are $(e, f, g) = (\ell, 1, \ell)$, resp. $(\ell, \ell, 1)$, resp. $(\ell^2, 1, 1)$, then those of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in B must be identical $(e, f, g) = (1, 1, \ell)$, resp. $(1, \ell, 1)$, resp. $(\ell, 1, 1)$, and unique prime decomposition enforces $\mathfrak{p}_1\mathcal{O}_B = \mathfrak{p}_2\mathcal{O}_B$. □

Corollary 3. *Let $\mu \in \{1, 2, 3, 4\}$ and $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$. Then the following* ***capitulation laws*** *for ideal classes hold independently of the combined cubic residue symbol $[p, q, r]_3$.*

1. *$[\mathfrak{p}]$ capitulates in B_5/k_μ, for $\mu = 1, 3$, and in B_8/k_μ, for $\mu = 2, 4$.*
2. *$[\mathfrak{q}]$ capitulates in B_6/k_μ, for $\mu = 1, 4$, and in B_9/k_μ, for $\mu = 2, 3$.*
3. *$[\mathfrak{r}]$ capitulates in B_7/k_μ, for $\mu = 1, 2$, and in B_{10}/k_μ, for $\mu = 3, 4$.*

Proof. We show that $[\mathfrak{p}] \in \mathrm{Cl}_3(k_1)$ capitulates in B_5. Everythig else is proved in the same way, always using Proposition 2 with $\ell = 3$. The bicyclic bicubic field $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$ is compositum of the cyclic cubic fields k_p and k_1. Since the conductor of k_p is p, the principal factor $A(k_p) = p$ is determined uniquely, and $p\mathcal{O}_{k_p} = \mathfrak{p}_0^3$ is totally ramified, whence $\mathfrak{p}_0 = \alpha\mathcal{O}_{k_p}$ with $\alpha \in k_p^\times$ is necessarily a principal ideal. Since the conductor of k_1 is $c = pqr$, the prime $p\mathcal{O}_{k_1} = \mathfrak{p}^3$ is also totally ramified, and Proposition 2 asserts that $\mathfrak{p}\mathcal{O}_{B_5} = \mathfrak{p}_0\mathcal{O}_{B_5}$, which is the principal ideal $\alpha\mathcal{O}_{B_5}$. Thus the class $[\mathfrak{p}]$ capitulates in B_5. □

Proposition 3. *If $\left(\frac{p}{q}\right)_3 = 1$ but $\left(\frac{q}{p}\right)_3 \neq 1$, then $\mathrm{Cl}_3(k_{pq}) \simeq (3)$, $\mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3)$, and* ***two principal factors*** *are given by $A(k_{pq}) = p$, $A(\tilde{k}_{pq}) = p$.*

Proof. If $p \to q$, then p splits in k_q, $p\mathcal{O}_{k_q} = \wp_1\wp_2\wp_3$, and $\mathrm{Cl}_3(k_{pq}) \simeq (3)$, according to Georges Gras [11]. The Hilbert 3-class field $\mathrm{F}_3^1(k_{pq})$ of k_{pq} with $[\mathrm{F}_3^1(k_{pq}) : k_{pq}] = 3$ coincides with the absolute 3-genus field $k^* = k_p \cdot k_q = k_{pq} \cdot \tilde{k}_{pq}$ of the doublet $(k_{pq}, \tilde{k}_{pq})$ with $[k^* : \mathbb{Q}] = 9$ and $[k^* : k_{pq}] = 3$.

Since the conductor of k_{pq} is pq, $p\mathcal{O}_{k_{pq}} = \mathfrak{p}^3$ is ramified in k_{pq}, but k^* is unramified over k_{pq}, and the decomposition invariants of p in k^* are $(e, f, g) = (3, 1, 3)$, those of $\mathfrak{p}$ in $k^* = \mathrm{F}_3^1(k_{pq})$ are $(e, f, g) = (1, 1, 3)$, i.e. $\mathfrak{p}$ splits completely in $\mathrm{F}_3^1(k_{pq})$,

By the decomposition law of the Hilbert 3-class field, $\mathfrak{p} = \alpha\mathcal{O}_{k_{pq}}$ is principal with $\alpha \in k_{pq}^\times$. Therefore the unique principal factor of k_{pq} is $A(k_{pq}) = p$. The same reasoning is true for $\tilde{k}_{pq}$. □

Proposition 4. *Let $\mu \in \{1, 2, 3, 4\}$, such that $\mathrm{Cl}_3(k_\mu) \simeq (3, 3)$.*
If $\left(\frac{p}{q}\right)_3 = 1$ and $\left(\frac{p}{r}\right)_3 = 1$, then ***the principal factor*** *of k_μ is $A(k_\mu) = p$.*

Proof. Since $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$, the Hilbert 3-class field $\mathrm{F}_3^1(k_\mu)$ of k_μ with $[\mathrm{F}_3^1(k_\mu) : k_\mu] = 9$ coincides with the absolute 3-genus field $k^* = k_p \cdot k_q \cdot k_r$ of the quartet $(k_1, \ldots, k_4)$ with $[k^* : \mathbb{Q}] = 27$ and $[k^* : k_\mu] = 9$.

Since the conductor of k_μ is $c = pqr$, $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$ is ramified in k_μ, but k^* is unramified over k_μ.

If $q \leftarrow p \rightarrow r$ is universally repelling, then p splits in k_q and in k_r, and the decomposition invariants of p in k^* are $(e,f,g) = (3,1,9)$, those of $\mathfrak{p}$ in $k^* = \mathrm{F}_3^1(k_\mu)$ are $(e,f,g) = (1,1,9)$, i.e. $\mathfrak{p}$ splits completely in $\mathrm{F}_3^1(k_\mu)$, and the decomposition law of the Hilbert 3-class field implies that $\mathfrak{p} = \alpha\mathcal{O}_{k_\mu}$ is principal with $\alpha \in k_\mu^\times$. Therefore the unique principal factor of k_μ is $A(k_\mu) = p$. □

5. Finite 3-groups of type (3,3)

In the following tables, we list those invariants of finite 3-groups G with elementary bicyclic commutator quotient $G/G' \simeq (3,3)$ which qualify metabelian groups $\mathfrak{M}$ as second 3-class groups $\mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ and non-metabelian groups $\mathfrak{G}$ as 3-class field tower groups $\mathrm{Gal}(\mathrm{F}_3^\infty(k)/k)$ of cyclic cubic number fields k. The process of searching for suitable groups in descendant trees with the strategy of pattern recognition [19] is governed by the *Artin pattern* $\mathrm{AP} = (\alpha, \varkappa)$ [17, p. 27], where $\alpha = \alpha_1$, respectively $\varkappa = \varkappa_1$, denotes the first layer of the transfer target type (TTT), respectively transfer kernel type (TKT). Additionally, we give the top layer α_2 of the TTT, which consists of the abelian quotient invariants of the commutator subgroup $\mathfrak{M}'$, corresponding to the 3-class group of the first Hilbert 3-class field $\mathrm{F}_3^1(k)$ of k. The *nuclear rank* ν is responsible for the search complexity. The p-multiplicator rank μ of a group G is precisely its *relation rank* $d_2(G) = \dim_{\mathbb{F}_3} \mathrm{H}^2(G, \mathbb{F}_3)$, which decides whether G is admissible as $\mathrm{Gal}(\mathrm{F}_3^\infty(k)/k)$, according to the Shafarevich Theorem [24], [17]. In the case of cyclic cubic fields k, it is limited by the *Shafarevich bound* $\mu \le \varrho + r + \theta$, where $\varrho = d_1(G) = \dim_{\mathbb{F}_3} \mathrm{H}^1(G, \mathbb{F}_3)$ denotes the *generator rank* of G, which coincides with the 3-class rank ϱ of k, $r = r_1 + r_2 - 1 = 2$ is the torsion free Dirichlet unit rank of the field k with signature $(r_1, r_2) = (3,0)$, and $\theta = 0$ indicates the absence of a (complex) primitive third root of unity in the totally real field k. Finally, $\pi(\mathfrak{M}) = \mathfrak{M}/\gamma_c(\mathfrak{M})$ denotes the parent of $\mathfrak{M}$, that is the quotient by the last non-trivial lower central with $c = \mathrm{cl}(\mathfrak{M})$.

Theorem 9. *Let k be a cyclic cubic number field with elementary bicyclic 3-class group $\mathrm{Cl}_3(k) \simeq (3,3)$. Denote by $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ the second 3-class group of k, and by $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k)/k)$ the 3-class field tower group of k. Then, the Artin pattern $(\alpha, \varkappa)$ of k identifies the groups $\mathfrak{M}$ and $\mathfrak{G}$, and determines the length $\ell_3(k)$ of the 3-class field tower of k, according to the following* ***deterministic laws****. (See the associated descendant tree $\mathcal{T}^1\langle 9,2\rangle$ in* [20, Fig. 6.1, p. 44]*.)*

1. *If $\alpha = [1,1,1,1]$, $\varkappa = (0000)$ (type* a.1*), then $\mathfrak{G} \simeq \langle 9,2\rangle$ and $\ell_3(k) = 1$.*
2. *If $\alpha \sim [11,2,2,2]$, $\varkappa \sim (1111)$ (type* A.1*), then $\mathfrak{G} \simeq \langle 27,4\rangle$.*
3. *If $\alpha \sim [111,11,11,11]$, $\varkappa \sim (2000)$ (type* a.3**), then $\mathfrak{G} \simeq \langle 81,7\rangle$.*
4. *If $\alpha \sim [21,11,11,11]$, $\varkappa \sim (2000)$ (type* a.3*), then $\mathfrak{G} \simeq \langle 81,8\rangle$.*
5. *If $\alpha \sim [21,11,11,11]$, $\varkappa \sim (1000)$ (type* a.2*), then $\mathfrak{G} \simeq \langle 81,10\rangle$.*
6. *If $\alpha \sim [22,11,11,11]$, $\varkappa \sim (2000)$ (type* a.3*), then $\mathfrak{G} \simeq \langle 243,25\rangle$.*
7. *If $\alpha \sim [22,11,11,11]$, $\varkappa \sim (1000)$ (type* a.2*), then $\mathfrak{G} \simeq \langle 243,27\rangle$.*

Except for the abelian tower in item (1), the tower is metabelian with $\ell_3(k) = 2$.

Proof. Generally, a cyclic cubic field k has signature $(r_1, r_2) = (3,0)$, torsion free unit rank $r = r_1 + r_2 - 1 = 2$, does not contain primitive third roots of unity, and thus possesses the maximal admissible relation rank $d_2 \le d_1 + r = 4$ for the group $\mathfrak{G}$, when its 3-class rank, i.e. the generator rank of $\mathfrak{G}$, is $d_1 = \varrho = 2$. Consequently, $\ell_3(k) \ge 3$ in the case of $d_2(\mathfrak{M}) \ge 5$.

For item (1), we have $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k) \simeq \langle 9,2\rangle \simeq (3,3) \simeq \mathrm{Cl}_3(k) \simeq \mathrm{Gal}(\mathrm{F}_3^1(k)/k)$, whence $\ell_3(k) = 1$. We always identify groups according to [5] and [10].

For item (2) to item (7), the group $\mathfrak{M}$ is of maximal class (coclass $\mathrm{cc}(\mathfrak{M}) = 1$), and thus coincides with $\mathfrak{G}$, whence $\ell_3(k) = 2$.

In each case, the Artin pattern $(\alpha, \varkappa)$ identifies $\mathfrak{M} = \mathfrak{G}$ uniquely, and the relation ranks are $d_2\langle 9,2\rangle = 3$, $d_2\langle 27,4\rangle = 2$, $d_2\langle 81,7\rangle = 3$, $d_2\langle 81,8\rangle = 3$, $d_2\langle 81,10\rangle = 3$, $d_2\langle 243,25\rangle = 3$, $d_2\langle 243,27\rangle = 3$, each of them less than 4. □

Corollary 4. *Under the assumptions of Theorem 9, the abelian type invariants α_2 of the 3-class group $\mathrm{Cl}_3(\mathrm{F}_3^1(k))$ of the first Hilbert 3-class field of k are required for the unambiguous identification of the following groups $\mathfrak{G}$ respectively $\mathfrak{M}$. (See the associated descendant tree $\mathcal{T}^2\langle 729, 40\rangle$ in [20, Fig. 6.2, p. 45].)*

If $\alpha \sim [21, 11, 11, 11]$, $\varkappa = (0000)$, a.1, *then* $\mathfrak{G} \simeq \begin{cases} \langle 81, 9\rangle & \text{for } \alpha_2 = [11], \\ \langle 243, 28..30\rangle & \text{for } \alpha_2 \sim [21]. \end{cases}$

If $\alpha \sim [21, 21, 111, 111]$, $\varkappa \sim (0043)$, b.10, *then* $\mathfrak{M} \simeq \begin{cases} \langle 729, 34..36\rangle & \text{for } \alpha_2 = [1111], \\ \langle 729, 37..39\rangle & \text{for } \alpha_2 \sim [211]. \end{cases}$

Proof. The Artin pattern $(\alpha, \varkappa)$ of k alone is not able to identify the groups $\mathfrak{M}$ and $\mathfrak{G}$ unambiguously. Ascione [1] uses the notation $\langle 729, 34\rangle = H$, $\langle 729, 35\rangle = I$, $\langle 729, 37\rangle = A$, $\langle 729, 38\rangle = C$. □

TABLE 2. Invariants of metabelian 3-groups $\mathfrak{M}$ with $\mathfrak{M}/\mathfrak{M}' \simeq (3, 3)$

$\mathfrak{M}$	cc	Type	$\varkappa$	α	α_2	ν	μ	$\pi(\mathfrak{M})$
$\langle 9, 2\rangle$	1	a.1	(0000)	1, 1, 1, 1	0	3	3	—
$\langle 27, 4\rangle$	1	A.1	(1111)	11, 2, 2, 2	1	0	2	$\langle 9, 2\rangle$
$\langle 81, 7\rangle$	1	a.3*	(2000)	111, 11, 11, 11	11	0	3	$\langle 27, 3\rangle$
$\langle 81, 8\rangle$	1	a.3	(2000)	21, 11, 11, 11	11	0	3	$\langle 27, 3\rangle$
$\langle 81, 9\rangle$	1	a.1	(0000)	21, 11, 11, 11	11	1	4	$\langle 27, 3\rangle$
$\langle 81, 10\rangle$	1	a.2	(1000)	21, 11, 11, 11	11	0	3	$\langle 27, 3\rangle$
$\langle 243, 25\rangle$	1	a.3	(2000)	22, 11, 11, 11	21	0	3	$\langle 81, 9\rangle$
$\langle 243, 27\rangle$	1	a.2	(1000)	22, 11, 11, 11	21	0	3	$\langle 81, 9\rangle$
$\langle 243, 28..30\rangle$	1	a.1	(0000)	21, 11, 11, 11	21	0	3	$\langle 81, 9\rangle$
$\langle 243, 3\rangle$	2	b.10	(0043)	21, 21, 111, 111	111	2	4	$\langle 27, 3\rangle$
$\langle 729, 34\rangle = H$	2	b.10	(0043)	21, 21, 111, 111	1111	2	5	$\langle 243, 3\rangle$
$\langle 729, 35\rangle = I$	2	b.10	(0043)	21, 21, 111, 111	1111	1	4	$\langle 243, 3\rangle$
$\langle 729, 37\rangle = A$	2	b.10	(0043)	21, 21, 111, 111	211	2	5	$\langle 243, 3\rangle$
$\langle 729, 38\rangle = C$	2	b.10	(0043)	21, 21, 111, 111	211	1	4	$\langle 243, 3\rangle$
$\langle 729, 40\rangle = B$	2	b.10	(0043)	22, 21, 111, 111	211	2	5	$\langle 243, 3\rangle$
$\langle 729, 41\rangle = D$	2	d.19	(4043)	22, 21, 111, 111	211	1	4	$\langle 243, 3\rangle$
$\langle 729, 42\rangle$	2	d.23	(1043)	22, 21, 111, 111	211	0	3	$\langle 243, 3\rangle$
$\langle 729, 43\rangle$	2	d.25	(2043)	22, 21, 111, 111	211	0	3	$\langle 243, 3\rangle$
$\langle 2187, 248\|249\rangle$	2	d.19	(4043)	32, 21, 111, 111	221	0	4	$\langle 729, 40\rangle$
$\langle 2187, 250\rangle$	2	d.23	(1043)	32, 21, 111, 111	221	0	4	$\langle 729, 40\rangle$
$\langle 2187, 251\|252\rangle$	2	d.25	(2043)	32, 21, 111, 111	221	0	4	$\langle 729, 40\rangle$
$\langle 2187, 253\rangle$	2	b.10	(0043)	22, 21, 111, 111	221	1	5	$\langle 729, 40\rangle$
$\langle 6561, 1989\rangle$	2	d.19	(4043)	33, 21, 111, 111	321	0	4	$\langle 2187, 247\rangle$
$\langle 243, 8\rangle$	2	c.21	(0231)	21, 21, 21, 21	111	1	3	$\langle 27, 3\rangle$
$\langle 729, 52\rangle = S$	2	G.16	(4231)	22, 21, 21, 21	211	1	3	$\langle 243, 8\rangle$
$\langle 729, 54\rangle = U$	2	c.21	(0231)	22, 21, 21, 21	211	2	4	$\langle 243, 8\rangle$
$\langle 2187, 301\|305\rangle$	2	G.16	(4231)	32, 21, 21, 21	221	1	4	$\langle 729, 54\rangle$
$\langle 2187, 303\rangle$	2	c.21	(0231)	32, 21, 21, 21	221	1	4	$\langle 729, 54\rangle$
$\langle 2187, 64\rangle = P_7$	3	b.10	(0043)	22, 22, 111, 111	2111	4	6	$\langle 243, 3\rangle$
$\langle 2187, 65\|67\rangle$	3	H.4	(3343)	22, 22, 111, 111	2111	3	5	$\langle 243, 3\rangle$
$\langle 2187, 66\|73\rangle$	3	F.11	(1143)	22, 22, 111, 111	2111	2	4	$\langle 243, 3\rangle$
$\langle 2187, 69\rangle$	3	G.16	(1243)	22, 22, 111, 111	2111	2	4	$\langle 243, 3\rangle$
$\langle 2187, 71\rangle$	3	G.19	(2143)	22, 22, 111, 111	2111	2	4	$\langle 243, 3\rangle$
$\langle 6561, 676\|677\rangle$	3	d.19	(4043)	32, 22, 111, 111	2211	0	5	$\langle 2187, 64\rangle$
$\langle 6561, 678\rangle$	3	d.23	(1043)	32, 22, 111, 111	2211	0	5	$\langle 2187, 64\rangle$
$\langle 6561, 679\|680\rangle$	3	d.25	(2043)	32, 22, 111, 111	2211	0	5	$\langle 2187, 64\rangle$
$\langle 6561, 693..698\rangle$	3	b.10	(0043)	22, 22, 111, 111	2211	0	5	$\langle 2187, 64\rangle$
$P_7 - \#2; 34\|35$	4	H.4	(3343)	32, 32, 111, 111	2221	1	5	$\langle 2187, 64\rangle$

In Table 2, we begin with metabelian groups $\mathfrak{M}$ of generator rank $d_1(\mathfrak{M}) = 2$. The Shafarevich bound [17, Thm. 5.1, p. 28] is given by $\mu \le \varrho + r + \theta = 2 + 2 + 0 = 4$. For order 6561 see [13].

TABLE 3. Invariants of non-metabelian 3-groups $\mathfrak{G}$ with $\mathfrak{G}/\mathfrak{G}' \simeq (3, 3)$

$\mathfrak{G}$	cc	Type	$\varkappa$	α	α_2	ν	μ	$\mathfrak{G}/\mathfrak{G}''$
$\langle 2187, 263..265\rangle$	2	d.19	(4043)	22, 21, 111, 111	211	0	3	$\langle 729, 41\rangle$
$\langle 2187, 307\|308\rangle$	2	c.21	(0231)	22, 21, 21, 21	211	0	3	$\langle 729, 54\rangle$
$\langle 6561, 619\|623\rangle$	3	G.16	(4231)	32, 21, 21, 21	221	1	3	$\langle 2187, 301\|305\rangle$

Capital letters for $\mathfrak{M}$ are due to Ascione [1]. For the metabelian groups $\mathfrak{M}$ with non-trivial cover $\mathrm{cov}(\mathfrak{M})$ [17, p. 30], we need non-metabelian groups $\mathfrak{G}$ in the cover, which are given in Table 3, where we begin with groups $\mathfrak{G}$ of generator rank $d_1(\mathfrak{G}) = 2$. For $d_1(\mathfrak{G}) = 3$, we refer to a forthcoming paper. Instead of the parent $\pi(\mathfrak{G})$, we give the metabelianization $\mathfrak{G}/\mathfrak{G}''$.

6. Categories I and II

Common feature of these two categories is the inhomogeneity of 3-class ranks of the four components in the quartet $(k_\mu)_{\mu=1}^4$ sharing the conductor $c = pqr$. In the present article, we restrict ourselves to 3, respectively 2, components with elementary bicyclic 3-class group $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$, for Category I, respectively Category II, and we postpone elementary tricyclic $\mathrm{Cl}_3(k_\mu) \simeq (3,3,3)$ to a future paper. All computations for examples were performed with Magma [6, 7, 12].

Definition 5. According to the 3-class numbers $h_3(k_\mu)$, a quartet $(k_\mu)_{\mu=1}^4$ of cyclic cubic fields with common conductor $c = pqr$ belonging to Category I or II is called

$$\begin{cases}\textbf{regular}\\ \textbf{singular}\\ \textbf{super-singular}\end{cases} \quad \text{if } \max\{h_3(k_\mu) \mid 1 \le \mu \le 4\} \begin{cases}= 27,\\ = 81,\\ \ge 243.\end{cases} \tag{6.1}$$

In a regular, respectively singular, respectively super-singular, quartet, there occurs a 3-class group $\mathrm{Cl}_3(k_\mu) \simeq (3,3,3)$, respectively $\mathrm{Cl}_3(k_\mu) \simeq (9,3,3)$, respectively $\mathrm{Cl}_3(k_\mu) \simeq (9,9,3)$, for some $1 \le \mu \le 4$.

6.1. **Category I, Graph 1.** Let $(k_1, \dots, k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 1 of Category I with combined cubic residue symbol $[p,q,r]_3 = \{p,q,r;\delta \equiv 0 \,(\mathrm{mod}\, 3)\}$.

Since there are no trivial cubic residue symbols among the three prime(power) divisors p,q,r of the conductor $c = pqr$, the principal factors of the subfields with $t = 2$ of the absolute genus field k^* must be divisible by both relevant primes, and we can use the general approach

$$\begin{aligned} A(k_{pq}) &= p^\ell q, \quad A(\tilde{k}_{pq}) = p^{-\ell} q,\\ A(k_{pr}) &= p^m r, \quad A(\tilde{k}_{pr}) = p^{-m} r, \text{ and}\\ A(k_{qr}) &= q^n r, \quad A(\tilde{k}_{qr}) = q^{-n} r, \end{aligned} \tag{6.2}$$

with $\ell, m, n \in \{-1, 1\}$, identifying $-1 \equiv 2 \,(\mathrm{mod}\, 3)$, since it is easier to manage: $\ell^2 = m^2 = n^2 = 1$.

Lemma 3. *The product $\ell \cdot m \cdot n = -1$ is negative (that is, either one or three among ℓ, m, n are negative) if and only if the compositum $L = k_{pq}k_{pr}k_{qr}$ satisfies the normalization $[L : \mathbb{Q}] = 9$:*

$$\begin{aligned} \ell \cdot m \cdot n = -1 &\iff [L : \mathbb{Q}] = 9,\\ \ell \cdot m \cdot n = +1 &\iff [L : \mathbb{Q}] = 27. \end{aligned} \tag{6.3}$$

Proof. By Theorem 8, the fields L and $\tilde{L} = \tilde{k}_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, satisfy a skew balance of their degrees $\in \{9, 27\}$ in the product $[L : \mathbb{Q}] \cdot [\tilde{L} : \mathbb{Q}] = 243$.

Suppose $lmn = +1$. Then we produce a contradiction by the assumption that $[L : \mathbb{Q}] = 9$ and $[\tilde{L} : \mathbb{Q}] = 27$. We define the compositum $K := \tilde{k}_{pq}\tilde{k}_{pr}$ of degree 9. Then K contains one of the fields k_μ, $\mu = 1, \dots, 4$, and either $\tilde{k}_{qr}$ or k_{qr}. In the former case, $K = \tilde{L}$ would have degree 27. So $K = \tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, and we calculate the following sub-determinants of the principal factor matrices M_L and M_K, with respect to the fields with $t = 2$ only (ignoring the field with $t = 3$):

$$\begin{vmatrix} \ell & 1 & 0\\ m & 0 & 1\\ 0 & n & 1 \end{vmatrix} = -\ell n - m = 0 \iff \begin{vmatrix} -\ell & 1 & 0\\ -m & 0 & 1\\ 0 & n & 1 \end{vmatrix} = \ell n + m = 0 \iff \ell n = -m.$$

However, $lmn = +1$ implies $\ell n = m$ and thus rank 3 of M_L and M_K. By (4.3), this gives indices of subfield units $(U_L : V_L) = 3$ and $(U_K : V_K) = 3$. At least one among L and K, say X, does not contain the critical field k_μ with $\varrho_3(k_\mu) = 3$, whence it is tame with $h_3(X) = (U_X : V_X) = 3$, in contradiction to $9 \mid h_3(X)$, by Proposition 1. Thus we must have $[L : \mathbb{Q}] = 27$.

With nearly identical arguments, it is easy to show that $lmn = -1$ implies $[L : \mathbb{Q}] = 9$. □

Lemma 4. ***(3-class ranks of components for*** I.1.***)*** *Without loss of generality, precisely three components k_2, k_3, k_4 of the quartet have elementary bicyclic 3-class groups $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$,*

*$2 \le \mu \le 4$, whereas the single remaining component k_1 has 3-class rank $\varrho_3(k_1) = 3$. In dependence on the **decisive principal factors** in Equation (6.2), the principal factors of k_μ are*

$$(6.4)\qquad \begin{aligned} A(k_2) &= pq^2r, & A(k_3) &= pqr, & A(k_4) &= pqr^2 && \text{if } (\ell,m,n) = (1,1,2),\\ A(k_2) &= p^2qr, & A(k_3) &= pqr^2, & A(k_4) &= pqr && \text{if } (\ell,m,n) = (1,2,1),\\ A(k_2) &= pqr, & A(k_3) &= pq^2r, & A(k_4) &= p^2qr && \text{if } (\ell,m,n) = (2,1,1),\\ A(k_2) &= pqr^2, & A(k_3) &= p^2qr, & A(k_4) &= pq^2r && \text{if } (\ell,m,n) = (2,2,2). \end{aligned}$$

*The **tame** condition $9 \mid h_3(B_j) = (U_j : V_j) \in \{9, 27\}$ with $r_j = 2$ is satisfied for $j \in \{2, 3, 4, 8, 9, 10\}$.*

*A further **decisive principal factor** $A(k_1) = p^{e_1}q^{e_2}r^{e_3}$ and the associated invariant counter $\mathcal{D} := \#\{1 \le i \le 3 \mid e_i \ne 0\}$ admit several conclusions for **wild** ranks:*

$$(6.5)\qquad r_5 = r_6 = r_7 = 3 \quad \text{iff} \quad \mathcal{D} = 2 \quad \text{iff} \quad A(k_1) \text{ has precisely two prime divisors.}$$

Proof. According to [2, Prop. 4.4, pp. 43–44], the required condition to distinguish the unique component k_1 with 3-class rank $\varrho(k_1) = 3$ in the quartet $(k_\mu)_{\mu=1}^4$ is the set of decomposition invariants $(e, f, g) = (3, 1, 3)$ simultaneously for p, q, r in the bicyclic bicubic field $B_1 = k_1k_{pq}k_{pr}k_{qr}$, that is,
p splits in k_{qr}, and thus also in B_j for $j \in \{1, 3, 8\}$,
q splits in k_{pr}, and thus also in B_j for $j \in \{1, 4, 9\}$,
r splits in k_{pq}, and thus also in B_j for $j \in \{1, 2, 10\}$.

Then, exactly the six fields $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, $B_8 = k_2k_4k_pk_{qr}$, $B_9 = k_2k_3k_qk_{pr}$, $B_{10} = k_3k_4k_rk_{pq}$ do not contain k_1, and satisfy the *tame* relation $9 \mid h_3(B_j) = (U_j : V_j) \in \{9, 27\}$ with ranks $r_j = 2$ for $j = 2, 3, 4, 8, 9, 10$, by Proposition 1.

This fact can be exploited for each tame bicyclic bicubic field B_j, by calculating the rank r_j with row operations on the associated principal factor matrix M_j and drawing conclusions for the exponents x_μ, y_μ, z_μ in the approach $A(k_\mu) = p^{x_\mu}q^{y_\mu}r^{z_\mu}$, $1 \le \mu \le 4$:

$$M_8 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 0 & n & 1 \end{pmatrix},\ M_9 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix},\ M_{10} = \begin{pmatrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ 0 & 0 & 1 \\ \ell & 1 & 0 \end{pmatrix}.$$

For $B_8 = k_2k_4k_pk_{qr}$, M_8 leads to decisive pivot elements $z_2 - ny_2$ and $z_4 - ny_4$ in the last column, for $B_9 = k_2k_3k_qk_{pr}$, M_9 leads to $z_2 - mx_2$ and $z_3 - mx_3$ in the last column, and for $B_{10} = k_3k_4k_rk_{pq}$, M_{10} leads to $y_3 - \ell x_3$ and $y_4 - \ell x_4$ in the middle column.

So $r_8 = r_9 = r_{10} = 2$ implies $ny_2 \equiv z_2$, $ny_4 \equiv z_4$, $z_2 \equiv mx_2$, $z_3 \equiv mx_3$, $\ell x_3 \equiv y_3$, $\ell x_4 \equiv y_4$. Or, in combined form, $mx_2 \equiv ny_2 \equiv z_2$, $mx_3 \equiv -ny_3 \equiv z_3$, $-mx_4 \equiv ny_4 \equiv z_4$. This yields (6.4).

Additionally, we use the remaining three tame ranks for

$$M_2 = \begin{pmatrix} x_2 & y_2 & z_2 \\ \ell & 1 & 0 \\ -m & 0 & 1 \\ 0 & -n & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} x_3 & y_3 & z_3 \\ -\ell & 1 & 0 \\ -m & 0 & 1 \\ 0 & n & 1 \end{pmatrix},\ M_4 = \begin{pmatrix} x_4 & y_4 & z_4 \\ -\ell & 1 & 0 \\ m & 0 & 1 \\ 0 & -n & 1 \end{pmatrix}.$$

For $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, M_2 leads to the decisive pivot elements $z_2 + mx_2 + ny_2$, $\ell m + n$ in the last column, for $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, M_3 leads to $z_3 + mx_3 - ny_3$, $-\ell m - n$ in the last column, and for $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, M_4 leads to $z_4 - mx_4 + ny_4$, $\ell m + n$ in the last column. So $r_2 = r_3 = r_4 = 2$ implies $mx_2 + ny_2 \equiv -z_2$, $mx_3 - ny_3 \equiv -z_3$, $-mx_4 + ny_4 \equiv -z_4$, since the other pivot elements vanish a priori, $\ell m + n = 0$, i.e. $\ell m = -n$, because $\ell mn = -1$ and $n^2 = 1$ in Lemma 3. The congruences follow already from those for $r_8 = r_9 = r_{10} = 2$.

For each *wild* bicyclic bicubic field B_j, $j \in \{1, 5, 6, 7\}$, the rank r_j is now calculated with row operations on the associated principal factor matrix M_j:

$$M_1 = \begin{pmatrix} x_1 & y_1 & z_1 \\ \ell & 1 & 0 \\ m & 0 & 1 \\ 0 & n & 1 \end{pmatrix},\ M_5 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ 1 & 0 & 0 \\ 0 & -n & 1 \end{pmatrix},\ M_6 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ 0 & 1 & 0 \\ -m & 0 & 1 \end{pmatrix},\ M_7 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 0 & 0 & 1 \\ -\ell & 1 & 0 \end{pmatrix}.$$

For $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, M_1 leads to the decisive pivot element $z_1 - mx_1 - ny_1$, since $-\ell m - n \equiv 0$. So, $r_1 = 2$ implies $z_1 \equiv mx_1 + ny_1$. For $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, M_5 leads to $z_1 + ny_1$, $z_3 + ny_3$ in the last column. So, $r_5 = 3$ iff either $-z_1 \not\equiv ny_1$ or $-z_3 \not\equiv ny_3$ modulo 3. For $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, M_6 leads to $z_1 + mx_1$, $z_4 + mx_4$. So, $r_6 = 3$ iff either $-z_1 \not\equiv mx_1$ or $-z_4 \not\equiv mx_4$. For $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, M_7 leads to $y_1 + \ell x_1$, $y_2 + \ell x_2$ in the middle column. So, $r_7 = 3$ iff either $-y_1 \not\equiv \ell x_1$ or $-y_2 \not\equiv \ell x_2$. For each of these three ranks, the *second* condition can *never* be satisfied.

Since at most one of the exponents x_1, y_1, z_1 may vanish, the new congruences immediately lead to (6.5). For instance, $z_1 = 0 \Rightarrow -z_1 = 0 \not\equiv ny_1$, $-z_1 = 0 \not\equiv mx_1 \Rightarrow r_5 = r_6 = 3$; but $0 = z_1 \equiv mx_1 + ny_1$ also implies $mx_1 \equiv -ny_1$, $mnx_1 \equiv -y_1$, $\ell x_1 \equiv y_1$ and thus $r_7 = 3$. Conversely, suppose $\mathcal{D} = 3$. If $-z_1 \not\equiv ny_1$, then $z_1 \equiv ny_1$, and $z_1 \equiv mx_1 + ny_1$ implies $mx_1 \equiv 0$, and thus the contradiction $x_1 = 0$. □

Proposition 5. ***(Sub-triplet with*** **3*-rank two for*** I.1*.)* *For fixed $\mu \in \{2,3,4\}$, let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be the prime ideals of k_μ over p, q, r, that is $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$, then the 3-class group of k_μ is generated by any two among the non-trivial classes $[\mathfrak{p}], [\mathfrak{q}], [\mathfrak{r}]$, that is,*

$$\mathrm{Cl}_3(k_\mu) = \langle [\mathfrak{p}], [\mathfrak{q}] \rangle = \langle [\mathfrak{p}], [\mathfrak{r}] \rangle = \langle [\mathfrak{q}], [\mathfrak{r}] \rangle \simeq (3,3). \tag{6.6}$$

The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic subfields B_i, $1 \le i \le 10$, of the common genus field k^* of the four components of the quartet $(k_1, \ldots, k_4)$. The unique $B_9 > k_\mu$, $\mu \in \{2,3\}$, has norm class group $N_{B_9/k_\mu}(\mathrm{Cl}_3(B_9)) = \langle [\mathfrak{q}] \rangle$, and potential ***fixed point*** *transfer kernel*

$$\ker(T_{B_9/k_\mu}) \ge \langle [\mathfrak{q}] \rangle.$$

The unique $B_{10} > k_\mu$, $\mu \in \{3,4\}$, has norm class group $N_{B_{10}/k_\mu}(\mathrm{Cl}_3(B_{10})) = \langle [\mathfrak{r}] \rangle$, and potential ***fixed point*** *transfer kernel*

$$\ker(T_{B_{10}/k_\mu}) \ge \langle [\mathfrak{r}] \rangle.$$

The unique $B_8 > k_\mu$, $\mu \in \{2,4\}$, has norm class group $N_{B_8/k_\mu}(\mathrm{Cl}_3(B_8)) = \langle [\mathfrak{q}\mathfrak{r}^n] \rangle$, and potential ***fixed point*** *transfer kernel*

$$\ker(T_{B_8/k_\mu}) \ge \langle [\mathfrak{q}^n \mathfrak{r}] \rangle.$$

The remaining $B_i > k_\mu$, $i \in \{5,6,7\}$, more precisely, $i = 7$ for $\mu = 2$, $i = 5$ for $\mu = 3$, and $i = 6$ for $\mu = 4$, have norm class group $\langle [\mathfrak{q}^2 \mathfrak{r}^{-n}] \rangle$ and a hidden or explicit ***transposition*** *transfer kernel, with respect to the corresponding μ.*

Proof. As mentioned in the proof of Lemma 4, q splits in B_9, r splits in B_{10}, and p splits in B_8, where $[\mathfrak{p}] = [\mathfrak{q}\mathfrak{r}^n]$, according to (6.4), independently of $n \in \{1,2\}$.
Now we use Corollary 3 and Proposition 2.
Since $\mathfrak{q}$ is principal in k_q, $[\mathfrak{q}]$ capitulates in $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$ and $B_9 = k_2 k_3 k_q k_{pr}$.
Since $\mathfrak{r}$ is principal in k_r, $[\mathfrak{r}]$ capitulates in $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$ and $B_{10} = k_3 k_4 k_r k_{pq}$.
Since $\mathfrak{q}^n \mathfrak{r}$ is principal in k_{qr}, $[\mathfrak{q}^n \mathfrak{r}]$ capitulates in $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$ and $B_8 = k_2 k_4 k_p k_{qr}$. □

In terms of capitulation targets in Corollary 2, Proposition 5 and parts of its proof are now summarized in Table 4 for the minimal transfer kernel type (mTKT) and $n = 2$, with transposition in **boldface** font. This essential new perspective admits progress beyond Ayadi's work [2].

TABLE 4. Norm class groups and minimal transfer kernels with $n = 2$ for Graph I.1

Base	k_2				k_3				k_4			
Ext	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$
TK	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$
$\varkappa$	**2**	**1**	3	4	**2**	**1**	3	4	**2**	**1**	3	4

Theorem 10. ***(Second*** **3*-class group for*** I.1.*) Let* $(k_1,\ldots,k_4)$ *be a quartet of cyclic cubic number fields sharing the common conductor* $c = pqr$ *belonging to Graph* 1 *of Category* I*, that is,* $[p,q,r]_3 = \{p,q,r;\delta \equiv 0\,(\mathrm{mod}\,3)\}$*. Without loss of generality, suppose that* $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$*, for* $\mu = 2,3,4$*, and* $\varrho_3(k_1) = 3$*.*

Then the ***minimal transfer kernel type*** *(mTKT) of* k_μ*,* $2 \le \mu \le 4$*, is* $\varkappa_0 = (4231)$*, type* G.16*, and other possible capitulation types in ascending partial order* $\varkappa_0 < \varkappa < \varkappa', \varkappa'' < \varkappa'''$ *are* $\varkappa = (0231)$*, type* c.21*,* $\varkappa' = (0001)$*, type* a.3*,* $\varkappa'' = (0200)$*, type* a.2*, and* $\varkappa''' = (0000)$*, type* a.1*.*

In order to identify the second 3*-class group* $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$*,* $2 \le \mu \le 4$*, let the* ***principal factor*** *of* k_1 *be* $A(k_1) = p^{e_1}q^{e_2}r^{e_3}$*, and define* $\mathcal{D} := \#\{1 \le i \le 3 \mid e_i \ne 0\}$*. In the* ***regular*** *situation where* $\mathrm{Cl}_3(k_1) \simeq (3,3,3)$ *is elementary tricyclic, we have*

$$\mathfrak{M} \simeq \begin{cases} \langle 81,8\rangle,\ \alpha = [11,11,11,21],\ \varkappa = (0001) & \text{once if } \mathcal{D}=2,\ \mathcal{N}=1, \\ \langle 81,10\rangle^2,\ \alpha = [11,21,11,11],\ \varkappa = (0200) & \text{twice if } \mathcal{D}=2,\ \mathcal{N}=1, \\ \langle 243,25\rangle,\ \alpha = [11,11,11,22],\ \varkappa = (0001) & \text{once if } \mathcal{D}=3,\ \mathcal{N}=1, \\ \langle 243,28..30\rangle^2,\ \alpha = [21,11,11,11],\ \varkappa = (0000) & \text{twice if } \mathcal{D}=3,\ \mathcal{N}=0, \end{cases} \tag{6.7}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j,\ I_j = 27\}$*. In the* ***(super-)singular*** *situation where* $81 \mid h(k_1)$ *and* $\mathrm{Cl}_3(k_1)$ *is non-elementary tricyclic, we have* $\mathfrak{M} \simeq$

$$\begin{cases} \langle 243,8\rangle^3,\ \alpha = [21,21,21,21],\ \varkappa = (0231) & \text{if } h_3(k_1) = 81,\ \mathcal{D}=2,\ \mathcal{N}=3, \\ \langle 729,54\rangle^3,\ \alpha = [22,21,21,21],\ \varkappa = (0231) & \text{if } h_3(k_1) = 81,\ \mathcal{D}=3,\ \mathcal{N}=3, \\ \langle 2187,301|305\rangle^3,\ \alpha = [32,21,21,21],\ \varkappa = (4231) & \text{if } h_3(k_1) = 81,\ \mathcal{D}=3,\ \mathcal{N}=4, \\ \langle 2187,303\rangle^3,\ \alpha = [32,21,21,21],\ \varkappa = (0231) & \text{if } h_3(k_1) = 243,\ \mathcal{D}=3,\ \mathcal{N}=3. \end{cases} \tag{6.8}$$

With exception of the last three rows, the 3*-class field tower has the group* $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$*,* $\ell_3(k_\mu) = 2$*, since* $d_2(\mathfrak{M}) \le 4$*. For the last three rows, the tower length is* $2 \le \ell_3(k_\mu) \le 3$ [17]*. (See the associated descendant tree* $\mathcal{T}^2\langle 243,8\rangle$ *in* [20, Fig. 6.4, p. 63]*.)*

Proof. In the *non-uniform regular* situations, we have $r_j = 2$, $h_3(B_j) = I_j \in \{9,27\}$ for the tame bicyclic bicubic fields $j \in \{2,3,4,8,9,10\}$. Now we use Lemma 2 and Lemma 4.

If $\mathcal{D} = 3$, then all tame indices of subfield units $I_j = 9$ are minimal, and the ranks of wild bicyclic bicubic fields are $r_j = 2$ for $j = 5,6,7$, but non-uniform indices two times $I_5 = I_6 = 9$, i.e. $\mathcal{N} = 0$, and one time $I_7 = 27$, i.e. $\mathcal{N} = 1$, corresponding to total capitulation twice and non-fixed point capitulation once (due to a hidden transposition). According to Theorem 9, the common $\alpha_2 = (21)$, and Corollary 4, the Artin pattern $\alpha = [21,11,11,11]$ and $\varkappa = (0000)$ determines three possible groups $\langle 243,28..30\rangle$, and $\alpha = [11,11,11,22]$, $\varkappa = (0001)$ uniquely leads to $\langle 243,25\rangle$.

If $\mathcal{D} = 2$, then tame indices of subfield units I_j are non-uniform, two times $I_j = 9$, for $j = 2,4,9,10$, and one time $I_j = 27$, for $j = 3,8$, the latter corresponding to fixed point capitulation twice, $j = 8$ over $\mu = 2,4$, and non-fixed point capitulation once, $j = 3$ over $\mu = 3$. So $\mathcal{N} = 1$, since the ranks of wild bicyclic bicubic fields are $r_j = 3$ with uniform index $I_j = 3$ for $j = 5,6,7$, corresponding to a total capitulation. The Artin pattern $\alpha = [11,21,11,11]$, $\varkappa = (0200)$ uniquely determines the group $\langle 81,10\rangle$, and $\alpha = [11,11,11,21]$, $\varkappa = (0001)$ uniquely leads to $\langle 81,8\rangle$.

In the *uniform singular* situation with TKT c.21, $\varkappa = (0231)$, $\mathcal{N} = 3$, the ATI decide about the group: $\alpha = [21,21,21,21]$ uniquely identifies $\langle 243,8\rangle$, $\alpha = [22,21,21,21]$ leads to $\langle 729,54\rangle$, and in the super-singular situation, $\alpha = [32,21,21,22]$ leads to $\langle 2187,303\rangle$. In contrast, for TKT G.16, $\varkappa = (4231)$, $\mathcal{N} = 4$, the ATI $\alpha = [32,21,21,21]$ lead to $\langle 2187,301|305\rangle$.

The regular groups are of maximal class, which guarantees length $\ell_3(k_\mu) = 2$ of the tower. The annihilator ideal of $\langle 243,8\rangle$ is $\mathfrak{L}$, which enforces $\ell_3(k_\mu) = 2$, according to Scholz and Taussky [23]. The (super-)singular groups $\langle 729,54\rangle$ and $\langle 2187,303\rangle$ have non-metabelian descendants. Although they satisfy the bound $d_2(\mathfrak{G}) \le 4$ for the relation rank, a tower with three stages could only be excluded by means of computationally expensive invariants $\alpha^{(2)}$ of second order. □

Corollary 5. ***(Non-uniformity of the sub-triplet for*** I.1.*) Only two components of the sub-triplet with* 3*-rank two share a common capitulation type* $\varkappa(k_\lambda) \sim \varkappa(k_\mu)$*, common abelian type invariants* $\alpha(k_\lambda) \sim \alpha(k_\mu)$*, and a common second* 3*-class group* $\mathrm{Gal}(\mathrm{F}_3^2(k_\lambda)/k_\lambda) \simeq \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$*.*

*The invariants of the third component k_ν **differ** in the **regular** situation* $\mathrm{Cl}_3(k_1) \simeq (3,3,3)$, *however, they **agree** in the **(super-)singular** situation* $81 \mid h(k_1)$. *Here,* $\{\lambda,\mu,\nu\} = \{2,3,4\}$.

Proof. This is an immediate consequence of Theorem 10. □

Example 1. Examples 1–9 are supplemented by [20, Tbl. 6.4–21, pp. 49–67]. The prototypes for Graph I.1, that is, the minimal conductors for each scenario in Theorem 10, are as follows.

There are **regular** cases: $c = 4\,977$ with symbol $\{9,7,79\}$ and, non-uniformly, $\mathfrak{G} = \mathfrak{M} = \langle 243,25\rangle$, $\langle 243,28..30\rangle^2$ (Corollary 4); $c = 11\,349$ with symbol $\{9,13,97\}$ and, non-uniformly, $\mathfrak{G} = \mathfrak{M} = \langle 81,8\rangle$, $\langle 81,10\rangle^2$.

Further, **singular** cases: $c = 28\,791$ with symbol $\{9,7,457\}$ and $\mathfrak{M} = \langle 729,54\rangle^3$; $c = 38\,727$ with symbol $\{9,13,331\}$ and $\mathfrak{G} = \mathfrak{M} = \langle 243,8\rangle^3$; and, with **considerable statistic delay**, there occurred $c = 417\,807$ with ordinal number 189, symbol $\{9,13,3571\}$ and $\mathfrak{M} = \langle 2187,301|305\rangle^3$.

And **super-singular** cases: $c = 67\,347$ with symbol $\{9,7,1069\}$ and $\mathfrak{M} = \langle 2187,303\rangle^3$; $c = 436\,267$ with symbol $\{13,37,907\}$ and $\mathfrak{M} = (\langle 6561,2050\rangle - \#1;3|5)^3$.

In Table 5, we summarize the prototypes of graph I.1. Data comprises ordinal number No., conductor c of k, combined cubic residue symbol $[p,q,r]_3$, regularity, resp. (super-)singularity, expressed by 3-valuation $v = v_3(\#\mathrm{Cl}(k_1))$ of class number of critical field k_1, critical exponents x,y,z in principal factor $A(k_1) = p^x q^y r^z$ and ℓ,m,n in $A(k_{pq}) = p^\ell q$, $A(k_{pr}) = p^m r$, $A(k_{qr}) = q^n r$, capitulation type of k, second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ of k, and length $\ell_3(k)$ of 3-class field tower of k. We put $R := \langle 6561,2050\rangle$ for abbreviation.

Table 5. Prototypes for Graph I.1

No.	c	p,q,r	v	$x,y,z;\ell,m,n$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	4 977	9, 7, 79	3	2, 1, 1; 1, 1, 2	a.3, a.1	$\langle 243,25\rangle, \langle 243,28..30\rangle^2$	$= 2$
3	11 349	9, 13, 97	3	0, 1, 1; 2, 1, 1	a.3, a.2	$\langle 81,8\rangle, \langle 81,10\rangle^2$	$= 2$
10	28 791	9, 7, 457	4	2, 1, 1; 1, 1, 2	c.21	$\langle 729,54\rangle^3$	≥ 2
14	38 727	9, 13, 331	4	1, 0, 1; 2, 1, 1	c.21	$\langle 243,8\rangle^3$	$= 2$
27	67 347	9, 7, 1069	5	2, 1, 1; 1, 1, 2	c.21	$\langle 2187,303\rangle^3$	≥ 2
189	417 807	9, 13, 3571	4	2, 2, 1; 2, 2, 2	G.16	$\langle 2187,301|305\rangle^3$	≥ 2
198	436 267	13, 37, 907	6	1, 1, 1; 2, 2, 2	G.16	$(R-\#1;3|5)^3$	≥ 2

6.2. **Category I, Graph 2.** Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 2 of Category I with combined cubic residue symbol $[p,q,r]_3 = \{q \leftarrow p \rightarrow r\}$.

Lemma 5. ***(3-class ranks of components for*** I.2.***)*** *Under the normalizing assumptions that q splits in k_{pr} and r splits in k_{pq}, precisely the three components k_2, k_3, k_4 of the quartet have elementary bicyclic* 3*-class group* $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$, $\mu = 2,3,4$, *of rank* 2, *whereas the remaining component has* 3*-class rank* $\varrho_3(k_1) = 3$. *Thus, the **tame** condition* $9 \mid h_3(B_j) = (U_j : V_j) \in \{9,27\}$, $r_j = 2$, *is satisfied for the bicyclic bicubic fields B_j with* $j \in \{2,3,4,8,9,10\}$.

Proof. p is universally repelling $\{q \leftarrow p \rightarrow r\}$. Since $p \rightarrow r$, p splits in k_r. Since $q \leftarrow p$, p splits in k_q. Thus p also splits in k_{qr} and $\tilde{k}_{qr}$. By the normalizing assumptions that q splits in k_{pr} and r splits in k_{pq}, the primes p,q,r share the common decomposition type $(e,f,g) = (3,1,3)$ in the bicyclic bicubic field $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, which implies that $\varrho_3(k_1) = 3$, according to [2, Prop. 4.4, pp. 43–44]. Finally, none among $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, $B_8 = k_2 k_4 k_p k_{qr}$, $B_9 = k_2 k_3 k_q k_{pr}$, $B_{10} = k_3 k_4 k_r k_{pq}$ contains k_1. □

Proposition 6. ***(Sub-triplet with*** 3***-rank two for*** I.2.***)*** *For fixed* $\mu \in \{2,3,4\}$, *let* $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ *be the prime ideals of k_μ over p,q,r, that is* $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$, *then the* 3*-class group of k_μ is generated by the non-trivial classes* $[\mathfrak{q}],[\mathfrak{r}]$, *that is,*

$$\mathrm{Cl}_3(k_\mu) = \langle [\mathfrak{q}],[\mathfrak{r}]\rangle \simeq (3,3). \tag{6.9}$$

The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic fields B_i, $1 \leq i \leq 10$.

In terms of ***decisive principal factors*** $A(k_1) = p^x q^y r^z$*,* $x, y, z \in \{0, 1, 2\}$*, and* $A(k_{qr}) = qr^n$*,* $n \in \{1, 2\}$*, the ranks of principal factor matrices of* ***wild*** *bicyclic bicubic fields are*

$$r_5 = 3 \text{ iff } -z \not\equiv ny \pmod 3,\ r_6 = 3 \text{ iff } z \neq 0 \text{ iff } r \mid A(k_1),\ r_7 = 3 \text{ iff } y \neq 0 \text{ iff } q \mid A(k_1). \tag{6.10}$$

The field $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$ *has norm class group* $N_{B_2/k_2}(\mathrm{Cl}_3(B_2)) = \langle[\mathfrak{r}]\rangle$*, and transfer kernel*

$$\ker(T_{B_2/k_2}) \geq \langle[\mathfrak{q}^2\mathfrak{r}^n]\rangle.$$

The field $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$ *has norm class group* $N_{B_4/k_4}(\mathrm{Cl}_3(B_2)) = \langle[\mathfrak{q}]\rangle$*, and transfer kernel*

$$\ker(T_{B_4/k_4}) \geq \langle[\mathfrak{q}^2\mathfrak{r}^n]\rangle.$$

The field $B_9 = k_2 k_3 k_q k_{pr}$*, which contains* k_2 *and* k_3*, has norm class group* $N_{B_9/k_\mu}(\mathrm{Cl}_3(B_9)) = \langle[\mathfrak{q}]\rangle$*, for* $\mu = 2, 3$*, and possible* ***fixed point*** *transfer kernel*

$$\ker(T_{B_9/k_\mu}) \geq \langle[\mathfrak{q}]\rangle. \tag{6.11}$$

The field $B_{10} = k_3 k_4 k_r k_{pq}$*, which contains* k_3 *and* k_4*, has norm class group* $N_{B_{10}/k_\mu}(\mathrm{Cl}_3(B_{10})) = \langle[\mathfrak{r}]\rangle$*, for* $\mu = 3, 4$*, and possible* ***fixed point*** *transfer kernel*

$$\ker(T_{B_{10}/k_\mu}) \geq \langle[\mathfrak{r}]\rangle. \tag{6.12}$$

The remaining two $B_i > k_\mu$*,* $i \in \{3, 5, 6, 7, 8\}$*, more precisely,* $i \in \{7, 8\}$ *for* $\mu = 2$*,* $i \in \{3, 5\}$ *for* $\mu = 3$*, and* $i \in \{6, 8\}$ *for* $\mu = 4$*, have norm class group* $\langle[\mathfrak{q}\mathfrak{r}]\rangle$ *respectively* $\langle[\mathfrak{q}\mathfrak{r}^2]\rangle$*. Among them, the tame extensions* $B_i > k_\mu$ *with either* $i = \mu = 3$ *or* $i = 8$*,* $\mu = 2, 4$*, have* ***partial*** *transfer kernel*

$$\ker(T_{B_i/k_\mu}) = \langle[\mathfrak{q}\mathfrak{r}]\rangle \tag{6.13}$$

of order 3*, giving rise to either a* ***transposition*** *or a* ***fixed point****.*

Proof. Since $p \to r$, two principal factors are $A(k_{pr}) = A(\tilde{k}_{pr}) = p$; and since $q \leftarrow p$, two further principal factors are $A(k_{pq}) = A(\tilde{k}_{pq}) = p$, by Proposition 3. Since p is universally repelling $\{q \leftarrow p \to r\}$, three further principal factors are $A(k_\mu) = p$ for $2 \leq \mu \leq 4$, by Proposition 4.

Thus, $[\mathfrak{p}] = 1$ is trivial, and the non-trivial classes $[\mathfrak{q}], [\mathfrak{r}]$ generate $\mathrm{Cl}_3(k_\mu) = \langle[\mathfrak{q}], [\mathfrak{r}]\rangle \simeq (3, 3)$.

Since q splits in k_{pr}, it also splits in $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, $B_9 = k_2 k_3 k_q k_{pr}$.

Since r splits in k_{pq}, it also splits in $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_{10} = k_3 k_4 k_r k_{pq}$.

Since the *tame* condition $9 \mid h_3(B_j) = (U_j : V_j)$ is satisfied for $j \in \{2, 3, 4, 8, 9, 10\}$, the rank of the corresponding principal factor matrix M_j must be $r_j = 2$. This can also be verified directly and has no further consequences.

We propose the principal factors $A(k_1) = p^x q^y r^z$ and and $A(k_{qr}) = qr^n$, $A(\tilde{k}_{qr}) = q^2 r^n$ with $n \in \{1, 2\}$. For each *wild* bicyclic bicubic field B_j, $j \in \{5, 6, 7\}$, the rank r_j is now calculated with row operations on the associated principal factor matrices M_j:

$$M_5 = \begin{pmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & n \end{pmatrix},\ M_6 = \begin{pmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ M_7 = \begin{pmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, M_5 leads to the decisive pivot element $z + ny$ in the last column. So, $r_5 = 3$ iff $-z \not\equiv ny$ modulo 3. For $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, M_6 leads to z. So, $r_6 = 3$ iff $z \neq 0$. For $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, M_7 leads to y in the middle column. So, $r_7 = 3$ iff $y \neq 0$.

Since $\mathfrak{q}$ is principal in k_q, $[\mathfrak{q}]$ capitulates in $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$ and $B_9 = k_2 k_3 k_q k_{pr}$, and since $\mathfrak{r}$ is principal in k_r, $[\mathfrak{r}]$ capitulates in $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$ and $B_{10} = k_3 k_4 k_r k_{pq}$, by Corollary 3.

Since $\mathfrak{q}\mathfrak{r}^n$ is principal in k_{qr}, $[\mathfrak{q}\mathfrak{r}^n]$ capitulates in $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, $B_8 = k_2 k_4 k_p k_{qr}$, and since $\mathfrak{q}^2\mathfrak{r}^n$ is principal in $\tilde{k}_{qr}$, $[\mathfrak{q}^2\mathfrak{r}^n]$ capitulates in $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, and $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, by Proposition 2.

In each case, the minimal subfield unit index $(U_j : V_j) = 3$ for $r_j = 3$ corresponds to the maximal unit norm index $(U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3$, associated to a *total* transfer kernel $\#\ker(T_{B_j/k_\mu}) = 9$, by Lemma 2.

The minimal unit norm index $(U(k_\mu) : N_{B_8/k_\mu}(U_8)) = 1$, associated to the partial transfer kernel $\ker(T_{B_8/k_\mu}) = \langle[\mathfrak{qr}]\rangle$, for $\mu = 2,4$, corresponds to the tame maximal subfield unit index $h_3(B_8) = (U_8 : V_8) = 27$, giving rise to type invariants $\mathrm{Cl}_3(B_8) \simeq (9,3)$. □

Using Corollary 2, Proposition 6 and parts of its proof are now summarized in Table 6 for the minimal transfer kernel type (mTKT) and $n = 1$, with transposition in **boldface** font.

TABLE 6. Norm class groups and minimal transfer kernels with $n = 1$ for Graph I.2

Base	k_2				k_3				k_4			
Ext	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{r}$	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{r}$
TK	$\mathfrak{qr}^2$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{qr}$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{qr}$	$\mathfrak{r}$
$\varkappa$	**2**	**1**	3	4	**2**	**1**	3	4	**2**	**1**	3	4

Theorem 11. ***(Second*** **3*-class group for*** I.2.*)* *To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $2 \le \mu \le 4$, let the* ***principal factors*** *of k_1, and k_{qr}, respectively $\tilde{k}_{qr}$, be $A(k_1) = p^x q^y r^z$, $x,y,z \in \{0,1,2\}$, and $A(k_{qr}) = qr^n$, respectively $A(\tilde{k}_{qr}) = q^2 r^n$, $n \in \{1,2\}$, and additionally assume the* ***regular*** *situation where $\mathrm{Cl}_3(k_1) \simeq (3,3,3)$.*

Then the ***minimal transfer kernel type*** *(mTKT) $\varkappa_0$ of k_μ, $1 \le \mu \le 4$, and other possible capitulation types in ascending partial order $\varkappa_0 < \varkappa < \varkappa', \varkappa''$, ending in two non-comparable types, are $\varkappa_0 = (2134)$, type* G.16*, $\varkappa = (0134)$, type* c.21*, $\varkappa' = (0004)$, type* a.2*, $\varkappa'' = (0100)$, type* a.3*, and the second 3-class group is $\mathfrak{M} \simeq$*

$$(6.14) \qquad \begin{cases} \langle 81,8\rangle,\ \alpha = [11,21,11,11],\ \varkappa = (0100) & \textit{once if } y \neq 0,\ z \neq 0,\ \mathcal{N} = 1, \\ \langle 81,10\rangle,\ \alpha = [11,11,11,21],\ \varkappa = (0004) & \textit{twice if } y \neq 0,\ z \neq 0,\ \mathcal{N} = 1, \\ \langle 243,8\rangle,\ \alpha = [21,21,21,21],\ \varkappa = (0134) & \textit{if } y = z = 0,\ \mathcal{N} = 3, \\ \langle 729,52\rangle,\ \alpha = [22,21,21,21],\ \varkappa = (2134) & \textit{if } y = z = 0,\ \mathcal{N} = 4, \end{cases}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j,\ I_j = 27\}$. Only for the leading three rows, the 3-class field tower has certainly the group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$ and length $\ell_3(k_\mu) = 2$, otherwise length $\ell_3(k_\mu) \ge 3$ cannot be excluded although $d_2(\mathfrak{M}) \le 4$.

Proof. Let $\mu \in \{2,3,4\}$.

The first scenario, $y \neq 0$, $z \neq 0$, and $-z \not\equiv ny$ modulo 3 is equivalent to $\mathcal{N} = 1$, since $r_j = 3$, $(U_j : V_j) = 3$, $h_3(B_j) = \frac{1}{3}h_3(k_1) = 9$, for the wild $j = 5,6,7$, and $h_3(B_j) = (U_j : V_j) = 9$, for the tame $j = 2,4,9,10$, whereas the distinguished tame $j = 3,8$ have $h_3(B_j) = (U_j : V_j) = 27$. This gives rise to Artin pattern either $\alpha = [11,11,11,21]$ $\varkappa = (0004)$, for $j = 8$, $\mu = 2,4$ (twice with fixed point), characteristic for $\langle 81,10\rangle$, or $\alpha = [11,21,11,11]$ $\varkappa = (0100)$, for $j = \mu = 3$ (only once with non-fixed point, due to a hidden transposition), characteristic for $\langle 81,8\rangle$.

The other two scenarios share $y = z = 0$, and thus also $-z = ny$, independently of n, which implies $r_j = 2$, $(U_j : V_j) \in \{9,27\}$, for $j = 5,6,7$, and $h_3(B_j) = (U_j : V_j) = 27$, for the tame $j = 2,4,9,10$, producing two fixed points at B_9 and B_{10}.

The second scenario with $\mathcal{N} = 3$ is supplemented by $(U_j : V_j) = 9$, $h_3(B_j) = h_3(k_1) = 27$, for $j = 5,6,7$, and *total* capitulation, $\#\ker(T_{B_j/k_\mu}) = 9$, for $\mu = 3,4,2$. This gives rise to $\alpha = [21,21,21,21]$, $\varkappa = (0134)$, characteristic for $\langle 243,8\rangle$ with annihilator ideal $\mathfrak{L}$ in the sense of Scholz and Taussky [23].

The third scenario with $\mathcal{N} = 4$ is supplemented by $(U_j : V_j) = 27$, $h_3(B_j) = 3h_3(k_1) = 81$, for $j = 5,6,7$ and *partial* non-fixed point capitulation. This gives rise to $\alpha = [22,21,21,21]$, $\varkappa = (2134)$, characteristic for $\langle 729,52\rangle$ with non-metabelian descendants. Here, the hidden *transposition* becomes explicit, between either B_2, B_7 or B_3, B_5 or B_4, B_6. □

Corollary 6. ***(Non-uniformity of the sub-triplet for*** I.2.*)* *The components of the sub-triplet with 3-rank two share a common capitulation type $\varkappa(k_\mu)$, common abelian type invariants $\alpha(k_\mu)$,* *and a common second* 3*-class group* $\mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$*, for* $\mu = 2,3,4$*, only if* $y = z = 0$*,* $\mathcal{N} = 3,4$*. For* $y \neq 0$*,* $z \neq 0$*,* $\mathcal{N} = 1$*, however, only two fields* k_2 *and* k_4 *share common invariants, whereas* k_3 *has different* $\varkappa(k_3)$ *and different* $\mathrm{Gal}(\mathrm{F}_3^2(k_3)/k_3)$*.*

Proof. This follows immediately from Theorem 11, whereas Table 6 with minimal transfer kernel type $\varkappa_0 = (2134)$ only shows the uniform situation, which can become non-uniform by superposition with total transfer kernels, when $\mathcal{N} = 1$. □

Example 2. Prototypes for Graph I.2, i.e., minimal conductors for each scenario in Theorem 11 have been found for each $\mathcal{N} \in \{1,3,4\}$.

Some are **regular**: $c = 8\,001$ with symbol $\{9 \leftarrow 127 \rightarrow 7\}$ and, non-uniformly, once $\mathfrak{G} = \mathfrak{M} = \langle 81,8\rangle$ but twice $\langle 81,10\rangle^2$; $c = 21\,049$ with symbol $\{7 \leftarrow 97 \rightarrow 31\}$ and uniformly three times $\mathfrak{G} = \mathfrak{M} = \langle 243,8\rangle^3$; and $c = 59\,031$ with symbol $\{9 \leftarrow 937 \rightarrow 7\}$ and $\mathfrak{M} = \langle 729,52\rangle^3$.

Others are **singular**: $c = 7\,657$ with symbol $\{13 \leftarrow 31 \rightarrow 19\}$ and $\mathfrak{G} = \mathfrak{M} = \langle 243,8\rangle^3$; and $c = 48\,393$ with symbol $\{9 \leftarrow 19 \rightarrow 283\}$ and $\mathfrak{M} = \langle 2187,301|305\rangle^3$.

The groups of order ≥ 729 with transfer kernel type G.16 have non-metabelian extensions.

In Table 7, we summarize the prototypes of Graph I.2 in the same way as in Table 5, except that two critical exponents y, z in principal factor $A(k_1) = p^x q^y r^z$ and n in $A(k_{qr}) = qr^n$ are sufficient.

Table 7. Prototypes for Graph I.2

No.	c	$q \leftarrow p \rightarrow r$	v	$y,z;n$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	7 657	$13 \leftarrow 31 \rightarrow 19$	4	1, 1; 2	c.21	$\langle 243,8\rangle^3$	$= 2$
2	8 001	$9 \leftarrow 127 \rightarrow 7$	3	1, 2; 1	a.3, a.2	$\langle 81,8\rangle, \langle 81,10\rangle^2$	$= 2$
12	21 049	$7 \leftarrow 97 \rightarrow 31$	3	0, 0; 1	c.21	$\langle 243,8\rangle^3$	$= 2$
27	48 393	$9 \leftarrow 19 \rightarrow 283$	4	0, 0; 2	G.16	$\langle 2187,301|305\rangle^3$	≥ 2
33	59 031	$9 \leftarrow 937 \rightarrow 7$	3	0, 0; 1	G.16	$\langle 729,52\rangle^3$	≥ 2

6.3. **Category II, Graph 1.** Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 1 of Category II with combined cubic residue symbol $[p,q,r]_3 = \{p \rightarrow q \leftarrow r\}$.

Lemma 6. ***(3-class ranks of components for*** II.1.***)*** *Under the normalizing assumption that* q *splits in* $\tilde{k}_{pr}$*, precisely the two components* k_2 *and* k_3 *of the quartet have elementary bicyclic* 3*-class group* $\mathrm{Cl}_3(k_2) \simeq \mathrm{Cl}_3(k_3) \simeq (3,3)$ *of rank* 2*, whereas the other two components have* 3*-class rank* $\varrho_3(k_1) = \varrho_3(k_4) = 3$*. Thus, the* ***tame*** *condition* $9 \mid h_3(B_j) = (U_j : V_j) \in \{9,27\}$*,* $r_j = 2$*, is only satisfied for the bicyclic bicubic fields* B_j *with* $j \in \{2,3,9\}$*.*

Proof. Since $p \rightarrow q$, p splits in k_q. Since $q \leftarrow r$, r splits in k_q, and also splits in $B_9 = k_2k_3k_qk_{pr}$. By the normalizing assumption that q splits in $\tilde{k}_{pr}$, it also splits in $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$ and $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$. The primes p,q,r share the common decomposition type $(e,f,g) = (3,1,3)$ in the bicyclic bicubic field $B_6 = k_1k_4k_q\tilde{k}_{pr}$, which implies that $\varrho_3(k_1) = \varrho_3(k_4) = 3$, according to [2, Prop. 4.4, pp. 43–44]. Finally, only $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $B_9 = k_2k_3k_qk_{pr}$ do not contain k_1, k_4. □

Proposition 7. ***(Sub-doublet with*** 3***-rank two for*** II.1.***)*** *For fixed* $\mu \in \{2,3\}$*, let* $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ *be the prime ideals of* k_μ *over* p,q,r*, that is* $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$*,* $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$*,* $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$*, then the* 3*-class group of* k_μ *is generated by the non-trivial classes* $[\mathfrak{q}],[\mathfrak{r}]$*, that is,*

$$\mathrm{Cl}_3(k_\mu) = \langle [\mathfrak{q}],[\mathfrak{r}]\rangle \simeq (3,3). \tag{6.15}$$

The unramified cyclic cubic relative extensions of k_μ *are among the absolutely bicyclic bicubic fields* B_i*,* $1 \leq i \leq 10$*. The unique* B_μ*,* $\mu \in \{2,3\}$*, which only contains* k_μ*, has norm class group* $N_{B_\mu/k_\mu}(\mathrm{Cl}_3(B_\mu)) = \langle[\mathfrak{q}]\rangle$*, transfer kernel*

$$\ker(T_{B_\mu/k_\mu}) \geq \langle[\mathfrak{r}]\rangle$$

and 3*-class group* $\mathrm{Cl}_3(B_\mu) = \langle[\mathfrak{Q}_1],[\mathfrak{Q}_2],[\mathfrak{Q}_3]\rangle \ge (3,3)$*, generated by the classes of the prime ideals of* B_μ *over* $\mathfrak{q}\mathcal{O}_{B_\mu} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$*. The unique* $B_9 = k_2k_3k_qk_{pr}$*, which contains* k_2 *and* k_3*, has norm class group* $N_{B_9/k_\mu}(\mathrm{Cl}_3(B_9)) = \langle[\mathfrak{r}]\rangle$*,* ***cyclic*** *transfer kernel*

$$\ker(T_{B_9/k_\mu}) = \langle[\mathfrak{q}]\rangle \tag{6.16}$$

of order 3*, and* ***elementary tricyclic*** 3*-class group* $\mathrm{Cl}_3(B_9) = \langle[\mathfrak{R}_1],[\mathfrak{R}_2],[\mathfrak{R}_3]\rangle \simeq (3,3,3)$*, generated by the classes of the prime ideals of* B_9 *over* $\mathfrak{r}\mathcal{O}_{B_9} = \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$*. The remaining two* $B_i > k_\mu$*,* $i \in \{5,7,8,10\}$*, more precisely,* $i \in \{7,8\}$ *for* $\mu = 2$*, and* $i \in \{5,10\}$ *for* $\mu = 3$*, have norm class group* $\langle[\mathfrak{q}\mathfrak{r}]\rangle$ *respectively* $\langle[\mathfrak{q}\mathfrak{r}^2]\rangle$*, and transfer kernel*

$$\ker(T_{B_i/k_\mu}) \ge \langle[\mathfrak{r}]\rangle.$$

In terms of ***decisive principal factors*** $A(k_\nu) = p^{x_\nu}q^{y_\nu}r^{z_\nu}$ *for* $\nu \in \{1,4\}$*, the ranks of principal factor matrices* M_i*,* $i \in \{1,4,5,7,8,10\}$*, of* ***wild*** *bicyclic bicubic fields are*

$$r_1 = r_5 = r_7 = 3 \text{ iff } y_1 \ne 0 \text{ iff } q \mid A(k_1) \text{ and } r_4 = r_8 = r_{10} = 3 \text{ iff } y_4 \ne 0 \text{ iff } q \mid A(k_4). \tag{6.17}$$

Proof. Since $p \to q$, two principal factors are $A(k_{pq}) = A(\tilde{k}_{pq}) = p$; since $q \leftarrow r$, two further principal factors are $A(k_{qr}) = A(\tilde{k}_{qr}) = r$; both by Proposition 3.

Since the *tame* condition $9 \mid h_3(B_j) = (U_j : V_j)$ is satisfied for $j \in \{2,3,9\}$, the rank of the corresponding principal factor matrix M_j must be $r_2 = r_3 = r_9 = 2$. We propose principal factors $A(k_\mu) = p^{x_\mu}q^{y_\mu}r^{z_\mu}$, for all $1 \le \mu \le 4$, and $A(k_{pr}) = pr^\ell$, $A(\tilde{k}_{pr}) = p^2r^\ell$ with $\ell \in \{1,2\}$.

For each bicyclic bicubic field B_j, the rank r_j is calculated with row operations on the associated principal factor matrices M_j:

$$M_2 = \begin{pmatrix} x_2 & y_2 & z_2 \\ 1 & 0 & 0 \\ 2 & 0 & \ell \\ 0 & 0 & 1 \end{pmatrix},\ M_3 = \begin{pmatrix} x_3 & y_3 & z_3 \\ 1 & 0 & 0 \\ 2 & 0 & \ell \\ 0 & 0 & 1 \end{pmatrix},\ M_9 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ 0 & 1 & 0 \\ 1 & 0 & \ell \end{pmatrix}.$$

For $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, M_2 leads to the decisive pivot element y_2 in the middle column, and similarly, for $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, M_3 leads to y_3. So, $r_2 = r_3 = 2$ enforces $y_2 = y_3 = 0$, i.e., $q \nmid A(k_2)$, $q \nmid A(k_3)$. However, for $B_9 = k_2k_3k_qk_{pr}$, M_9 leads to $z_2 - \ell x_2$ and $z_3 - \ell x_3$. So, $r_9 = 2$ enforces $z_2 \equiv \ell x_2$ and $z_3 \equiv \ell x_3$ modulo 3, i.e., $A(k_2) = A(k_3) = pr^\ell$.

For every *wild* bicyclic bicubic field B_j, $j \in \{1,4,5,6,7,8,10\}$, the rank r_j is calculated by row operations on the matrices M_j, using $A(k_2) = A(k_3) = pr^\ell$:

$$M_1 = \begin{pmatrix} x_1 & y_1 & z_1 \\ 1 & 0 & 0 \\ 1 & 0 & \ell \\ 0 & 0 & 1 \end{pmatrix},\ M_5 = \begin{pmatrix} x_1 & y_1 & z_1 \\ 1 & 0 & \ell \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_7 = \begin{pmatrix} x_1 & y_1 & z_1 \\ 1 & 0 & \ell \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $B_1 = k_1k_{pq}k_{pr}k_{qr}$, M_1 leads to the decisive pivot element y_1 in the middle column, similarly, for $B_5 = k_1k_3k_p\tilde{k}_{qr}$, M_5 leads to y_1, and similarly, for $B_7 = k_1k_2k_r\tilde{k}_{pq}$, M_7 leads to y_1. So, $r_1 = r_5 = r_7 = 3$ iff $y_1 \ne 0$ iff $q \mid A(k_1)$. Next we consider:

$$M_4 = \begin{pmatrix} x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 1 & 0 & \ell \\ 0 & 0 & 1 \end{pmatrix},\ M_8 = \begin{pmatrix} 1 & 0 & \ell \\ x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_{10} = \begin{pmatrix} 1 & 0 & \ell \\ x_4 & y_4 & z_4 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, M_4 leads to the decisive pivot element y_4 in the middle column, similarly, for $B_8 = k_2k_4k_pk_{qr}$, M_8 leads to y_4, and similarly, for $B_{10} = k_3k_4k_rk_{pq}$, M_{10} leads to y_4. So, $r_4 = r_8 = r_{10} = 3$ iff $y_4 \ne 0$ iff $q \mid A(k_4)$.

By Lemma 2, the minimal subfield unit index $(U_j : V_j) = 3$ for $r_j = 3$ corresponds to the maximal unit norm index $(U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3$, associated to a *total* transfer kernel $\#\ker(T_{B_j/k_\mu}) = 9$.

Since q splits in $\tilde{k}_{pr}$, it also splits in $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $\mathfrak{q}\mathcal{O}_{B_\mu} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$.

Since r splits in k_q, it also splits in $B_9 = k_2k_3k_qk_{pr}$, $\mathfrak{r}\mathcal{O}_{B_9} = \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$.

.

Since $\mathfrak{r}$ is principal in k_r, k_{qr}, $\tilde{k}_{qr}$, $[\mathfrak{r}]$ capitulates in $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, $B_8 = k_2 k_4 k_p k_{qr}$, $B_{10} = k_3 k_4 k_r k_{pq}$; since $\mathfrak{q}$ is principal in k_q, $[\mathfrak{q}]$ capitulates in $B_9 = k_2 k_3 k_q k_{pr}$ (Proposition 2). This gives a *transposition*, either $(2,9)$ or $(3,9)$.

The minimal unit norm index $(U(k_\mu) : N_{B_9/k_\mu}(U_9)) = 1$, associated to the partial transfer kernel $\ker(T_{B_9/k_\mu}) = \langle[\mathfrak{q}]\rangle$, corresponds to the maximal subfield unit index $h_3(B_9) = (U_9 : V_9) = 27$, giving rise to the *elementary tricyclic* type invariants $\mathrm{Cl}_3(B_9) = \langle[\mathfrak{R}_1], [\mathfrak{R}_2], [\mathfrak{R}_3]\rangle \simeq (3,3,3)$. □

Using Corollary 2, Proposition 7 and parts of its proof are now summarized in Table 8 with transposition in **bold** font.

TABLE 8. Norm class groups and minimal transfer kernels for Graph II.1

Base	k_2				k_3			
Ext	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}
NCG	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$
TK	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$
$\varkappa$	**4**	4	4	**1**	**3**	3	**1**	3

Theorem 12. ***(Second* 3*-class group for* II.1*.)*** *Let $(k_1, \ldots, k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph* 1 *of Category* II *with combined cubic residue symbol $[p,q,r]_3 = \{p \to q \leftarrow r\}$. Without loss of generality, suppose that q splits in $\tilde{k}_{pr}$, and thus $\mathrm{Cl}_3(k_2) \simeq \mathrm{Cl}_3(k_3) \simeq (3,3)$, and $\varrho_3(k_1) = \varrho_3(k_4) = 3$.*

Then the ***minimal transfer kernel type*** *(mTKT) of k_μ, $2 \le \mu \le 3$, is $\varkappa_0 = (2111)$, type* H.4*, and the other possible capitulation types in ascending order $\varkappa_0 < \varkappa' < \varkappa'' < \varkappa'''$ are $\varkappa' = (2110)$, type* d.19*, $\varkappa'' = (2100)$, type* b.10*, and $\varkappa''' = (2000)$, type* a.3**.*

To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $2 \le \mu \le 3$, let the ***decisive principal factors*** *of k_ν, $\nu \in \{1,4\}$, be $A(k_\nu) = p^{x_\nu} q^{y_\nu} r^{z_\nu}$, and additionally assume the* ***regular situation*** *where both $\mathrm{Cl}_3(k_1) \simeq \mathrm{Cl}_3(k_4) \simeq (3,3,3)$ are elementary tricyclic. Then*

$$(6.18) \qquad \mathfrak{M} \simeq \begin{cases} \langle 81,7\rangle, \ \alpha = [111,11,11,11], \ \varkappa = (2000) & \text{if } y_1 \ne 0, \ y_4 \ne 0, \ \mathcal{N} = 1, \\ \langle 729,34..39\rangle, \ \alpha = [111,111,21,21], \ \varkappa = (2100) & \text{if } y_1 = y_4 = 0, \ \mathcal{N} = 2, \\ \langle 729,41\rangle, \ \alpha = [111,111,22,21], \ \varkappa = (2110) & \text{if } y_1 = y_4 = 0, \ \mathcal{N} = 3, \\ \langle 6561,714..719|738..743\rangle \text{ or} & \\ \langle 2187,65|67\rangle, \ \alpha = [111,111,22,22], \ \varkappa = (2111) & \text{if } y_1 = y_4 = 0, \ \mathcal{N} = 4, \end{cases}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j, \ I_j = 27\}$. Only in the leading row, the 3-class field tower has warranted group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$, with length $\ell_3(k_\mu) = 2$. Otherwise, although the relation rank $d_2(\mathfrak{M}) \le 4$ is always admissible, tower length $\ell_3(k_\mu) \ge 3$ cannot be excluded.

Proof. We give the proof for k_3 with unramified cyclic cubic extensions B_3, B_5, B_9, B_{10}, (The proof for k_2 with unramified cyclic cubic extensions B_2, B_7, B_8, B_9 is similar.) We know that the tame ranks are $r_2 = r_3 = r_9 = 2$, and thus $I_2, I_3, I_9 \in \{9, 27\}$, in particular, $I_9 = 27$, whence certainly $\mathcal{N} \ge 1$. Further, the wild ranks are $r_1 = r_5 = r_7 = 3$ iff $y_1 \ne 0$, and $r_4 = r_8 = r_{10} = 3$ iff $y_4 \ne 0$.

In the **regular situation** where the 3-class groups of k_1 and k_4 are elementary tricyclic, tight bounds arise for the abelian quotient invariants α of the group $\mathfrak{M}$:

The first scenario, $y_1 \ne 0$, $y_4 \ne 0$, is equivalent to $\mathcal{N} = 1$, $h_3(B_5) = h_3(B_7) = \frac{1}{3} h_3(k_1) = 9$, $h_3(B_8) = h_3(B_{10}) = \frac{1}{3} h_3(k_4) = 9$, $h_3(B_2) = I_2 = h_3(B_3) = I_3 = 9$, $h_3(B_9) = I_9 = 27$, that is $\alpha = [111,11,11,11]$ and consequently $\varkappa = (2000)$, since $\langle 81,7\rangle$ is unique with this α.

The other three scenarios share $y_1 = y_4 = 0$, and an explicit transposition between B_3 and B_9, giving rise to $\varkappa = (21**)$, and common $h_3(B_3) = I_3 = 27$, $\alpha = [111,111,*,*]$.

The second scenario with $\mathcal{N} = 2$ is supplemented by $h_3(B_5) = h_3(k_1) = 27$, $h_3(B_{10}) = h_3(k_4) = 27$, giving rise to $\alpha = [111,111,21,21]$, $\varkappa = (2100)$, characteristic for $\langle 729,34..39\rangle$ (Cor. 4).

The third scenario with $\mathcal{N} = 3$ is supplemented by $h_3(B_5) = 3h_3(k_1) = 81$, $h_3(B_{10}) = h_3(k_4) = 27$, giving rise to $\alpha = [111, 111, 22, 21]$, $\varkappa = (2110)$, characteristic for $\langle 729, 41\rangle$.

The fourth scenario with $\mathcal{N} = 4$ is supplemented by $h_3(B_5) = 3h_3(k_1) = 81$, $h_3(B_{10}) = 3h_3(k_4) = 81$, giving rise to $\alpha = [111, 111, 22, 22]$, $\varkappa = (2111)$, characteristic for either $\langle 2187, 65|67\rangle$ or $\langle 6561, 714..719|738..743\rangle$ with coclass cc $= 3$. If $d_2(\mathfrak{M}) = 5$, then tower length must be $\ell_3(k_\mu) \geq 3$. For this minimal capitulation type H.4, $\varkappa = (2111)$, all transfer kernels are cyclic of order 3, and the minimal unit norm indices correspond to maximal subfield unit indices. □

Corollary 7. ***(Uniformity of the sub-doublet for*** II.1.***)*** *The components of the sub-doublet with* 3*-rank two share a common capitulation type* $\varkappa(k_2) \sim \varkappa(k_3)$*, common abelian type invariants* $\alpha(k_2) \sim \alpha(k_3)$*, and a common second* 3*-class group* $\mathrm{Gal}(\mathrm{F}_3^2(k_2)/k_2) \simeq \mathrm{Gal}(\mathrm{F}_3^2(k_3)/k_3)$*.*

Proof. This is an immediate consequence of Theorem 12 and Table 8. □

Example 3. Prototypes for Graph II.1, i.e., minimal conductors for each scenario in Theorem 12 have been detected for all $N \in \{1, 2, 3, 4\}$.

There are **regular** cases: $c = 3\,913$ with symbol $\{13 \to 7 \leftarrow 43\}$ and $\mathfrak{G} = \mathfrak{M} = \langle 81, 7\rangle$; $c = 22\,581$ with symbol $\{9 \to 193 \leftarrow 13\}$ and $\mathfrak{M} = \langle 729, 41\rangle$; $c = 25\,929$ with symbol $\{9 \to 67 \leftarrow 43\}$ and $\mathfrak{M} = \langle 729, 34..36\rangle$ (Corollary 4); $c = 74\,043$ with symbol $\{19 \to 9 \leftarrow 433\}$ and either $\mathfrak{M} = \langle 2187, 65|67\rangle$ with $d_2(\mathfrak{M}) = 5$ or $\mathfrak{M} = \langle 6561, 714..719|738..743\rangle$ with $d_2(\mathfrak{M}) = 4$; and $c = 82\,327$ with symbol $\{7 \to 19 \leftarrow 619\}$ and $\mathfrak{M} = \langle 729, 37..39\rangle$ (Corollary 4).

We also have **singular** cases: $c = 30\,457$ with symbol $\{7 \to 19 \leftarrow 229\}$ and $\mathfrak{M} = \langle 729, 37..39\rangle$ (Corollary 4); $c = 34\,029$ with symbol $\{19 \to 9 \leftarrow 199\}$ and $\mathfrak{M} = \langle 2187, 248|249\rangle$; $c = 41\,839$ with symbol $\{43 \to 7 \leftarrow 139\}$ and $\mathfrak{M} = \langle 6561, 693..698\rangle$.

Finally, there is the **super-singular** $c = 83\,817$ with symbol $\{9 \to 67 \leftarrow 139\}$ and $\mathfrak{M} = \langle 6561, 693..698\rangle$.

With exception of $\langle 81, 7\rangle$, all groups have non-metabelian descendants, respectively extensions.

In Table 9, we summarize the prototypes of Graph II.1 in the same manner as in Table 5, except that regularity, resp. (super-)singularity, is expressed by 3-valuations $v_\nu = v_3(\#\mathrm{Cl}(k_\nu))$ of class numbers of critical fields k_ν, $\nu = 1, 4$, and critical exponents are y_ν in principal factors $A(k_\nu) = p^{x_\nu} q^{y_\nu} r^{z_\nu}$, $\nu = 1, 4$.

Table 9. Prototypes for Graph II.1

No.	c	$p \to q \leftarrow r$	$v1, v4$	y_1, y_4	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	3 913	$13 \to 7 \leftarrow 43$	3, 3	1, 1	a.3*	$\langle 81, 7\rangle^2$	$= 2$
9	22 581	$9 \to 193 \leftarrow 13$	3, 3	0, 0	d.19	$\langle 729, 41\rangle^2$	≥ 2
11	25 929	$9 \to 67 \leftarrow 43$	3, 3	0, 0	b.10	$\langle 729, 34..36\rangle^2$	≥ 2
15	30 457	$7 \to 19 \leftarrow 229$	4, 4	1, 1	b.10	$\langle 729, 37..39\rangle^2$	≥ 2
18	34 029	$19 \to 9 \leftarrow 199$	4, 4	1, 0	d.19	$\langle 2187, 248\vert 249\rangle^2$	≥ 2
23	41 839	$43 \to 7 \leftarrow 139$	4, 4	0, 0	b.10	$\langle 6561, 693..698\rangle^2$	≥ 2
35	74 043	$19 \to 9 \leftarrow 433$	3, 3	0, 0	H.4	$\langle 2187, 65\vert 67\rangle^2$	≥ 3
					or	$\langle 6561, 714..719\vert 738..743\rangle$	≥ 2
39	82 327	$7 \to 19 \leftarrow 619$	3, 3	0, 0	b.10	$\langle 729, 37..39\rangle^2$	≥ 2
42	83 817	$9 \to 67 \leftarrow 139$	5, 4	0, 1	b.10	$\langle 6561, 693..698\rangle^2$	≥ 2

6.4. **Category II, Graph 2.** Let $(k_1, \ldots, k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 2 of Category II with combined cubic residue symbol $[p, q, r]_3 = \{q \leftarrow p \to r \leftarrow q\}$.

Lemma 7. ***(3-class ranks of components for*** II.2.***)*** *Under the normalizing assumption that* r *splits in* k_{pq}*, precisely the two components* k_1 *and* k_2 *of the quartet have elementary bicyclic* 3*-class group* $\mathrm{Cl}_3(k_1) \simeq \mathrm{Cl}_3(k_2) \simeq (3, 3)$ *of rank* 2*, whereas the other two components have* 3*-class rank* $\varrho_3(k_3) = \varrho_3(k_4) = 3$*. Thus, the* ***tame*** *condition* $9 \mid h_3(B_j) = (U_j : V_j) \in \{9, 27\}$*,* $r_j = 2$*, is only satisfied for the bicyclic bicubic fields* B_j *with* $j \in \{1, 2, 7\}$*.*

Proof. Since $p \to r$, p splits in k_r. Since $r \leftarrow q$, q splits in k_r, and also splits in $B_7 = k_1k_2k_r\tilde{k}_{pq}$. By the normalizing assumption that r splits in k_{pq}, it also splits in $B_1 = k_1k_{pq}k_{pr}k_{qr}$ and $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$. The primes p, q, r share the common decomposition type $(e, f, g) = (3, 1, 3)$ in the bicyclic bicubic field $B_{10} = k_3k_4k_rk_{pq}$, which implies that $\varrho_3(k_3) = \varrho_3(k_4) = 3$, according to [2, Prop. 4.4, pp. 43–44]. Finally, only $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_7 = k_1k_2k_r\tilde{k}_{pq}$ do not contain k_3, k_4. □

Proposition 8. ***(Sub-doublet with*** **3*-rank two for*** II.2.***)*** *For fixed $\mu \in \{1, 2\}$, let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be the prime ideals of k_μ over p, q, r, that is $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$, then the* ***principal factor*** *of k_μ is $A(k_\mu) = p$, with $[\mathfrak{p}] = 1$, and the 3-class group of k_μ is*

$$\mathrm{Cl}_3(k_\mu) = \langle [\mathfrak{q}], [\mathfrak{r}] \rangle \simeq (3, 3). \tag{6.19}$$

The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic fields B_i, $1 \le i \le 10$. The unique B_μ, $\mu \in \{1, 2\}$, which only contains k_μ, has norm class group $N_{B_\mu/k_\mu}(\mathrm{Cl}_3(B_\mu)) = \langle [\mathfrak{r}] \rangle$, transfer kernel

$$\ker(T_{B_\mu/k_\mu}) \ge \langle [\mathfrak{q}] \rangle$$

and 3-class group $\mathrm{Cl}_3(B_\mu) = \langle [\mathfrak{R}_1], [\mathfrak{R}_2], [\mathfrak{R}_3] \rangle \ge (3, 3)$, generated by the classes of the prime ideals of B_μ over $\mathfrak{r}\mathcal{O}_{B_\mu} = \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$. The unique $B_7 = k_1k_2k_r\tilde{k}_{pq}$, which contains k_1 and k_2, has norm class group $N_{B_7/k_\mu}(\mathrm{Cl}_3(B_7)) = \langle [\mathfrak{q}] \rangle$, ***cyclic*** *transfer kernel*

$$\ker(T_{B_7/k_\mu}) = \langle [\mathfrak{r}] \rangle \tag{6.20}$$

of order 3, and ***elementary tricyclic*** *3-class group $\mathrm{Cl}_3(B_7) = \langle [\mathfrak{Q}_1], [\mathfrak{Q}_2], [\mathfrak{Q}_3] \rangle \simeq (3, 3, 3)$, generated by the classes of the prime ideals of B_7 over $\mathfrak{q}\mathcal{O}_{B_7} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$. The remaining two $B_i > k_\mu$, $i \in \{5, 6, 8, 9\}$, more precisely, $i \in \{5, 6\}$ for $\mu = 1$, and $i \in \{8, 9\}$ for $\mu = 2$, have norm class group $\langle [\mathfrak{q}\mathfrak{r}] \rangle$ respectively $\langle [\mathfrak{q}\mathfrak{r}^2] \rangle$, and transfer kernel*

$$\ker(T_{B_i/k_\mu}) \ge \langle [\mathfrak{q}] \rangle.$$

In terms of ***decisive principal factors*** *$A(k_\nu) = p^{x_\nu}q^{y_\nu}r^{z_\nu}$ for $\nu \in \{3, 4\}$, the ranks of principal factor matrices M_i, $i \in \{3, 4, 5, 6, 8, 9\}$, of* ***wild*** *bicyclic bicubic fields are*

$$r_3 = r_5 = r_9 = 3 \text{ iff } z_3 \ne 0 \text{ iff } r \mid A(k_3) \text{ and } r_4 = r_6 = r_8 = 3 \text{ iff } z_4 \ne 0 \text{ iff } r \mid A(k_4). \tag{6.21}$$

Proof. Since $q \leftarrow p$, two principal factors are $A(k_{pq}) = A(\tilde{k}_{pq}) = p$; since $p \to r$, two further principal factors are $A(k_{pr}) = A(\tilde{k}_{pr}) = p$; since $r \leftarrow q$, two further principal factors are $A(k_{qr}) = A(\tilde{k}_{qr}) = q$; each by Proposition 3. Since $q \leftarrow p \to r$ is universally repelling, we have $A(k_1) = A(k_2) = p$, by Proposition 4.

Thus $\mathfrak{p} = \alpha\mathcal{O}_{k_\mu}$ is a principal ideal with trivial class $[\mathfrak{p}] = 1$, for $\mu \in \{1, 2\}$, whereas the classes $[\mathfrak{q}], [\mathfrak{r}]$ are non-trivial. We propose $A(k_\nu) = p^{x_\nu}q^{y_\nu}r^{z_\nu}$ for $\nu \in \{3, 4\}$.

Since the *tame* condition $9 \mid h_3(B_j) = (U_j : V_j)$ is satisfied for $j \in \{1, 2, 7\}$, the rank of the corresponding principal factor matrix M_j must be $r_1 = r_2 = r_7 = 2$. Due to the principal factors $A(k_1) = A(k_2) = p$, this also follows by direct calculation, but has no further consequences. For every *wild* bicyclic bicubic field B_j, $j \in \{3, 4, 5, 6, 8, 9, 10\}$, the rank r_j is calculated with row operations on the associated principal factor matrices M_j:

$$M_3 = \begin{pmatrix} x_3 & y_3 & z_3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, M_5 = \begin{pmatrix} 1 & 0 & 0 \\ x_3 & y_3 & z_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, M_9 = \begin{pmatrix} 1 & 0 & 0 \\ x_3 & y_3 & z_3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, M_3 leads to the decisive pivot element z_3 in the last column, similarly, for $B_5 = k_1k_3k_p\tilde{k}_{qr}$, M_5 leads to z_3, and similarly, for $B_9 = k_2k_3k_qk_{pr}$, M_9 leads to z_3. So, $r_3 = r_5 = r_9 = 3$ iff $z_3 \ne 0$ iff $r \mid A(k_3)$. Next we consider:

$$M_4 = \begin{pmatrix} x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 1 & 0 & 0 \\ x_4 & y_4 & z_4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, M_8 = \begin{pmatrix} 1 & 0 & 0 \\ x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, M_4 leads to the decisive pivot element z_4 in the last column, similarly, for $B_6 = k_1k_4k_q\tilde{k}_{pr}$, M_6 leads to z_4, and similarly, for $B_8 = k_2k_4k_pk_{qr}$, M_8 leads to z_4. So, $r_4 = r_6 = r_8 = 3$ iff $z_4 \neq 0$ iff $r \mid A(k_4)$.

By Lemma 2, the minimal subfield unit index $(U_j : V_j) = 3$ for $r_j = 3$ corresponds to the maximal unit norm index $(U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 3$, associated to a *total* transfer kernel $\#\ker(T_{B_j/k_\mu}) = 9$.

As mentioned in the proof of Lemma 7 already:
Since r splits in k_{pq}, it also splits in $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, i.e., $\mathfrak{r}\mathcal{O}_{B_\mu} = \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$.
Since q splits in k_r, it also splits in $B_7 = k_1k_2k_r\tilde{k}_{pq}$, i.e., $\mathfrak{q}\mathcal{O}_{B_7} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$.

Since $\mathfrak{q}$ is principal in k_q, k_{qr}, $\tilde{k}_{qr}$, $[\mathfrak{q}]$ capitulates in $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_5 = k_1k_3k_p\tilde{k}_{qr}$, $B_6 = k_1k_4k_q\tilde{k}_{pr}$, $B_8 = k_2k_4k_pk_{qr}$, $B_9 = k_2k_3k_qk_{pr}$; since $\mathfrak{r}$ is principal in k_r, $[\mathfrak{r}]$ capitulates in $B_7 = k_1k_2k_r\tilde{k}_{pq}$ (Proposition 2). This gives a *transposition*, either $(1,7)$ or $(2,7)$.

The minimal unit norm index $(U(k_\mu) : N_{B_7/k_\mu}(U_7)) = 1$, associated to the *partial* transfer kernel $\ker(T_{B_7/k_\mu}) = \langle[\mathfrak{r}]\rangle$, corresponds to the maximal subfield unit index $h_3(B_7) = (U_7 : V_7) = 27$, giving rise to the *elementary tricyclic* type invariants $\mathrm{Cl}_3(B_7) = \langle[\mathfrak{Q}_1], [\mathfrak{Q}_2], [\mathfrak{Q}_3]\rangle \simeq (3,3,3)$. □

In terms of capitulation targets in Corollary 2, Proposition 8 and parts of its proof are now summarized in Table 10 with transposition in **bold** font.

Table 10. Norm class groups and minimal transfer kernels for Graph II.2

Base	k_1				k_2			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9
NCG	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$
TK	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}$
$\varkappa$	**4**	4	4	**1**	**2**	**1**	2	2

Theorem 13. ***(Second*** **3*-class group for*** II.2.***)*** *Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 2 of Category* II *with combined cubic residue symbol $[p,q,r]_3 = \{q \leftarrow p \rightarrow r \leftarrow q\}$. Without loss of generality, suppose that r splits in k_{pq}, and thus $\mathrm{Cl}_3(k_1) \simeq \mathrm{Cl}_3(k_2) \simeq (3,3)$, and $\varrho_3(k_3) = \varrho_3(k_4) = 3$.*

*Then the **minimal transfer kernel type** (mTKT) of k_μ, $1 \le \mu \le 2$, is $\varkappa_0 = (2111)$, type* H.4*, and the other possible capitulation types in ascending order $\varkappa_0 < \varkappa' < \varkappa'' < \varkappa'''$ are $\varkappa' = (2110)$, type* d.19*, $\varkappa'' = (2100)$, type* b.10*, and $\varkappa''' = (2000)$, type* a.3**.*

*To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $1 \le \mu \le 2$, let the decisive **principal factors** of k_ν, $3 \le \nu \le 4$, be $A(k_\nu) = p^{x_\nu}q^{y_\nu}r^{z_\nu}$, and additionally assume the **regular** situation where both $\mathrm{Cl}_3(k_3) \simeq \mathrm{Cl}_3(k_4) \simeq (3,3,3)$ are elementary tricyclic. Then*

$$(6.22) \qquad \mathfrak{M} \simeq \begin{cases} \langle 81,7\rangle,\ \alpha = [111,11,11,11],\ \varkappa = (2000) & \text{if } z_3 \neq 0,\ z_4 \neq 0,\ \mathcal{N} = 1, \\ \langle 729,34..39\rangle,\ \alpha = [111,111,21,21],\ \varkappa = (2100) & \text{if } z_3 = z_4 = 0,\ \mathcal{N} = 2, \\ \langle 729,41\rangle,\ \alpha = [111,111,22,21],\ \varkappa = (2110) & \text{if } z_3 = z_4 = 0,\ \mathcal{N} = 3, \\ \langle 6561,714..719|738..743\rangle \text{ or} & \\ \langle 2187,65|67\rangle,\ \alpha = [111,111,22,22],\ \varkappa = (2111) & \text{if } z_3 = z_4 = 0,\ \mathcal{N} = 4, \end{cases}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j,\ I_j = 27\}$. Only in the leading row, the 3-class field tower has warranted group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$, with length $\ell_3(k_\mu) = 2$. Otherwise, even if the relation rank $d_2(\mathfrak{M}) \le 4$ is admissible, tower length $\ell_3(k_\mu) \ge 3$ cannot be excluded.

Proof. We give the proof for k_1 with unramified cyclic cubic extensions B_1, B_5, B_6, B_7, (The proof for k_2 with unramified cyclic cubic extensions B_2, B_7, B_8, B_9 is similar.) We know that the tame ranks are $r_1 = r_2 = r_7 = 2$, and thus $I_1, I_2, I_7 \in \{9, 27\}$, in particular, $I_7 = 27$, whence certainly $\mathcal{N} \ge 1$. Further, the wild ranks are $r_4 = r_6 = r_8 = 3$ iff $z_4 \neq 0$, and $r_3 = r_5 = r_9 = 3$ iff $z_3 \neq 0$.

In the **regular situation** where the 3-class groups of k_3 and k_4 are elementary tricyclic, tight bounds arise for the abelian quotient invariants α of the group $\mathfrak{M}$:

The first scenario, $z_3 \neq 0$, $z_4 \neq 0$, is equivalent to $\mathcal{N} = 1$, $h_3(B_5) = h_3(B_9) = \frac{1}{3}h_3(k_3) = 9$, $h_3(B_6) = h_3(B_8) = \frac{1}{3}h_3(k_4) = 9$, $h_3(B_1) = I_1 = h_3(B_2) = I_2 = 9$, $h_3(B_7) = I_7 = 27$, that is $\alpha = [111, 11, 11, 11]$ and consequently $\varkappa = (2000)$, since $\langle 81, 7\rangle$ is unique with this α.

The other three scenarios share $z_3 = z_4 = 0$, and an explicit transposition between B_1 and B_7, giving rise to $\varkappa = (21{*}{*})$, and common $h_3(B_1) = I_1 = 27$, $\alpha = [111, 111, *, *]$.

The second scenario with $\mathcal{N} = 2$ is supplemented by $h_3(B_5) = h_3(k_3) = 27$, $h_3(B_6) = h_3(k_4) = 27$, giving rise to $\alpha = [111, 111, 21, 21]$, $\varkappa = (2100)$, characteristic for $\langle 729, 34..39\rangle$ (Cor. 4).

The third scenario with $\mathcal{N} = 3$ is supplemented by $h_3(B_5) = 3h_3(k_3) = 81$, $h_3(B_6) = h_3(k_4) = 27$, giving rise to $\alpha = [111, 111, 22, 21]$, $\varkappa = (2110)$, characteristic for $\langle 729, 41\rangle$.

The fourth scenario with $\mathcal{N} = 4$ is supplemented by $h_3(B_5) = 3h_3(k_3) = 81$, $h_3(B_6) = 3h_3(k_4) = 81$, giving rise to $\alpha = [111, 111, 22, 22]$, $\varkappa = (2111)$, characteristic for $\langle 2187, 65|67\rangle$ or $\langle 6561, 714..719|738..743\rangle$ with coclass cc $= 3$. If $d_2(\mathfrak{M}) = 5$, then $\ell_3(k_\mu) \geq 3$. □

Corollary 8. ***(Uniformity of the sub-doublet for*** II.2.***)*** *The components of the sub-doublet with* 3*-rank two share a common capitulation type* $\varkappa(k_1) \sim \varkappa(k_2)$*, common abelian type invariants* $\alpha(k_1) \sim \alpha(k_2)$*, and a common second* 3*-class group* $\mathrm{Gal}(\mathrm{F}_3^2(k_1)/k_1) \simeq \mathrm{Gal}(\mathrm{F}_3^2(k_2)/k_2)$.

Proof. This follows immediately from Theorem 13 and Table 10. □

Example 4. Prototypes for Graph II.2, i.e., minimal conductors for each scenario in Theorem 13 have been found for each $\mathcal{N} \in \{1, 2, 3, 4\}$.

There are **regular** cases: $c = 6\,327$ with symbol $\{19 \to 9 \leftarrow 37 \to 19\}$ and $\mathfrak{G} = \mathfrak{M} = \langle 81, 7\rangle$; $c = 41\,629$ with symbol $\{19 \to 313 \leftarrow 7 \to 19\}$ and $\mathfrak{M} = \langle 729, 34..36\rangle$ (Corollary 4); $c = 56\,547$ with symbol $\{61 \to 103 \leftarrow 9 \to 61\}$ and $\mathfrak{M} = \langle 729, 41\rangle$; and, with **considerable statistic delay**, $c = 389\,329$ with ordinal number 207, symbol $\{19 \to 661 \leftarrow 31 \to 19\}$ and either $\mathfrak{M} = \langle 2187, 65|67\rangle$ with $d_2(\mathfrak{M}) = 5$ or $\mathfrak{M} = \langle 6561, 714..719|738..743\rangle$ with $d_2(\mathfrak{M}) = 4$.

Further, there are **singular** cases: $c = 27\,873$ with symbol $\{19 \to 9 \leftarrow 163 \to 19\}$ and $\mathfrak{M} = \langle 729, 34..36\rangle$ (Corollary 4); $c = 29\,197$ with symbol $\{43 \to 7 \leftarrow 97 \to 43\}$ and $\mathfrak{M} = \langle 2187, 253\rangle$; and $c = 63\,511$ with symbol $\{43 \to 7 \leftarrow 211 \to 43\}$ and $\mathfrak{M} = \langle 729, 37..39\rangle$ (Corollary 4).

Finally, there is the **super-singular** $c = 66\,157$ with symbol $\{13 \to 7 \leftarrow 727 \to 13\}$ and $\mathfrak{M} = \langle 6561, 1989\rangle$.

With exception of $\langle 81, 7\rangle$, all groups have non-metabelian descendants, respectively extensions.

In Table 11, we summarize the prototypes of Graph II.2 in the same way as in Table 5, except that regularity, resp. (super-)singularity, is expressed by 3-valuations $v_\nu = v_3(\#\mathrm{Cl}(k_\nu))$ of class numbers of critical fields k_ν, $\nu = 3, 4$, and critical exponents are z_ν in principal factors $A(k_\nu) = p^{x_\nu} q^{y_\nu} r^{z_\nu}$, $\nu = 3, 4$.

Table 11. Prototypes for Graph II.2

No.	c	$q \to r \leftarrow p \to q$	$v3, v4$	z_3, z_4	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	6 327	$19 \to 9 \leftarrow 37 \to 19$	3, 3	1, 1	a.3*	$\langle 81, 7\rangle^2$	$= 2$
8	27 873	$19 \to 9 \leftarrow 163 \to 19$	4, 4	1, 1	b.10	$\langle 729, 34..36\rangle^2$	≥ 2
10	29 197	$43 \to 7 \leftarrow 97 \to 43$	4, 4	0, 1	b.10	$\langle 2187, 253\rangle^2$	≥ 3
14	41 629	$19 \to 313 \leftarrow 7 \to 19$	3, 3	0, 0	b.10	$\langle 729, 34..36\rangle^2$	≥ 2
23	56 547	$61 \to 103 \leftarrow 9 \to 61$	3, 3	0, 0	d.19	$\langle 729, 41\rangle^2$	≥ 2
28	63 511	$43 \to 7 \leftarrow 211 \to 43$	4, 4	1, 1	b.10	$\langle 729, 37..39\rangle^2$	≥ 2
31	66 157	$13 \to 7 \leftarrow 727 \to 13$	5, 4	1, 0	d.19	$\langle 6561, 1989\rangle^2$	≥ 2
207	389 329	$19 \to 661 \leftarrow 31 \to 19$	3, 3	0, 0	H.4	$\langle 2187, 65\vert 67\rangle^2$	≥ 3
					or	$\langle 6561, 714..719\vert 738..743\rangle$	≥ 2

7. Category III, Graphs 1–4

Let the combined cubic residue symbol $[p, q, r]_3$ of three prime(power)s dividing the conductor $c = pqr$ be either $\{p, q, r; \delta \not\equiv 0\,(\mathrm{mod}\,3)\}$ or $\{p \to q; r\}$ or $\{p \to q \to r\}$ or $\{p \to q \to r \to p\}$. The symbol does not contain any mutual cubic residues. We verify a conjecture in [20, Cnj. 1, p. 48].

Theorem 14. *A cyclic cubic field* k *with conductor* $c = pqr$*, divisible by exactly three prime(power)s* p, q, r*, has an abelian* 3*-class field tower with group* $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k)/k) \simeq \langle 9, 2\rangle$*,* $\alpha = [1, 1, 1, 1]$*,* $\varkappa = (0000)$*, if and only if the primes* p, q, r *form one of the four Graphs* 1–4 *of Category* III*.*

Proof. Ayadi [2, Thm. 4.1, pp. 76–77] has proved the sufficiency of the condition. He does not claim explicitly that the condition is also necessary. However, his techniques are able to prove both directions. Recall that for both Graphs 1–2 of the Categories I and II, there is at least one component of the quartet $(k_\mu)_{\mu=1}^4$ with 3-class rank $\varrho(k_\mu) = 3$, and that for all Graphs 5–9 of Category III, two primes $p \leftrightarrow q$ are mutual cubic residues, according to Theorem 2. In contrast, precisely for the Graphs 1–4 of Category III, the symbol $[p, q, r]_3$ does not contain any mutual cubic residues, and all four components have 3-class rank $\varrho(k_\mu) = 2$ and elementary bicyclic 3-class group $\mathrm{Cl}_3(k_\mu) \simeq (3, 3)$, whence these are the only cases where all bicyclic bicubic fields B_j, $1 \le j \le 10$, satisfy the *tame* relation $h_3(B_j) = (U_j : V_j) = 3$ with matrix rank $r_j = 3$. This is equivalent with abelian type invariants $\alpha(k_\mu) = [1, 1, 1, 1]$ for all $1 \le \mu \le 4$. By the strategy of pattern recognition [19], this enforces the abelian group $\mathfrak{G} \simeq \langle 9, 2\rangle \simeq (3, 3)$, which is the unique 3-group G with $G/G' \simeq (3, 3)$ and abelian quotient invariants $\alpha(G) = [1, 1, 1, 1]$. □

For the prototypes of Graphs $1, \ldots, 4$ of Category III see [20, Tbl. 6.3, p. 48]. Systematic tables have been presented at `http://www.algebra.at/ResearchFrontier2013ThreeByThree.htm` in sections §§ 2.1–2.2.

8. Category III, Graphs 5–9

In this section, the combined cubic residue symbol $[p, q, r]_3$ of three prime(power)s dividing the conductor $c = pqr$ contains a unique pair $p \leftrightarrow q$ of mutual cubic residues.

Consequently, the **decisive principal factors**

$$A(k_{pq}) = p^m q^n, \quad A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}} \tag{8.1}$$

must be assumed with variable exponents in $\{0, 1, 2\}$, such that $(m, n) \ne (0, 0)$ and $(\tilde{m}, \tilde{n}) \ne (0, 0)$. Concerning 3-class groups of cyclic cubic subfields $k < k^*$ with $t = 2$, an elementary cyclic group $\mathrm{Cl}_3(k) \simeq (3)$ is warranted for $k \in \{k_{pr}, \tilde{k}_{pr}, k_{qr}, \tilde{k}_{qr}\}$. For the critical fields $k \in \{k_{pq}, \tilde{k}_{pq}\}$, however, we must distinguish the **regular** situation $\mathrm{Cl}_3(k_f^*) \simeq (3, 3)$ in terms of the sub-genus field $k_f^* = k_{pq} \cdot \tilde{k}_{pq}$ with partial conductor $f = pq$ which divides $c = pqr$, where $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3, 3)$ and equality $(m, n) = (\tilde{m}, \tilde{n})$ is warranted, as opposed to the **singular** situation $\mathrm{Cl}_3(k_f^*) \simeq (3, 3, 3)$, and the **super-singular** situation $81 \mid h_3(k_f^*)$, where usually $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (9, 3)$.

For doublets $(k_{pq}, \tilde{k}_{pq})$ with conductor $f = pq$ and non-elementary bicyclic 3-class group, a distinction arises from the 3-valuation $v^* := v_3(h(k_f^*))$ of the class number of the 3-genus field k_f^*:

Definition 6. A quartet $(k_\mu)_{1 \le \mu \le 4}$ with conductor $c = pqr$ and its sub-doublet $(k_{pq}, \tilde{k}_{pq})$ of cyclic cubic fields with common partial conductor $f = pq$ is called

$$\begin{cases} \textbf{regular} & \text{if } v^* \in \{0, 1, 2\}, \\ \textbf{singular} & \text{if } v^* = 3, \\ \textbf{super-singular} & \text{if } v^* \in \{4, 5, 6, \ldots\}. \end{cases} \tag{8.2}$$

Let $(k_1, \ldots, k_4)$ be the quartet of cyclic cubic number fields sharing the common discriminant $d = c^2$ with conductor $c = pqr$, divisible by exactly three primes $\equiv 1 \pmod 3$ (one among them may be the prime power 3^2), and belonging to one of the Graphs 5–9 of Category III. According to Theorem 2, $\mathrm{Cl}_3(k_\mu) \simeq (3, 3)$ and thus $h_3(k_\mu) = 9$, for $1 \le \mu \le 4$.

Due to these facts, the class number relation $243 \cdot h_3(B_j) = (U_j : V_j) \cdot 9 \cdot 9 \cdot 1 \cdot 3$ for $j \in \{5, 6, 8, 9\}$ implies that there are precisely four **tame** bicyclic bicubic fields, $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, $B_8 = k_2 k_4 k_p k_{qr}$, $B_9 = k_2 k_3 k_q k_{pr}$, satisfying $9 \mid h_3(B_j) = (U_j : V_j)$, for each $j \in \{5, 6, 8, 9\}$, and so we must have the matrix ranks $r_5 = r_6 = r_8 = r_9 = 2$ with indices $(U_j : V_j) \in \{9, 27\}$.

In contrast, each of the six **wild** bicyclic bicubic fields, $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_{10} = k_3 k_4 k_r k_{pq}$, $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, with $h_3(B_j) > (U_j : V_j)$, either contains k_{pq} or $\tilde{k}_{pq}$. The class number relation (4.2) implies

$$243 \cdot h_3(B_j) = (U_j : V_j) \cdot \begin{cases} 9 \cdot h_3(k_{pq}) \cdot 3 \cdot 3 & \text{for } j = 1, 2, \\ 9 \cdot 9 \cdot 1 \cdot h_3(k_{pq}) & \text{for } j = 10, \\ 9 \cdot h_3(\tilde{k}_{pq}) \cdot 3 \cdot 3 & \text{for } j = 3, 4, \\ 9 \cdot 9 \cdot 1 \cdot h_3(\tilde{k}_{pq}) & \text{for } j = 7. \end{cases}$$

Summarized, in dependence on the index $I_j := (U_j : V_j)$ of subfield units and the rank r_j,

$$h_3(B_j) = \begin{cases} h_3(k_{pq}) & \text{for } j = 1, 2, 10,\ I_j = 3,\ r_j = 3, \\ 3 \cdot h_3(k_{pq}) & \text{for } j = 1, 2, 10,\ I_j = 9,\ r_j = 2, \\ 9 \cdot h_3(k_{pq}) & \text{for } j = 1, 2, 10,\ I_j = 27,\ r_j = 2, \\ h_3(\tilde{k}_{pq}) & \text{for } j = 3, 4, 7,\ I_j = 3,\ r_j = 3, \\ 3 \cdot h_3(\tilde{k}_{pq}) & \text{for } j = 3, 4, 7,\ I_j = 9,\ r_j = 2, \\ 9 \cdot h_3(\tilde{k}_{pq}) & \text{for } j = 3, 4, 7,\ I_j = 27,\ r_j = 2, \end{cases} \tag{8.3}$$

with $h_3(k_{pq}) = h_3(\tilde{k}_{pq}) = 9$ in the regular situation, and $h_3(k_{pq}), h_3(\tilde{k}_{pq}) \geq 27$ in the singular or super-singular situation. Formula (8.3) supplements Corollary 1 in the case $p \leftrightarrow q$.

Lemma 8. ***(3-class ranks of components.)*** *All four components k_μ, $1 \leq \mu \leq 4$, of the quartet have elementary bicyclic 3-class group $\mathrm{Cl}_3(k_\mu) \simeq (3, 3)$. The condition $9 \mid h_3(B_j) = (U_j : V_j) \in \{9, 27\}$, $r_j = 2$, is satisfied for $j \in \{5, 6, 8, 9\}$, the so-called **tame** extensions.*

Proof. This is a consequence of the definition of Graph 5–9 in Category III and the rank distribution in Theorem 2. The fields B_j with $j \in \{5, 6, 8, 9\}$ neither contain k_{pq} nor $\tilde{k}_{pq}$. □

All computations for examples in the following subsections were performed with Magma [6, 7, 12].

8.1. **Category III, Graph 5.** In this section, the combined cubic residue symbol of three prime(power)s dividing the conductor $c = pqr$ is assumed to be $[p, q, r]_3 = \{p \leftrightarrow q; r\}$.

Since there are no trivial cubic residue symbols among the three prime(power) divisors p, q, r of the conductor $c = pqr$, except $p \leftrightarrow q$ with overall assumption (8.1), the principal factors of the subfields $k \in \{k_{pr}, \tilde{k}_{pr}, k_{qr}, \tilde{k}_{qr}\}$ with $t = 2$ of the absolute genus field k^* must be divisible by both relevant primes, and we can use the general approach

$$\begin{aligned} A(k_{pr}) &= p^\ell r, \quad A(\tilde{k}_{pr}) = p^{-\ell} r, \text{ and} \\ A(k_{qr}) &= q^s r, \quad A(\tilde{k}_{qr}) = q^{-s} r, \end{aligned} \tag{8.4}$$

with $\ell, s \in \{-1, 1\}$, identifying $-1 \equiv 2 \pmod 3$, since it is easier to manage: $\ell^2 = s^2 = 1$.

Lemma 9. *In dependence on the **decisive principal factors** in Equation (8.4), the **principal factors** of the quartet $(k_\mu)_{\mu=1}^4$ sharing common conductor $c = pqr$ with Graph* III.5 *are given by*

$$\begin{aligned} A(k_1) &= pqr^2, & A(k_2) &= pqr, & A(k_3) &= pq^2r, & A(k_4) &= p^2qr & &\text{if } (\ell, s) = (1, 1), \\ A(k_1) &= p^2qr, & A(k_2) &= pq^2r, & A(k_3) &= pqr, & A(k_4) &= pqr^2 & &\text{if } (\ell, s) = (1, 2), \\ A(k_1) &= pq^2r, & A(k_2) &= p^2qr, & A(k_3) &= pqr^2, & A(k_4) &= pqr & &\text{if } (\ell, s) = (2, 1), \\ A(k_1) &= pqr, & A(k_2) &= pqr^2, & A(k_3) &= p^2qr, & A(k_4) &= pq^2r & &\text{if } (\ell, s) = (2, 2), \\ A(k_1) &= p^\ell q^s r^{-1}, & A(k_2) &= p^\ell q^s r, & A(k_3) &= p^\ell q^{-s} r, & A(k_4) &= p^{-\ell} q^s r & &\text{generally.} \end{aligned} \tag{8.5}$$

Proof. We implement the general approach (8.4). From the ranks $r_j = 2$ for $j = 5, 6, 8, 9$, there arise constraints for the exponents in the proposal $A(k_\mu) = p^{x_\mu} q^{y_\mu} r^{z_\mu}$, $1 \leq \mu \leq 4$, with the aid of principal factor matrices. For these *tame* bicyclic bicubic fields B_j, $j \in \{5, 6, 8, 9\}$, the rank r_j is calculated with row operations on the associated matrix M_j:

$$M_5=\begin{pmatrix}x_1&y_1&z_1\\x_3&y_3&z_3\\1&0&0\\0&-s&1\end{pmatrix},\ M_6=\begin{pmatrix}x_1&y_1&z_1\\x_4&y_4&z_4\\0&1&0\\-\ell&0&1\end{pmatrix},\ M_8=\begin{pmatrix}x_2&y_2&z_2\\x_4&y_4&z_4\\1&0&0\\0&s&1\end{pmatrix},\ M_9=\begin{pmatrix}x_2&y_2&z_2\\x_3&y_3&z_3\\0&1&0\\\ell&0&1\end{pmatrix}.$$

For $B_5=k_1k_3k_p\tilde{k}_{qr}$, M_5 leads to the decisive pivot elements z_1+sy_1 and z_3+sy_3 in the last column, similarly, for $B_6=k_1k_4k_q\tilde{k}_{pr}$, M_6 leads to $z_1+\ell x_1$ and $z_4+\ell x_4$, similarly, for $B_8=k_2k_4k_pk_{qr}$, M_8 leads to z_2-sy_2 and z_4-sy_4, and similarly, for $B_9=k_2k_3k_qk_{pr}$, M_9 leads to $z_2-\ell x_2$ and $z_3-\ell x_3$. So, $r_5=r_6=r_8=r_9=2$ implies $\ell x_1\equiv sy_1\equiv -z_1$, $\ell x_2\equiv sy_2\equiv z_2$, $\ell x_3\equiv -sy_3\equiv z_3$, $-\ell x_4\equiv sy_4\equiv z_4$, and consequently (8.5). □

Proposition 9. ***(Quartet with** 3**-rank two for*** III.5.*)* *Let $(k_\mu)_{\mu=1}^4$ be a quartet with common conductor $c=pqr$, whose combined cubic residue symbol belongs to Graph* 5 *of Category* III. *Then the ranks of principal factor matrices of* ***tame*** *bicyclic bicubic fields are $r_j=2$ for $j=5,6,8,9$. In terms of exponents of primes in four variable principal factors, $A(k_{pq})=p^mq^n$, $A(\tilde{k}_{pq})=p^{\tilde{m}}q^{\tilde{n}}$, from* (8.1)*, and $A(k_{pr})=p^\ell r$, $A(k_{qr})=q^sr$, from* (8.4)*, the ranks of principal factor matrices of* ***wild*** *bicyclic bicubic fields are given by*

$$(8.6)\qquad r_1=r_2=r_{10}=3 \text{ iff } \ell m\not\equiv -sn \,(\mathrm{mod}\, 3)\quad \text{and}\quad r_3=r_4=r_7=3 \text{ iff } \ell\tilde{m}\not\equiv s\tilde{n}\,(\mathrm{mod}\,3).$$

Proof. Up to this point, the parameters $m,n,\tilde{m},\tilde{n}$ did not come into the play yet. They decide about the rank r_j of the associated principal factor matrices M_j of the *wild* bicyclic bicubic fields B_j, $j\in\{1,2,3,4,7,10\}$. Hence, we perform row operations on these matrices:

$$M_1=\begin{pmatrix}\ell&s&-1\\m&n&0\\\ell&0&1\\0&s&1\end{pmatrix},\ M_2=\begin{pmatrix}\ell&s&1\\m&n&0\\-\ell&0&1\\0&-s&1\end{pmatrix},\ M_{10}=\begin{pmatrix}\ell&-s&1\\-\ell&s&1\\0&0&1\\m&n&0\end{pmatrix}.$$

For $B_1=k_1k_{pq}k_{pr}k_{qr}$, M_1 leads to the decisive pivot element $-\ell m-sn$ in the last column, similarly, for $B_2=k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, M_2 leads to $\ell m+sn$, and similarly, for $B_{10}=k_3k_4k_rk_{pq}$, M_{10} leads to $n+\ell sm$ in the middle column. So, $r_1=r_2=r_{10}=3$ iff $\ell m\not\equiv -sn$, by viewing the pivot elements modulo 3. Next we consider:

$$M_3=\begin{pmatrix}\ell&-s&1\\\tilde{m}&\tilde{n}&0\\-\ell&0&1\\0&s&2\end{pmatrix},\ M_4=\begin{pmatrix}-\ell&s&1\\\tilde{m}&\tilde{n}&0\\\ell&0&1\\0&-s&1\end{pmatrix},\ M_7=\begin{pmatrix}\ell&s&-1\\\ell&s&1\\0&0&1\\\tilde{m}&\tilde{n}&0\end{pmatrix}.$$

For $B_3=k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, M_3 leads to $\ell\tilde{m}-s\tilde{n}$, similarly, for $B_4=k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, M_4 leads to $-\ell\tilde{m}+s\tilde{n}$, and similarly, for $B_7=k_1k_2k_r\tilde{k}_{pq}$, M_7 leads to $\tilde{n}-\ell s\tilde{m}$ in the middle column. So, $r_3=r_4=r_7=3$ iff $\ell\tilde{m}\not\equiv s\tilde{n}$. □

In Ayadi's Thesis [2, p. 80], only the special case $\ell=s=-1$ is elaborated. As mentioned above already, the condition $(m,n)=(\tilde{m},\tilde{n})$ is warranted in the *regular* situation $9\|h(k_{pq})$. In any situation, at least one of the following two rank equations, which imply a *total* transfer kernel, is satisfied — in many cases even both simultaneously:

$$(8.7)\qquad \begin{aligned} r_1=r_2=r_{10}=3 &\text{ for } (m,n)\in\{(0,1),(1,0)\},\\ r_3=r_4=r_7=3 &\text{ for } (\tilde{m},\tilde{n})\in\{(0,1),(1,0)\}.\end{aligned}$$

Theorem 15. ***(Second** 3**-class groups for*** III.5*)* *There are several* ***minimal transfer kernel types*** *(mTKT) $\varkappa_0$ of k_μ, $1\le\mu\le 4$, and other possible capitulation types in ascending order $\varkappa_0<\varkappa<\varkappa'<\varkappa''<\varkappa'''$, either $\varkappa_0=(2134)$, type* G.16*, $\varkappa=(2130)$, type* d.23*, or $\varkappa_0=(2143)$, type* G.19*, $\varkappa=(2140)$, type* d.25*, ending in $\varkappa'=(2100)$, type* b.10*, $\varkappa''=(2000)$, type* a.3* *or* a.3*, or $\varkappa''=(0004)$, type* a.2*, and the maximal $\varkappa'''=(0000)$, type* a.1.

In terms of the counter $\mathcal{N}^*:=\#\{1\le j\le 10\mid (U_j:V_j)=27\}$, of ***maximal indices of subfield units*** *$I_j=(U_j:V_j)$ for all ten bicyclic bicubic fields $B_j<k^*$ with conductor $c=pqr$,*

the second 3-class groups $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$ *are given in the following way as uniform or non-uniform quartets, with abbreviation* $P_7 := \langle 2187, 64\rangle$*:*

$$(8.8)\qquad \mathfrak{M} = \begin{cases} \langle 243, 28..30\rangle^4,\ \alpha = [21,11,11,11],\ \varkappa = (0000) & \textit{if } \mathcal{N}^* = 0, \\ \langle 243, 27\rangle,\ \alpha = [11,11,11,22],\ \varkappa = (0004) & \textit{once if } \mathcal{N}^* = 1, \\ \langle 243, 28..30\rangle^3,\ \alpha = [21,11,11,11],\ \varkappa = (0000) & \textit{thrice if } \mathcal{N}^* = 1, \\ \langle 81, 7\rangle^4,\ \alpha = [111,11,11,11],\ \varkappa = (2000) & \textit{if } \mathcal{N}^* = 2, \\ \langle 243, 27\rangle^2,\ \alpha = [11,11,11,22],\ \varkappa = (0004) & \textit{twice if } \mathcal{N}^* = 3, \\ \langle 243, 25\rangle^2,\ \alpha = [22,11,11,11],\ \varkappa = (2000) & \textit{twice if } \mathcal{N}^* = 3, \\ \langle 729, 34..39\rangle^4,\ \alpha = [111,111,21,21],\ \varkappa = (2100) & \textit{if } \mathcal{N}^* = 4, \\ \langle 2187, 250\rangle^2,\ \alpha = [111,111,32,21],\ \varkappa = (2130) & \textit{twice if } \mathcal{N}^* = 7, \\ \langle 2187, 251|252\rangle^2,\ \alpha = [111,111,32,21],\ \varkappa = (2140) & \textit{twice if } \mathcal{N}^* = 7, \\ (P_7 - \#2; 40|48)^2,\ \alpha = [111,111,32,32],\ \varkappa = (2134) & \textit{twice if } \mathcal{N}^* = 10, \\ (P_7 - \#2; 42|45|49)^2,\ \alpha = [111,111,32,32],\ \varkappa = (2143) & \textit{twice if } \mathcal{N}^* = 10. \end{cases}$$

The leading six rows concern the ***regular*** *situation* $\mathrm{Cl}_3(k_{pq}) \simeq (3,3)$*. In particular, the condition* $(m,n) \in \{(0,1),(1,0)\}$ *for* $\langle 81,7\rangle^4$ *is equivalent to the extra special group* $\mathrm{Gal}(\mathrm{F}_3^2(k_{pq})/k_{pq}) \simeq \langle 27,4\rangle$*, whereas* $\mathrm{Gal}(\mathrm{F}_3^2(k_{pq})/k_{pq}) \simeq \langle 9,2\rangle$ *is abelian for all other pairs* (m,n)*. The trailing rows concern the* ***(super-)singular*** *situation with* $h_3(k_{pq}) = h_3(\tilde{k}_{pq}) = 27$*. With exception of the trailing rows, the* 3*-class field tower has length* $\ell_3(K_\mu) = 2$ *and group* $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$*.*

Proof. Let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be the prime ideals of k_μ over p, q, r.

Since p splits in k_q, it also splits in $B_6 = k_1k_4k_q\tilde{k}_{pr}$ and $B_9 = k_2k_3k_qk_{pr}$.

Since q splits in k_p, it also splits in $B_5 = k_1k_3k_p\tilde{k}_{qr}$ and $B_8 = k_2k_4k_pk_{qr}$. By Corollary 3,

since $\mathfrak{q}$ is principal ideal in k_q, the class $[\mathfrak{q}]$ capitulates in $B_6 = k_1k_4k_q\tilde{k}_{pr}$ and $B_9 = k_2k_3k_qk_{pr}$;

since $\mathfrak{r}$ is principal ideal in k_r, the class $[\mathfrak{r}]$ capitulates in $B_7 = k_1k_2k_r\tilde{k}_{pq}$ and $B_{10} = k_3k_4k_rk_{pq}$.

Since $\mathfrak{q}\mathfrak{r}$ is principal ideal in $\tilde{k}_{qr}$, the class $[\mathfrak{q}\mathfrak{r}]$ capitulates in $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, and $B_5 = k_1k_3k_p\tilde{k}_{qr}$, by Proposition 2. Since $A(k_1) = pqr$ and $A(k_3) = p^2qr$, $[\mathfrak{q}\mathfrak{r}]$ generates the same subgroup as $[\mathfrak{p}]$ in $\ker(T_{B_5/k_\mu})$, $\mu = 1,3$.

Since $\mathfrak{q}\mathfrak{r}^2$ is principal ideal in k_{qr}, the class $[\mathfrak{q}\mathfrak{r}^2]$ capitulates in $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, and $B_8 = k_2k_4k_pk_{qr}$, by Proposition 2. Since $A(k_2) = pqr^2$ and $A(k_4) = pq^2r$, $[\mathfrak{q}\mathfrak{r}^2]$ generates the same subgroup as $[\mathfrak{p}]$ in $\ker(T_{B_8/k_\mu})$, $\mu = 2,4$.

The 3-class group of k_μ is always $\mathrm{Cl}_3 = \langle[\mathfrak{q}],[\mathfrak{r}]\rangle$. It contains the **norm class groups** of $B_j > k_\mu$ as subgroups of index 3: $N_{B_j/k_\mu}\mathrm{Cl}_3(B_j)$ is always generated by $[\mathfrak{q}]$ for $j = 5,8$, due to the above mentioned splitting of q. See also Table 12.

We recall that equality $(m,n) = (\tilde{m},\tilde{n})$ is warranted for the **regular** situation $\mathrm{Cl}_3(k_{pq}) \simeq (3,3)$, and there is an equivalence involving the counter $\mathcal{P}$ in Theorem 4: $\mathrm{Gal}(\mathrm{F}_3^2(k_{pq})/k_{pq}) \simeq \langle 27,4\rangle$ iff $\mathcal{P} = 1$ iff (either $m = 0$ or $n = 0$) iff $(m,n) \in \{(0,1),(1,0)\}$. Let $I_j := (U_j : V_j)$.

$\mathcal{N}^* = 0$ implies wild ranks $r_j = 2$ and $I_j = 9$, $h_3(B_j) = 3 \cdot h_3(k_{pq}) = 3 \cdot 9 = 27$ for $j \in \{1,2,10\}$, but $r_j = 3$, $I_j = 3$, $h_3(B_j) = h_3(\tilde{k}_{pq}) = 9$ for $j \in \{3,4,7\}$, according to Equation (8.3), and tame indices $I_j = 9$ for all $j \in \{5,6,8,9\}$. The uniform minimal indices I_j of subgroup units correspond to maximal norm unit indices $(U(k_\mu) : N_{B_j/k_\mu}U(B_j)) = 3$ and thus to total capitulations whenever $k_\mu < B_j$ is a subfield for $1 \le j \le 10$, $1 \le \mu \le 4$. According to Theorem 9 and Corollary 4, the resulting abelian type invariants $\alpha = [21,11,11,11]$ and transfer kernel type $\varkappa = (0000)$, that is the **Artin pattern** $(\alpha,\varkappa)$, identify three possible groups $\mathfrak{M} \simeq \langle 243, 28..30\rangle$, since $\langle 81,9\rangle$ must be cancelled, due to wrong second layer α_2.

An exception arises for $\mathcal{N}^* = 1$, which causes non-uniformity with $I_1 = 27$, $h_3(B_1) = 9 \cdot h_3(k_{pq}) = 9 \cdot 9 = 81$, as opposed to the remaining $I_2 = I_{10} = 9$. (Everything else is like $\mathcal{N}^* = 0$.) Thus $(U(k_1) : N_{B_1/k_1}U(B_1)) = 1$, and here we have a fixed point capitulation, $\ker(T_{B_1/k_1}) = \langle[\mathfrak{q}\mathfrak{r}^2]\rangle$. The corresponding abelian type invariants $\alpha = [11,11,11,22]$ and transfer kernel type $\varkappa = (0004)$ uniquely identify the group $\langle 243,27\rangle$ for $\mu = 1$. The remaining three groups are $\langle 243, 28..30\rangle^3$.

For $\mathcal{N}^\ast = 2$ and $(m,n) \in \{(0,1),(1,0)\}$, the relations $m \not\equiv -n$ and $m \not\equiv n$ imply $r_j = 3$, $I_j = 3$, $h_3(B_j) = h_3(\tilde{k}_{pq}) = 9$ for all wild bicyclic cicubic fields B_j, $j \in \{1,2,3,4,7,10\}$. For $j \in \{5,8\}$, we have $I_j = 9$, but for $j \in \{6,9\}$, the maximal index $I_j = 27$ is attained and enables an elementary tricyclic 3-class group $\mathrm{Cl}_3(B_j) = \langle \mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\rangle \simeq (3,3,3)$ generated by the prime ideals lying over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \mathfrak{P}_3$. Here, we have a non-fixed point capitulation $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{q}]\rangle$. The transposition is hidden by total capitulation in B_5 and B_8 with norm class group generated by $[\mathfrak{q}]$. The abelian type invariants $\alpha = [111,11,11,11]$ and transfer kernel type $\varkappa = (2000)$ uniquely identify the group $\langle 81,7\rangle \simeq \mathrm{Syl}_3(A_9)$ for $\mu = 1$.

$\mathcal{N}^\ast = 3$ implies $r_j = 3$ and thus wild indices $I_j = 3$, $h_3(B_j) = h_3(\tilde{k}_{pq}) = 9$ for $j \in \{1,2,10\}$. We also have tame indices $I_j = 9$ for $j \in \{5,6,8,9\}$. However, we have $r_j = 2$ and remaining wild indices $I_j = 27$, $h_3(B_j) = 9 \cdot h_3(k_{pq}) = 9 \cdot 9 = 81$ for $j \in \{3,4,7\}$, according to Equation (8.3). There arises a fixed point capitulation, $\ker(T_{B_7/k_\mu}) = \langle[\mathfrak{r}]\rangle$ and, non-uniformly, a non-fixed point capitulation, $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{p}]\rangle$ for $j = 3,4$ with norm class groups also generated by $[\mathfrak{r}]$. The corresponding abelian type invariants $\alpha = [11,11,11,22]$ and transfer kernel type $\varkappa = (0004)$, respectively $\varkappa = (0003)$, uniquely identify the two groups $\langle 243,27\rangle^2$, respectively the remaining two groups $\langle 243,25\rangle^2$.

For $\mathcal{N}^\ast = 4$ and the simplest singular or super-singular situation with $h_3(k_{pq}) = h_3(\tilde{k}_{pq}) = 27$, $\mathcal{P} = 1$ implies $r_j = 3$, $I_j = 3$, $h_3(B_j) = 27$ for all wild $j \in \{1,2,3,4,7,10\}$, and uniformly $h_3(B_j) = I_j = 27$ for all tame $j \in \{5,6,8,9\}$. The latter correspond to elementary tricyclic 3-class groups $\mathrm{Cl}_3(B_j) = \langle \mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3\rangle \simeq (3,3,3)$ generated by the prime ideals lying over $\mathfrak{q}\mathcal{O}_{B_j} = \mathfrak{Q}_1 \cdot \mathfrak{Q}_2 \cdot \mathfrak{Q}_3$ for $j = 5,8$, and $\mathrm{Cl}_3(B_j) = \langle \mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\rangle \simeq (3,3,3)$ generated by the prime ideals lying over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \mathfrak{P}_3$ for $j = 6,9$. Here, we have a non-fixed point capitulation $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{p}]\rangle$ for $j = 5,8$, and $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{q}]\rangle$ for $j = 6,9$. The transposition is not hidden by total capitulation and characteristic for uniform transfer kernel type b.10. According to Theorem 9 and Corollary 4, the abelian type invariants $\alpha = [111,111,21,21]$ and the transfer kernel type $\varkappa = (2100)$, identify six possible groups $\mathfrak{M} \simeq \langle 729, 34..39\rangle$.

For $\mathcal{N}^\ast = 7$, only three wild indices $r_j = 3$, $I_j = 3$, $h_3(B_j) = 27$ for $j = 3,4,7$ are not maximal. The TKTs are not uniform, $\varkappa = (2130)$, type d.23, twice and $\varkappa = (2140)$, type d.25, twice.

For $\mathcal{N}^\ast = 10$, all tame and wild indices are maximal $I_j = 27$ for $1 \le j \le 10$, which implies non-uniform minimal TKTs $\varkappa_0 = (2134)$, type G.16, twice and $\varkappa_0 = (2143)$, type G.19, twice. □

Corollary 9. ***(Non-uniformity of the quartet for*** III.5.***)*** *For $\mathcal{N}^\ast = 1$, only a sub-triplet of the quartet shares a common capitulation type $\varkappa(k_\mu)$, abelian type invariants $\alpha(k_\mu)$, and second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$. The invariants of the fourth component* ***differ****. For $\mathcal{N}^\ast \in \{3,7,10\}$, only two pairs of components of the quartet share a common capitulation type $\varkappa(k_\mu)$, and second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, whereas the abelian type invariants $\alpha(k_\mu)$ are uniform. However, the four components* ***agree*** *in all situations with even $\mathcal{N}^\ast \in \{0,2,4\}$.*

Proof. This is an immediate consequence of Theorem 15 and Table 12. □

In terms of capitulation targets in Corollary 2, Theorem 15 and parts of its proof are now summarized in Table 12 with transpositions in **bold** font.

Table 12. Norm class groups and minimal transfer kernels for Graph III.5

Base	k_1				k_2				k_3				k_4			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}$
TK	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{q}\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{q}\mathfrak{r}^2$	$\mathfrak{r}$
$\varkappa$	1	**3**	**2**	4	1	2	**4**	**3**	**4**	**3**	**2**	**1**	**4**	**3**	**2**	**1**

Example 5. The prototypes for Graph III.5, i.e., the cases in Theorem are four **regular** situations, $c = 14\,049$ with $\{7 \leftrightarrow 223; 9\}$ and $\mathcal{N}^\ast = 0$; $c = 17\,073$ with $\{9 \leftrightarrow 271; 7\}$ and $\mathcal{N}^\ast = 2$; $c = 20\,367$ with $\{9 \leftrightarrow 73; 31\}$ and $\mathcal{N}^\ast = 1$; $c = 21\,231$ with $\{7 \leftrightarrow 337; 9\}$ and $\mathcal{N}^\ast = 3$; and the **singular**

situation $c = 42\,399$ with $\{7 \leftrightarrow 673; 9\}$ and $\mathcal{N}^* = 4$. Here, we have distinct $(m, n) = (0, 1)$, but $(\tilde{m}, \tilde{n}) = (1, 0)$. There is also a **super-singular** prototype $c = 48\,447$ with $\{7 \leftrightarrow 769; 9\}$ and $\mathcal{N}^* = 4$, phenomenologically completely identical with the singular prototype, except that $(m, n) = (\tilde{m}, \tilde{n}) = (0, 1)$. With **considerable statistic delay**, there appeared $\mathcal{N}^* \in \{7, 10\}$.

In Table 13, we summarize the prototypes of graph III.5. Data comprises ordinal number No., conductor c of k, combined cubic residue symbol $[p, q, r]_3$, regularity, resp. (super-)singularity, expressed by 3-valuation $v^* = v_3(\#\mathrm{Cl}(k^*))$ of class number of absolute 3-genus field k^*, 3-valuation $v = v_3(\#\mathrm{Cl}(k_{pq}))$, respectively $\tilde{v} = v_3(\#\mathrm{Cl}(\tilde{k}_{pq}))$, of class number of critical field k_{pq}, respectively $\tilde{k}_{pq}$, critical exponents m, n in principal factor $A(k_{pq}) = p^m q^n$, resp. $\tilde{m}, \tilde{n}$ in $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, resp. ℓ in $A(k_{pr}) = p^\ell r$, resp. s in $A(k_{qr}) = q^s r$, capitulation type of k, second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ of k, and length $\ell_3(k)$ of 3-class field tower of k. For abbreviation we put
$P_7 := \langle 2187, 64\rangle$, $R_4^4 := P_7 - \#2; 54$, $R_5^4 := P_7 - \#2; 57$, $R_6^4 := P_7 - \#2; 59$,
$S_4^4 := R_4^4 - \#1; 8 - \#1; 3|7$, $U_5^4 := R_5^4 - \#1; 1 - \#1; 3|6$, $V_6^4 := R_6^4 - \#1; 6 - \#1; 2|6$.
See the tables and tree diagrams in [18, §§ 11.3–11.4, pp. 108–116, Tbl. 4–5, Fig. 9–11].

Table 13. Prototypes for Graph III.5

No.	c	$p \leftrightarrow q, r$	v^*	v	$\tilde{v}$	m, n	$\tilde{m}, \tilde{n}$	ℓ	s	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	14 049	$7 \leftrightarrow 223, 9$	1	2	2	2, 1	2, 1	1	1	a.1	$\langle 243, 28..30\rangle^4$	$= 2$
2	17 073	$9 \leftrightarrow 271, 7$	2	2	2	0, 1	0, 1	1	1	a.3*	$\langle 81, 7\rangle^4$	$= 2$
3	20 367	$9 \leftrightarrow 73, 31$	1	2	2	2, 1	2, 1	2	2	a.2, a.1	$\langle 243, 27\rangle, \langle 243, 28..30\rangle^3$	$= 2$
4	21 231	$7 \leftrightarrow 337, 9$	1	2	2	1, 1	1, 1	1	1	a.2, a.3	$\langle 243, 27\rangle^2, \langle 243, 25\rangle^2$	$= 2$
13	42 399	$7 \leftrightarrow 673, 9$	3	3	3	0, 1	1, 0	1	2	b.10	$\langle 729, 37..39\rangle^4$	≥ 2
16	48 447	$7 \leftrightarrow 769, 9$	4	3	3	0, 1	0, 1	1	1	b.10	$\langle 729, 37..39\rangle^4$	≥ 2
39	100 503	$13 \leftrightarrow 859, 9$	3	3	3	1, 0	1, 1	2	1	b.10	$\langle 729, 34..36\rangle^4$	≥ 2
67	145 593	$7 \leftrightarrow 2311, 9$	4	3	3	2, 1	2, 1	1	1	d.23, d.25	$\langle 2187, 250\rangle^2, \langle 2187, 251 \vert 252\rangle^2$	≥ 2
128	256 669	$37 \leftrightarrow 991, 7$	6	5	3	1, 1	2, 1	2	1	G.16, G.19	$(S_4^4)^2, (U_5^4 \vert V_6^4)^2$	≥ 3

8.2. **Category III, Graph 6.** Let the combined cubic residue symbol of three primes dividing the conductor $c = pqr$ be $[p, q, r]_3 = \{r \leftarrow p \leftrightarrow q\}$.

Proposition 10. ***(Quartet with*** **3*-rank two for*** **III.6*.)*** *For fixed $\mu \in \{1, 2, 3, 4\}$, let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be the prime ideals of k_μ over p, q, r, that is $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$, then the* ***principal factor*** *of k_μ is $A(k_\mu) = p$, and the 3-class group of k_μ is,*

$$\mathrm{Cl}_3(k_\mu) = \langle [\mathfrak{q}], [\mathfrak{r}]\rangle \simeq (3, 3). \tag{8.9}$$

The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic fields B_i, $1 \leq i \leq 10$. The ***tame*** *extensions with $9 \mid h_3(B_i) = (U_i : V_i) \in \{9, 27\}$ are B_i with $i = 5, 6, 8, 9$, since they neither contain k_{pq} nor $\tilde{k}_{pq}$. For each μ, there are two tame extensions B_j/k_μ, B_ℓ/k_μ with the following properties. The first, B_j with $j \in \{6, 9\}$, has norm class group $N_{B_j/k_\mu}(\mathrm{Cl}_3(B_j)) = \langle [\mathfrak{q}\mathfrak{r}^s]\rangle$ with $s \in \{1, 2\}$,* ***cyclic*** *transfer kernel*

$$\ker(T_{B_j/k_\mu}) = \langle [\mathfrak{q}]\rangle \tag{8.10}$$

of order 3, and ***elementary tricyclic*** *3-class group $\mathrm{Cl}_3(B_j) = \langle [\mathfrak{Q}\mathfrak{R}^s\mathfrak{P}_1], [\mathfrak{Q}\mathfrak{R}^s\mathfrak{P}_2], [\mathfrak{Q}\mathfrak{R}^s\mathfrak{P}_3]\rangle \simeq (3, 3, 3)$, generated by the classes of the prime ideals of B_j over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$, $\mathfrak{q}\mathcal{O}_{B_j} = \mathfrak{Q}$, $\mathfrak{r}\mathcal{O}_{B_j} = \mathfrak{R}$. The second, B_ℓ with $\ell \in \{5, 8\}$, has norm class group $N_{B_\ell/k_\mu}(\mathrm{Cl}_3(B_\ell)) = \langle [\mathfrak{q}]\rangle$, transfer kernel*

$$\ker(T_{B_\ell/k_\mu}) \geq \langle [\mathfrak{q}\mathfrak{r}^s]\rangle, \tag{8.11}$$

and 3-class group $\mathrm{Cl}_3(B_\ell) = \langle [\mathfrak{Q}_1], [\mathfrak{Q}_2], [\mathfrak{Q}_3]\rangle \geq (3, 3)$, generated by the classes of the prime ideals of B_ℓ over $\mathfrak{q}\mathcal{O}_{B_\ell} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$. The pair (j, ℓ) forms a hidden or actual ***transposition*** *of the transfer kernel type $\varkappa(k_\mu)$. The remaining two $B_i > k_\mu$, $i \neq j$, $i \neq \ell$, have norm class group $\langle [\mathfrak{r}]\rangle$, respectively $\langle [\mathfrak{q}^2\mathfrak{r}^s]\rangle$, and transfer kernel*

$$\ker(T_{B_i/k_\mu}) \geq \langle [\mathfrak{r}]\rangle, \ \textit{or} \ \geq \langle [\mathfrak{q}^2\mathfrak{r}^s]\rangle,$$

*providing the option of either two possible **fixed points** or a further **transposition** in the transfer kernel type $\varkappa(k_\mu)$. In terms of n and $\tilde{n}$ in $A(k_{pq}) = p^m q^n$ and $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, the ranks of the **wild** extensions are*

$$r_1 = r_2 = r_{10} = 3 \text{ iff } n \neq 0 \text{ iff } q \mid A(k_{pq}) \text{ and } r_3 = r_4 = r_7 = 3 \text{ iff } \tilde{n} \neq 0 \text{ iff } q \mid A(\tilde{k}_{pq}). \quad (8.12)$$

Proof. By Proposition 3, principal factors are $A(k_{pr}) = A(\tilde{k}_{pr}) = p$, since $r \leftarrow p$. Further, by Proposition 4, $A(k_\mu) = p$, for all $1 \leq \mu \leq 4$, since p is universally repelling $r \leftarrow p \rightarrow q$. Since $\mathfrak{p} = \alpha\mathcal{O}_{k_\mu}$ is a principal ideal, its class $[\mathfrak{p}] = 1$ is trivial, whereas the classes $[\mathfrak{q}], [\mathfrak{r}]$ are non-trivial.

Assume the principal factors $A(k_{qr}) = qr^2$ and $A(\tilde{k}_{qr}) = qr$. The parameters $m, n, \tilde{m}, \tilde{n}$, proposed for all Graphs 5–9 of Category III, decide about the rank r_j of the associated principal factor matrices M_j of the *wild* bicyclic bicubic fields B_j, $j \in \{1, 2, 3, 4, 7, 10\}$. As usual, we perform row operations on these matrices:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ m & n & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ m & n & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ m & n & 0 \end{pmatrix}.$$

For $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, M_1 leads to the decisive pivot element $-2n$ in the last column, similarly for $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, M_2 leads to $-n$, and similarly for $B_{10} = k_3 k_4 k_r k_{pq}$, M_{10} leads to n in the middle column. So, $r_1 = r_2 = r_{10} = 3$ iff $n \neq 0$, by viewing the pivot elements modulo 3. Next:

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{m} & \tilde{n} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{m} & \tilde{n} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \tilde{m} & \tilde{n} & 0 \end{pmatrix}.$$

For $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, M_3 leads to $-2\tilde{n}$, similarly for $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, M_4 leads to $-\tilde{n}$, and similarly for $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, M_7 leads to $\tilde{n}$ in the middle column. So, $r_3 = r_4 = r_7 = 3$ iff $\tilde{n} \neq 0$.

In the regular case $h_3(k_{pq}) = h_3(\tilde{k}_{pq}) = 9$, where $(m, n) = (\tilde{m}, \tilde{n})$, the condition $n \neq 0$, that is $q \mid A(k_{pq})$, is certainly satisfied when $\mathcal{P} = 2$ or equivalently $\mathrm{Gal}(\mathrm{F}_3^2(k_{pq})/k_{pq}) \simeq \langle 9, 2 \rangle$, according to Theorem 4. However, when $\mathcal{P} = 1$ or equivalently $\mathrm{Gal}(\mathrm{F}_3^2(k_{pq})/k_{pq}) \simeq \langle 27, 4 \rangle$, then we may either have $q \mid A(k_{pq})$ and still $n \neq 0$, or $p \mid A(k_{pq})$, $n = 0$, with completely different consequence $r_1 = r_2 = r_{10} = r_3 = r_4 = r_7 = 2$. In the singular and super-singular cases, both pairs of parameters, (m, n) and $(\tilde{m}, \tilde{n})$, more precisely only n and $\tilde{n}$, must be taken into consideration, separately. See also the proof of Theorem 16. ☐

In terms of capitulation targets in Corollary 2, Theorem 16 and parts of its proof are now summarized in Table 14 with transpositions in **bold** font.

TABLE 14. Norm class groups and minimal transfer kernels for Graph III.6

Base	k_1				k_2				k_3				k_4			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{qr}$	$\mathfrak{qr}^2$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{qr}^2$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{qr}$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{r}$
TK	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{r}$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{qr}^2$	$\mathfrak{r}$
$\varkappa$	**4**	**3**	**2**	**1**	**2**	**1**	**4**	**3**	1	**3**	**2**	4	1	**3**	**2**	4

Theorem 16. ***(Second** 3**-class group for*** III.6.*)* *To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $1 \leq \mu \leq 4$, let the **principal factor** of k_{pq}, respectively $\tilde{k}_{pq}$, be $A(k_{pq}) = p^m q^n$, respectively $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, and additionally assume the **regular** situation where both $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3, 3)$ are elementary bicyclic, whence $(m, n) = (\tilde{m}, \tilde{n})$.*

*Then there are several **minimal transfer kernel types** (mTKT) $\varkappa_0$ of k_μ, $1 \leq \mu \leq 4$, and other possible capitulation types in ascending order $\varkappa_0 < \varkappa < \varkappa' < \varkappa''$, ending in the mandatory $\varkappa'' = (2000)$, type* a.3*, *either $\varkappa_0 = (2134)$, type* G.16, *$\varkappa = (2130)$, type* d.23, *$\varkappa' = (2100)$, type* b.10, *or $\varkappa_0 = (2143)$, type* G.19, *$\varkappa = (2140)$, type* d.25, *and again $\varkappa' = (2100)$, type* b.10.

*In the **regular** situation, the second 3-class group is* $\mathfrak{M} \simeq$
(8.13)
$$\begin{cases} \langle 81,7\rangle^4,\ \alpha=[111,11,11,11]^4,\ \varkappa=(2000)^4 & \text{if } n\ne 0,\ \mathcal{N}=1,\\ \langle 729,34..39\rangle^4,\ \alpha=[111,111,21,21]^4,\ \varkappa=(2100)^4 & \text{if } n=0,\ \mathcal{N}=2,\\ \langle 729,43\rangle^2,\langle 729,42\rangle^2,\ \alpha=[111,111,22,21]^4,\ \varkappa=(2140)^2,(2130)^2 & \text{if } n=0,\ \mathcal{N}=3,\\ \langle 2187,71\rangle^2,\langle 2187,69\rangle^2\ \alpha=[111,111,22,22]^4,\ \varkappa=(2143)^2,(2134)^2 & \text{if } n=0,\ \mathcal{N}=4, \end{cases}$$

where $\mathcal{N} := \#\{1\le w\le 10 \mid k_\mu < B_w,\ I_w = 27\}$*. Only in the first case, the 3-class field tower has certainly the group* $\mathfrak{G}=\mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu)\simeq\mathfrak{M}$ *and length* $\ell_3(k_\mu)=2$*, otherwise* $\ell_3(k_\mu)\ge 3$ *cannot be excluded, even if* $d_2(\mathfrak{M})\le 4$*.*

*In the **singular** situation, the second 3-class group is* $\mathfrak{M} \simeq$
(8.14)
$$\Big\{\langle 2187,251|252\rangle^2,\langle 2187,250\rangle^2,\ \alpha=[111,111,32,21]^4,\ \varkappa=(2140)^2,(2130)^2 \quad \text{if } n=\tilde{n}=0,\ \mathcal{N}=3.$$

*In the **super-singular** situation, no statement is possible, since the order of* $\mathfrak{M}$ *may increase unboundedly.*

Proof. In the *regular* situation $\mathrm{Cl}_3(k_{pq})=\mathrm{Cl}_3(\tilde{k}_{pq})=(3,3)$, exponents (m,n) and $(\tilde{m},\tilde{n})$ of principal factors $A(k_{pq})=p^mq^n$ and $A(\tilde{k}_{pq})=p^{\tilde{m}}q^{\tilde{n}}$ are equal. Let $\mathcal{P}$ be the number of primes dividing $A(k_{pq})$. According to the proof of Proposition 10, ranks r_w and indices $I_w:=(U_w:V_w)$ of subfield units for *wild* extensions are given by $r_w=3$, $I_w=3$ iff $n\ne 0$, for $w\in\{1,2,10\}$, and $r_w=3$, $I_w=3$ iff $\tilde{n}\ne 0$, for $w\in\{3,4,7\}$, in particular, certainly for $\mathcal{P}=2$.

This implies 3-class numbers $h_3(B_w)=h_3(k_{pq})=h_3(\tilde{k}_{pq})=9$ and 3-class groups $\mathrm{Cl}_3(B_w)\simeq(3,3)$, for $w\in\{1,2,3,4,7,10\}$, whenever $n\ne 0$, i.e. $q\mid A(k_{pq})$, a remarkable distinction of the prime q against the primes p,r. We point out that this can occur not only for $\mathcal{P}=2$, but also for $\mathcal{P}=1$, provided that $A(k_{pq})=q$, $n=1$, and not $A(k_{pq})=p$, $m=1$.

Indices of *tame* extensions with $9\mid h_3(B_w)=I_w\in\{9,27\}$ and $r_w=2$ are non-uniform: corresponding to a unique elementary tricyclic $\mathrm{Cl}_3(B_w)\simeq(3,3,3)$, we must have $I_w=27$ for $w\in\{6,9\}$ with norm class group $N_{B_w/k_\mu}\mathrm{Cl}_3(B_w)$ either $\langle[\mathfrak{qr}]\rangle$ or $\langle[\mathfrak{qr}^2]\rangle$, but corresponding to the remaining bicyclic $\mathrm{Cl}_3(B_w)\simeq(3,3)$, the index $I_w=9$ takes the minimal value for $w\in\{5,8\}$ with norm class group $N_{B_w/k_\mu}\mathrm{Cl}_3(B_w)=\langle[\mathfrak{q}]\rangle$. Thus $\mathcal{N}=1$ and the resulting Artin pattern $\alpha=[111,11,11,11]$ uniquely identifies the group $\mathfrak{G}=\mathfrak{M}\simeq\langle 81,7\rangle$ of maximal class.

Now we come to $n=0$, whence necessarily $\mathcal{P}=1$. Then $r_w=2$ and $I_w\in\{9,27\}$ for the wild extensions $w\in\{1,2,3,4,7,10\}$. Indices of tame extensions now become uniform, corresponding to a pair of elementary tricyclic $\mathrm{Cl}_3(B_w)\simeq(3,3,3)$, which enforces $I_w=27$ for $w\in\{5,6,8,9\}$, i.e., $\mathcal{N}\ge 2$. The number $\mathcal{N}$ of maximal unit indices decides about the group $\mathfrak{M}$: If $\mathcal{N}=2$, then for all $w\in\{1,2,3,4,7,10\}$: $I_w=9$, $h_3(B_w)=3\cdot h_3(k_{pq})=27$, and $\mathrm{Cl}_3\simeq(9,3)$, according to the laws for 3-groups of coclass ≥ 2 [16, pp. 289–292]. The Artin pattern $\alpha=[111,111,21,21]$ identifies the possible groups $\mathfrak{M}\simeq\langle 729,34..39\rangle$. If $\mathcal{N}=3$, then $I_w=27$ for $w\in\{3,4,7\}$, but $I_w=9$ for $w\in\{1,2,10\}$. The Artin pattern $\alpha=[111,111,22,21]$ together with $\varkappa=(2140)^2$, $\varkappa=(2130)^2$, according to Table 14, identifies the possible groups $\mathfrak{M}\simeq\langle 2187,251|252\rangle^2,\langle 2187,250\rangle^2$ of coclass 2. If $\mathcal{N}=4$, then for all $w\in\{1,2,3,4,7,10\}$: $I_w=27$, $h_3(B_w)=9\cdot h_3(k_{pq})=81$, and $\mathrm{Cl}_3\simeq(9,9)$. The Artin pattern $\alpha=[111,111,22,22]$ together with $\varkappa=(2143)^2$, $\varkappa=(2134)^2$, according to Table 14, identifies the possible groups $\mathfrak{M}\simeq\langle 2187,71\rangle^2,\langle 2187,69\rangle^2$ of coclass 3. □

Corollary 10. ***(Non-uniformity of the quartet for*** III.6.***)*** *Only for* $\mathcal{N}\le 2$*, the components of the quartet, all with 3-rank two, share a common capitulation type* $\varkappa(k_\mu)$*, common abelian type invariants* $\alpha(k_\mu)$*, and a common second 3-class group* $\mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$*, for* $1\le\mu\le 4$*. For* $\mathcal{N}\ge 3$*, the quartet splits into two sub-doublets and thus becomes non-uniform.*

Proof. This is an immediate consequence of Theorem 16 and Table 14. □

Example 6. Prototypes for Graph III.6, that is, minimal conductors for each scenario in Theorem 16 are the following.

There are the **regular** cases $c=8\,541$ with symbol $\{9\leftrightarrow 73\rightarrow 13\}$, $(m,n)=(1,2)$; $c=9\,373$ with symbol $\{103\leftrightarrow 13\rightarrow 7\}$, $(m,n)=(1,1)$; $c=56\,329$ with symbol $\{619\leftrightarrow 13\rightarrow 7\}$,

$(m,n) = (0,1)$, all uniformly with $\mathfrak{G} = \mathfrak{M} = \langle 81,7\rangle^4$, in contrast to $c = 142\,519$ with symbol $\{19 \leftrightarrow 577 \to 13\}$, $(m,n) = (1,0)$, and uniform $\mathfrak{M} = \langle 729, 37..39\rangle^4$; $c = 152\,893$ with symbol $\{13 \leftrightarrow 619 \to 19\}$, $(m,n) = (1,0)$, and uniform $\mathfrak{M} = \langle 729, 34..36\rangle^4$; $c = 163\,681$ with symbol $\{67 \leftrightarrow 349 \to 7\}$, $(m,n) = (1,0)$, and non-uniform $\mathfrak{M} = \langle 729, 42\rangle^2, \langle 729, 43\rangle^2$; $c = 193\,059$ with symbol $\{1129 \leftrightarrow 19 \to 9\}$, $(m,n) = (1,0)$, and non-uniform $\mathfrak{M} = \langle 2187, 69\rangle^2, \langle 2187, 71\rangle^2$ with two distinct minimal transfer kernel types.

Further, the **singular** cases $c = 78\,169$ with symbol $\{859 \leftrightarrow 13 \to 7\}$, $(m,n) = (1,0)$, $(\tilde{m},\tilde{n}) = (1,1)$, and non-uniform $\mathfrak{M} = \langle 2187, 250\rangle^2, \langle 2187, 251|252\rangle^2$; $c = 142\,947$ with symbol $\{9 \leftrightarrow 2269 \to 7\}$, $(m,n) = (1,0)$, $(\tilde{m},\tilde{n}) = (0,1)$, and uniform $\mathfrak{M} = \langle 2187, 253\rangle^4$.

Finally, the **super-singular** cases $c = 102\,277$ with symbol $\{769 \leftrightarrow 7 \to 19\}$, $(m,n) = (\tilde{m},\tilde{n}) = (0,1)$, and uniform $\mathfrak{M} = \langle 729, 37..39\rangle^4$; $c = 199\,171$ with symbol $\{7 \leftrightarrow 769 \to 37\}$, $(m,n) = (\tilde{m},\tilde{n}) = (1,0)$, and uniform $\mathfrak{M} = \langle 6561, 693..698\rangle^4$.

In Table 15, we summarize the prototypes of Graph III.6 in the same way as in Table 13.

TABLE 15. Prototypes for Graph III.6

No.	c	$r \leftarrow p \leftrightarrow q$	v^*	v	m,n	$\tilde{v}$	$\tilde{m},\tilde{n}$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	8 541	$13 \leftarrow 73 \leftrightarrow 9$	1	2	1, 2	2	1, 2	a.3*	$\langle 81,7\rangle^4$	$= 2$
2	9 373	$7 \leftarrow 13 \leftrightarrow 103$	1	2	1, 1	2	1, 1	a.3*	$\langle 81,7\rangle^4$	$= 2$
20	56 329	$7 \leftarrow 13 \leftrightarrow 619$	2	2	0, 1	2	0, 1	a.3*	$\langle 81,7\rangle^4$	$= 2$
29	78 169	$7 \leftarrow 13 \leftrightarrow 859$	3	3	1, 0	3	1, 1	d.23	$\langle 2187,250\rangle^2$	≥ 2
								d.25	$\langle 2187,251\|252\rangle^2$	≥ 2
34	102 277	$19 \leftarrow 7 \leftrightarrow 769$	4	3	0, 1	3	0, 1	b.10	$\langle 729,37..39\rangle^4$	≥ 2
52	142 519	$13 \leftarrow 577 \leftrightarrow 19$	2	2	1, 0	2	1, 0	b.10	$\langle 729,37..39\rangle^4$	≥ 2
54	142 947	$7 \leftarrow 2269 \leftrightarrow 9$	3	3	1, 0	3	0, 1	b.10	$\langle 2187,253\rangle^4$	≥ 2
56	152 893	$19 \leftarrow 619 \leftrightarrow 13$	2	2	1, 0	2	1, 0	b.10	$\langle 729,34..36\rangle^4$	≥ 2
58	163 681	$7 \leftarrow 349 \leftrightarrow 67$	2	2	1, 0	2	1, 0	d.23	$\langle 729,42\rangle^2$	≥ 2
								d.25	$\langle 729,43\rangle^2$	≥ 2
71	193 059	$9 \leftarrow 19 \leftrightarrow 1129$	2	2	1, 0	2	1, 0	G.16	$\langle 2187,69\rangle^2$	≥ 2
								G.19	$\langle 2187,71\rangle^2$	≥ 2
75	199 171	$37 \leftarrow 769 \leftrightarrow 7$	4	3	1, 0	3	1, 0	b.10	$\langle 6561,693..698\rangle^4$	≥ 3

8.3. **Category III, Graph 7.** Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 7 of Category III with combined cubic residue symbol $[p,q,r]_3 = \{r \to p \leftrightarrow q\}$.

Proposition 11. ***(Quartet with** 3**-rank two for** III.7.)* *For fixed $\mu \in \{1,2,3,4\}$, let $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ be the prime ideals of k_μ over p,q,r, that is, $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$. Under the normalizing assumption $A(k_{qr}) = qr^2$, $A(\tilde{k}_{qr}) = qr$, the **principal factors** of k_μ are*

$$A(k_1) = A(k_3) = qr \quad \text{and} \quad A(k_2) = A(k_4) = qr^2, \tag{8.15}$$

and the 3-class group of k_μ is

$$\mathrm{Cl}_3(k_\mu) = \langle[\mathfrak{p}],[\mathfrak{q}]\rangle = \langle[\mathfrak{p}],[\mathfrak{r}]\rangle \simeq (3,3). \tag{8.16}$$

*The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic fields B_i, $1 \leq i \leq 10$. The **tame** extensions with $9 \mid h_3(B_i) = (U_i : V_i) \in \{9,27\}$ are B_i with $i = 5,6,8,9$, since they neither contain k_{pq} nor $\tilde{k}_{pq}$. For each μ, there are two tame extensions B_j/k_μ, B_ℓ/k_μ with the following properties. The first, B_j with $j \in \{6,9\}$, has norm class group $N_{B_j/k_\mu}(\mathrm{Cl}_3(B_j)) = \langle[\mathfrak{p}]\rangle$, transfer kernel*

$$\ker(T_{B_j/k_\mu}) \geq \langle[\mathfrak{q}]\rangle, \tag{8.17}$$

*and 3-class group $\mathrm{Cl}_3(B_j) = \langle[\mathfrak{P}_1],[\mathfrak{P}_2],[\mathfrak{P}_3]\rangle \geq (3,3)$, generated by the classes of the prime ideals of B_j over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$. The second, B_ℓ with $\ell \in \{5,8\}$, has norm class group $N_{B_\ell/k_\mu}(\mathrm{Cl}_3(B_\ell)) = \langle[\mathfrak{q}]\rangle$, **cyclic** transfer kernel*

$$\ker(T_{B_\ell/k_\mu}) = \langle[\mathfrak{p}]\rangle \tag{8.18}$$

*of order 3, and **elementary tricyclic** 3-class group $\mathrm{Cl}_3(B_\ell) = \langle[\mathfrak{Q}_1],[\mathfrak{Q}_2],[\mathfrak{Q}_3]\rangle \simeq (3,3,3)$, generated by the classes of the prime ideals of B_ℓ over $\mathfrak{q}\mathcal{O}_{B_\ell} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$. The pair (j,ℓ) forms a* *hidden or actual* ***transposition*** *of the transfer kernel type* $\varkappa(k_\mu)$*. The remaining two* $B_i > k_\mu$*,* $i \neq j$*,* $i \neq \ell$*, have norm class group* $\langle[\mathfrak{pq}]\rangle$*, respectively* $\langle[\mathfrak{pq}^2]\rangle$*, and transfer kernel*

$$\ker(T_{B_i/k_\mu}) \geq \langle[\mathfrak{q}]\rangle,$$

providing the option of two possible ***repetitions*** *in the transfer kernel type* $\varkappa(k_\mu)$*.*
In terms of n *and* $\tilde{n}$ *in* $A(k_{pq}) = p^m q^n$ *and* $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$*, the ranks of the* ***wild*** *extensions are*

$$(8.19)\quad r_1 = r_2 = r_{10} = 3 \text{ iff } m \neq 0 \text{ iff } p \mid A(k_{pq}) \text{ and } r_3 = r_4 = r_7 = 3 \text{ iff } \tilde{m} \neq 0 \text{ iff } p \mid A(\tilde{k}_{pq}).$$

Proof. By Proposition 3, the symbol $r \to p$ implies principal factors $A(k_{pr}) = A(\tilde{k}_{pr}) = r$.

We assume principal factors $A(k_\mu) = p^{x_\mu} q^{y_\mu} r^{z_\mu}$, for $1 \leq \mu \leq 4$, and $A(k_{qr}) = qr^2$, $A(\tilde{k}_{qr}) = qr$.

We generally have the *tame* matrix ranks $r_5 = r_6 = r_8 = r_9 = 2$ and draw conclusions by explicit calculations. For these bicyclic bicubic fields B_j, $j \in \{5,6,8,9\}$, the rank r_j is calculated with row operations on the associated principal factor matrices M_j:

$$M_5 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ M_6 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_8 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix},\ M_9 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, M_5 leads to the decisive pivot elements $z_1 - y_1$ and $z_3 - y_3$, similarly, for $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, M_6 leads to x_1 and x_4, similarly, for $B_8 = k_2 k_4 k_p k_{qr}$, M_8 leads to $z_2 - 2y_2$ and $z_4 - 2y_4$, and similarly, for $B_9 = k_2 k_3 k_q k_{pr}$, M_9 leads to x_2 and x_3. So, $r_5 = r_6 = 2$ implies $z_1 = y_1$, $z_3 = y_3$, $x_1 = x_4 = 0$, and $r_8 = r_9 = 2$ implies $z_2 = 2y_2$, $z_4 = 2y_4$, $x_2 = x_3 = 0$, i.e. $A(k_1) = A(k_3) = qr$ and $A(k_2) = A(k_4) = qr^2$.

A consequence of these principal factors is the coincidence of the subgroups of $\mathrm{Cl}_3(k_\mu)$ generated by the classes $[\mathfrak{q}]$ and $[\mathfrak{r}]$ in k_μ, $\mu = 1, \ldots, 4$. By Corollary 3,
since $\mathfrak{p}$ is principal ideal in k_p, the class $[\mathfrak{p}]$ capitulates in $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$ and $B_8 = k_2 k_4 k_p k_{qr}$;
since $\mathfrak{q}$ is principal ideal in k_q, the class $[\mathfrak{q}]$ capitulates in $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$ and $B_9 = k_2 k_3 k_q k_{pr}$;
since $\mathfrak{r}$ is principal ideal in k_r, the class $[\mathfrak{r}]$, and thus $[\mathfrak{q}]$, capitulates in $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$ and $B_{10} = k_3 k_4 k_r k_{pq}$.

Moreover, since $\mathfrak{r}$ is principal ideal in k_{pr} and $\tilde{k}_{pr}$, the class $[\mathfrak{r}]$, and thus $[\mathfrak{q}]$, also capitulates in $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$,
$B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, and $B_9 = k_2 k_3 k_q k_{pr}$, by Proposition 2.

The parameters $m, n, \tilde{m}, \tilde{n}$, proposed for all Graphs 5–9 of Category III, decide about the rank r_j of the associated principal factor matrices M_j of the *wild* bicyclic bicubic fields B_j, $j \in \{1,2,3,4,7,10\}$. As usual, we perform row operations on these matrices:

$$M_1 = \begin{pmatrix} 0 & 1 & 1 \\ m & n & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 1 & 2 \\ m & n & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ M_{10} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ m & n & 0 \end{pmatrix}.$$

For $B_1 = k_1 k_{pq} k_{pr} k_{qr}$, M_1 leads to the decisive pivot element m in the first column, similarly, for $B_2 = k_2 k_{pq} \tilde{k}_{pr} \tilde{k}_{qr}$, M_2 leads to m, and similarly, for $B_{10} = k_3 k_4 k_r k_{pq}$, M_{10} leads to m in the first column. So, $r_1 = r_2 = r_{10} = 3$ iff $m \neq 0$. Next we consider:

$$M_3 = \begin{pmatrix} 0 & 1 & 1 \\ \tilde{m} & \tilde{n} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix},\ M_4 = \begin{pmatrix} 0 & 1 & 2 \\ \tilde{m} & \tilde{n} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ M_7 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ \tilde{m} & \tilde{n} & 0 \end{pmatrix}.$$

For $B_3 = k_3 \tilde{k}_{pq} \tilde{k}_{pr} k_{qr}$, M_3 leads to $\tilde{m}$, similarly, for $B_4 = k_4 \tilde{k}_{pq} k_{pr} \tilde{k}_{qr}$, M_4 leads to $\tilde{m}$, and similarly, for $B_7 = k_1 k_2 k_r \tilde{k}_{pq}$, M_7 leads to $\tilde{m}$ in the first column. So, $r_3 = r_4 = r_7 = 3$ iff $\tilde{m} \neq 0$.

Since r splits in k_p, it also splits in $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, $B_8 = k_2 k_4 k_p k_{qr}$.

Since q splits in k_p, it also splits in $B_5 = k_1 k_3 k_p \tilde{k}_{qr}$, $B_8 = k_2 k_4 k_p k_{qr}$.

Since p splits in k_q, it also splits in $B_6 = k_1 k_4 k_q \tilde{k}_{pr}$, $B_9 = k_2 k_3 k_q k_{pr}$. □

In terms of capitulation targets in Corollary 2, Theorem 17 and parts of its proof are now summarized in Table 16 with transpositions in **bold** font.

TABLE 16. Norm class groups and minimal transfer kernels for Graph III.7

Base	k_1				k_2				k_3				k_4			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}\mathfrak{q}$
TK	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$
$\varkappa$	2	**3**	**2**	2	3	3	**4**	**3**	2	**3**	**2**	2	3	**3**	**2**	3

Theorem 17. ***(Second*** **3*-class group for*** III.7.***)*** *Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 7 of Category* III *with combined cubic residue symbol $[p,q,r]_3 = \{q \leftrightarrow p \to r\}$.*

*Then the **minimal transfer kernel type** (mTKT) of k_μ, $1 \le \mu \le 4$, is $\varkappa_0 = (2111)$, type* H.4*, and the other possible capitulation types in ascending order $\varkappa_0 < \varkappa' < \varkappa'' < \varkappa'''$ are $\varkappa' = (2110)$, type* d.19*, $\varkappa'' = (2100)$, type* b.10*, and $\varkappa''' = (2000)$, type* a.3**.*

*To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $1 \le \mu \le 4$, let the **decisive principal factors** be $A(k_{pq}) = p^m q^n$, $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, and additionally assume the **regular situation** where both $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3,3)$ are elementary bicyclic, whence $(m,n) = (\tilde{m},\tilde{n})$. Then*

$$\mathfrak{M} \simeq \begin{cases} \langle 81,7\rangle, \ \alpha = [111,11,11,11], \ \varkappa = (2000) & \text{if } m \ne 0, \ \mathcal{N} = 1, \\ \langle 729,34..39\rangle, \ \alpha = [111,111,21,21], \ \varkappa = (2100) & \text{if } m = 0, \ \mathcal{N} = 2, \\ \langle 729,41\rangle, \ \alpha = [111,111,22,21], \ \varkappa = (2110) & \text{if } m = 0, \ \mathcal{N} = 3, \\ \langle 2187,65|67\rangle, \ \alpha = [111,111,22,22], \ \varkappa = (2111) & \text{if } m = 0, \ \mathcal{N} = 4, \end{cases} \tag{8.20}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j, \ I_j = 27\}$. Only in the leading row, the 3-class field tower has warranted group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$, with length $\ell_3(k_\mu) = 2$. Otherwise, tower length $\ell_3(k_\mu) \ge 3$ cannot be excluded, even if $d_2(\mathfrak{M}) \le 4$.

*In **(super-)singular situations**, the group $\mathfrak{M}$ must be of coclass $\mathrm{cc}(\mathfrak{M}) \ge 2$, and capitulation of type $\varkappa''' = (2000)$ is impossible.*

Proof. We know that the tame ranks are $r_5 = r_6 = r_8 = r_9 = 2$, and thus $I_5, I_6, I_8, I_9 \in \{9, 27\}$, in particular, $I_5 = I_8 = 27$, whence certainly $\mathcal{N} \ge 1$. Further, the wild ranks are $r_1 = r_2 = r_{10} = 3$ iff $m \ne 0$, and $r_3 = r_4 = r_7 = 3$ iff $\tilde{m} \ne 0$.

In the **regular situation** where the 3-class groups of k_{pq} and $\tilde{k}_{pq}$ are elementary bicyclic, tight bounds arise for the abelian quotient invariants α of the group $\mathfrak{M}$:

The first scenario, $m \ne 0$, is equivalent to $\mathcal{N} = 1$, with wild ranks $h_3(B_j) = h_3(k_{pq}) = 9$, for $j = 1,2,10$, $h_3(B_j) = h_3(\tilde{k}_{pq}) = 9$, for $j = 3,4,7$, and tame ranks $h_3(B_j) = I_j = 9$, for $j = 6,9$, $h_3(B_j) = I_j = 27$, for $j = 5,8$, that is $\alpha = [111,11,11,11]$ and consequently $\varkappa = (2000)$, since $\langle 81,7\rangle$ is unique with this α.

The other three scenarios share $m = 0$, and an explicit transposition between B_5, B_6, respectively B_5, B_9, and B_6, B_8, respectively B_8, B_9, giving rise to $\varkappa = (21**)$, and common $h_3(B_j) = I_j = 27$, for $j = 5,6,8,9$, implying $\alpha = [111,111,*,*]$.

The second scenario with $\mathcal{N} = 2$ is supplemented by $I_j = 9$, $h_3(B_j) = 3 \cdot h_3(k_{pq}) = 27$, for $j = 1,2,10$. $I_j = 9$, $h_3(B_j) = 3 \cdot h_3(\tilde{k}_{pq}) = 27$, for $j = 3,4,7$, giving rise to $\alpha = [111,111,21,21]$, $\varkappa = (2100)$, characteristic for $\langle 729,34..39\rangle$ (Cor. 4).

The third scenario with $\mathcal{N} = 3$ is supplemented by $I_j = 27$, $h_3(B_j) = 9 \cdot h_3(k_{pq}) = 81$, for $j = 1,2,10$, but still $I_j = 9$, $h_3(B_j) = 3 \cdot h_3(\tilde{k}_{pq}) = 27$, for $j = 3,4,7$, giving rise to $\alpha = [111,111,22,21]$, $\varkappa = (2110)$, characteristic for $\langle 729,41\rangle$.

The fourth scenario with $\mathcal{N} = 4$ is supplemented by $I_j = 27$, $h_3(B_j) = 9 \cdot h_3(k_{pq}) = 81$, for $j = 1,2,10$, $I_j = 27$, $h_3(B_j) = 9 \cdot h_3(k_{pq}) = 81$, for $j = 3,4,7$, giving rise to $\alpha = [111,111,22,22]$,

$\varkappa = (2111)$, characteristic for either $\langle 2187, 65|67\rangle$ or $\langle 6561, 714..719|738..743\rangle$ with coclass $\mathrm{cc} = 3$. If $d_2(\mathfrak{M}) = 5$, then tower length must be $\ell_3(k_\mu) \ge 3$ For this minimal capitulation type H.4, $\varkappa = (2111)$, all transfer kernels are cyclic of order 3, and the minimal unit norm indices correspond to maximal subfield unit indices.

In **(super-)singular situations**, the 3-class groups of k_{pq} and $\tilde{k}_{pq}$ are non-elementary bicyclic, and even in the simplest case $m \ne 0$, $\tilde{m} \ne 0$, we have $I_j = 3,\ 27 \mid h_3(B_j) = h_3(k_{pq})$, for $j = 1, 2, 10$, $I_j = 3,\ 27 \mid h_3(B_j) = h_3(\tilde{k}_{pq})$, for $j = 3, 4, 7$, which prohibits the occurrence of abelian type invariants (11), required for 3-groups of coclass $\mathrm{cc}(\mathfrak{M}) = 1$ (maximal class). □

Corollary 11. ***(Uniformity of the quartet for*** III.7.***)*** *The components of the quartet, all with 3-rank two, share a common capitulation type $\varkappa(k_\mu)$, common abelian type invariants $\alpha(k_\mu)$, and a common second 3-class group $\mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, for $1 \le \mu \le 4$.*

Proof. This is a consequence of Theorem 17 and Table 16. □

Example 7. We have found prototypes for Graph III.7 in the form of minimal conductors for each scenario in Theorem 17 as follows. There are **regular** cases: $c = 4\,599$ with symbol $\{9 \leftrightarrow 73 \leftarrow 7\}$, $v^* = 1$, and $\mathfrak{G} = \mathfrak{M} = \langle 81, 7\rangle^4$; $c = 31\,707$ with symbol $\{9 \leftrightarrow 271 \leftarrow 13\}$, $v^* = 2$, and $\mathfrak{G} = \mathfrak{M} = \langle 81, 7\rangle^4$; $c = 76\,741$ with symbol $\{577 \leftrightarrow 19 \leftarrow 7\}$, $v^* = 2$, and $\mathfrak{M} = \langle 2187, 65|67\rangle^4$ of elevated coclass 3; and $c = 90\,243$ with symbol $\{271 \leftrightarrow 9 \leftarrow 37\}$, $v^* = 2$, and $\mathfrak{M} = \langle 729, 41\rangle^4$. There is also a **singular** case $c = 61\,243$ with symbol $\{673 \leftrightarrow 7 \leftarrow 13\}$, $v^* = 3$, and $\mathfrak{M} = \langle 2187, 253\rangle^4$; and **super-singular** cases $c = 69\,979$ with symbol $\{769 \leftrightarrow 7 \leftarrow 13\}$, $v^* = 4$, and $\mathfrak{M} = \langle 6561, 676|677\rangle^4$ of elevated coclass 3; and $c = 86\,821$ with symbol $\{79 \leftrightarrow 157 \leftarrow 7\}$, $v^* = 4$, and $\mathfrak{M} = \langle 729, 37..39\rangle^4$.

In Table 17, we summarize the prototypes of Graph III.7 in the same way as in Table 13.

Table 17. Prototypes for Graph III.7

No.	c	$q \leftrightarrow p \leftarrow r$	v^*	v	m, n	$\tilde{v}$	$\tilde{m}, \tilde{n}$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	4 599	$9 \leftrightarrow 73 \leftarrow 7$	1	2	1, 2	2	1, 2	a.3*	$\langle 81, 7\rangle$	$= 2$
2	12 051	$13 \leftrightarrow 103 \leftarrow 9$	1	2	1, 1	2	1, 1	a.3*	$\langle 81, 7\rangle$	$= 2$
6	31 707	$9 \leftrightarrow 271 \leftarrow 13$	2	2	1, 0	2	1, 0	a.3*	$\langle 81, 7\rangle$	$= 2$
21	76 741	$577 \leftrightarrow 19 \leftarrow 7$	2	2	0, 1	2	0, 1	H.4	$\langle 2187, 65\|67\rangle$	≥ 3
27	90 243	$271 \leftrightarrow 9 \leftarrow 37$	2	2	0, 1	2	0, 1	d.19	$\langle 729, 41\rangle$	≥ 2
13	61 243	$673 \leftrightarrow 7 \leftarrow 13$	3	3	0, 1	3	1, 0	b.10	$\langle 2187, 253\rangle$	≥ 2
17	69 979	$769 \leftrightarrow 7 \leftarrow 13$	4	3	0, 1	3	0, 1	d.19	$\langle 6561, 676\|677\rangle$	≥ 3
25	86 821	$79 \leftrightarrow 157 \leftarrow 7$	4	3	1, 1	3	1, 2	b.10	$\langle 729, 37..39\rangle$	≥ 2

8.4. **Category III, Graph 8.** Let $(k_1, \ldots, k_4)$ be a quartet of cyclic cubic number fields belonging to Graph 8 of Category III with combined cubic residue symbol $[p, q, r]_3 = \{r \to p \leftrightarrow q \leftarrow r\}$ of three prime(power)s dividing the conductor $c = pqr$.

Proposition 12. ***(Quartet with 3-rank two for*** III.8.***)*** *For fixed $\mu \in \{1, 2, 3, 4\}$, let $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ be the prime ideals of k_μ over p, q, r, that is, $p\mathcal{O}_{k_\mu} = \mathfrak{p}^3$, $q\mathcal{O}_{k_\mu} = \mathfrak{q}^3$, $r\mathcal{O}_{k_\mu} = \mathfrak{r}^3$, then the* ***principal factor*** *of k_μ is $A(k_\mu) = r$, and the 3-class group of k_μ is*

$$\mathrm{Cl}_3(k_\mu) = \langle[\mathfrak{p}], [\mathfrak{q}]\rangle \simeq (3, 3). \tag{8.21}$$

The unramified cyclic cubic relative extensions of k_μ are among the absolutely bicyclic bicubic fields B_i, $1 \le i \le 10$. The ***wild*** *ranks for $i = 1, 2, 3, 4, 7, 10$ are $r_i = 2$, independently of $m, n, \tilde{m}, \tilde{n}$. For each μ, there are two tame extensions B_j/k_μ, B_ℓ/k_μ with the following properties.*
The first, B_j, has norm class group $N_{B_j/k_\mu}(\mathrm{Cl}_3(B_j)) = \langle[\mathfrak{p}]\rangle$, ***cyclic*** *transfer kernel*

$$\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{q}]\rangle \tag{8.22}$$

of order 3, and ***elementary tricyclic*** *3-class group $\mathrm{Cl}_3(B_j) = \langle[\mathfrak{P}_1], [\mathfrak{P}_2], [\mathfrak{P}_3]\rangle \simeq (3, 3, 3)$, generated by the classes of the prime ideals of B_j over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$.*
The second, B_ℓ, has norm class group $N_{B_\ell/k_\mu}(\mathrm{Cl}_3(B_\ell)) = \langle[\mathfrak{q}]\rangle$, ***cyclic*** *transfer kernel*

$$\ker(T_{B_\ell/k_\mu}) = \langle[\mathfrak{p}]\rangle \tag{8.23}$$

of order 3, and ***elementary tricyclic*** *3-class group $\mathrm{Cl}_3(B_\ell) = \langle[\mathfrak{Q}_1], [\mathfrak{Q}_2], [\mathfrak{Q}_3]\rangle \simeq (3, 3, 3)$, generated by the classes of the prime ideals of B_ℓ over $\mathfrak{q}\mathcal{O}_{B_\ell} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$.*

The pair (j,ℓ) forms a mandatory ***transposition*** *of the transfer kernel type $\varkappa(k_\mu)$. The remaining two $B_i > k_\mu$, $i \ne j$, $i \ne \ell$, have norm class group $\langle[\mathfrak{pq}]\rangle$, respectively $\langle[\mathfrak{pq}^2]\rangle$, necessarily* ***non-elementary bicyclic*** *3-class group of order $27 \mid h_3(B_i) = \frac{(U_i:V_i)}{3}h_3(k_{pq})$, respectively $\frac{(U_i:V_i)}{3}h_3(\tilde{k}_{pq})$, and transfer kernel*

$$\ker(T_{B_i/k_\mu}) \ge \langle[\mathfrak{p}^m\mathfrak{q}^n]\rangle, \text{ respectively } \ge \langle[\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}]\rangle,$$

providing the option of a possible ***fixed point*** *in the transfer kernel type $\varkappa(k_\mu)$.*

Proof. Since $r \to p$, there are two principal factors $A(k_{pr}) = A(\tilde{k}_{pr}) = r$. Since $q \leftarrow r$, there are two further principal factors $A(k_{qr}) = A(\tilde{k}_{qr}) = r$. Since $q \leftarrow r \to p$ is universally repelling, we also have four other principal factors $A(k_\mu) = r$, for all $1 \le \mu \le 4$ according to [2, Prop. 4.6, p. 49]. Since $\mathfrak{r} = \alpha\mathcal{O}_{k_\mu}$ is a principal ideal, its class $[\mathfrak{r}] = 1$ is trivial, whereas the classes $[\mathfrak{p}],[\mathfrak{q}]$ are non-trivial and generate $\mathrm{Cl}_3(k_\mu)$.

There are four *tame* bicyclic bicubic fields, $B_5 = k_1k_3k_p\tilde{k}_{qr}$, $B_6 = k_1k_4k_q\tilde{k}_{pr}$, $B_8 = k_2k_4k_pk_{qr}$, $B_9 = k_2k_3k_qk_{pr}$, satisfying $9 \mid h_3(B_i) = (U_i : V_i)$, for $i \in \{5,6,8,9\}$. Consequently, we must have the indices $I_i = (U_i : V_i) \in \{9,27\}$, and thus the matrix ranks $r_5 = r_6 = r_8 = r_9 = 2$.

On the other hand, there are six *wild* bicyclic bicubic fields, $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, $B_7 = k_1k_2k_r\tilde{k}_{pq}$, $B_{10} = k_3k_4k_rk_{pq}$, with $h_3(B_i) > (U_i : V_i)$.

For these bicyclic bicubic fields B_i, $i \in \{1,2,3,4,7,10\}$, the rank r_i is calculated with row operations on the associated principal factor matrices M_i:

$$M_1 = M_2 = \begin{pmatrix} 0 & 0 & 1 \\ m & n & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ M_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ m & n & 0 \end{pmatrix},\ M_3 = M_4 = \begin{pmatrix} 0 & 0 & 1 \\ \tilde{m} & \tilde{n} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ M_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ \tilde{m} & \tilde{n} & 0 \end{pmatrix}.$$

For $B_1 = k_1k_{pq}k_{pr}k_{qr}$ and $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $M_1 = M_2$ immediately leads to rank $r_1 = r_2 = 2$, since $(m,n) \ne (0,0)$, and similarly, for $B_{10} = k_3k_4k_rk_{pq}$, M_{10} leads to rank $r_{10} = 2$.

For $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, and $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, $M_3 = M_4$ immediately leads to rank $r_3 = r_4 = 2$, since $(\tilde{m},\tilde{n}) \ne (0,0)$, and similarly, for $B_7 = k_1k_2k_r\tilde{k}_{pq}$, M_7 leads to rank $r_7 = 2$.
So Graph 8 of Category III is the **unique** situation where $r_i = 2$, for all $1 \le i \le 10$, without any conditions, and thus $h_3(B_i) = \frac{I_i}{3}h_3(k_{pq})$, for $i \in \{1,2,10\}$, and $h_3(B_i) = \frac{I_i}{3}h_3(\tilde{k}_{pq})$, for $i \in \{3,4,7\}$, where $I_i = (U_i : V_i) \in \{9,27\}$, and $9 \mid h_3(k_{pq})$, $9 \mid h_3(\tilde{k}_{pq})$.

In each case, the minimal subfield unit index $(U_i : V_i) = 9$ corresponds to the maximal unit norm index $(U(k_\mu) : N_{B_i/k_\mu}(U_i)) = 3$, associated with a *total* transfer kernel $\#\ker(T_{B_i/k_\mu}) = 9$, whenever $k_\mu < B_i$ $1 \le \mu \le 4$, $1 \le i \le 10$.

According to Theorem 8, the unramified cyclic cubic relative extensions of k_μ among the absolutely bicyclic bicubic subfields of the 3-genus field $k^* = k_pk_qk_r$ are B_1, B_5, B_6, B_7, for $\mu = 1$, B_2, B_7, B_8, B_9, for $\mu = 2$, B_3, B_5, B_9, B_{10}, for $\mu = 3$, and B_4, B_6, B_8, B_{10}, for $\mu = 4$.

Since p splits in k_q, it also splits in $B_6 = k_1k_4k_q\tilde{k}_{pr}$ and $B_9 = k_2k_3k_qk_{pr}$.

Since $\mathfrak{q}$ is principal in k_q, $[\mathfrak{q}]$ capitulates in $B_6 = k_1k_4k_q\tilde{k}_{pr}$ and $B_9 = k_2k_3k_qk_{pr}$.

For $(\mu,j) \in \{(1,6),(4,6),(2,9),(3,9)\}$, the minimal unit norm index $(U(k_\mu) : N_{B_j/k_\mu}(U_j)) = 1$, associated to the *partial* transfer kernel $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{q}]\rangle$, corresponds to the maximal subfield unit index $h_3(B_j) = (U_j : V_j) = 27$, giving rise to the characteristic abelian type invariants $\mathrm{Cl}_3(B_j) = \langle[\mathfrak{P}_1],[\mathfrak{P}_2],[\mathfrak{P}_3]\rangle \simeq (3,3,3)$ generated by the classes of the prime ideals of B_j over $\mathfrak{p}\mathcal{O}_{B_j} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$. The field B_j, which contains k_μ, has norm class group $N_{B_j/k_\mu}(\mathrm{Cl}_3(B_j)) = \langle[\mathfrak{p}]\rangle$.

Since q splits in k_p, it also splits in $B_5 = k_1k_3k_p\tilde{k}_{qr}$ and $B_8 = k_2k_4k_pk_{qr}$.

Since $\mathfrak{p}$ is principal in k_p, $[\mathfrak{p}]$ capitulates in $B_5 = k_1k_3k_p\tilde{k}_{qr}$ and $B_8 = k_2k_4k_pk_{qr}$.

For $(\mu,\ell) \in \{(1,5),(3,5),(2,8),(4,8)\}$, the minimal unit norm index $(U(k_\mu) : N_{B_\ell/k_\mu}(U_\ell)) = 1$, associated to the *partial* transfer kernel $\ker(T_{B_\ell/k_\mu}) = \langle[\mathfrak{p}]\rangle$, corresponds to the maximal subfield unit index $h_3(B_\ell) = (U_\ell : V_\ell) = 27$, giving rise to the characteristic abelian type invariants $\mathrm{Cl}_3(B_\ell) = \langle[\mathfrak{Q}_1],[\mathfrak{Q}_2],[\mathfrak{Q}_3]\rangle \simeq (3,3,3)$ generated by the classes of the prime ideals of B_ℓ over $\mathfrak{q}\mathcal{O}_{B_\ell} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$. The field B_ℓ, which contains k_μ, has norm class group $N_{B_\ell/k_\mu}(\mathrm{Cl}_3(B_\ell)) = \langle[\mathfrak{q}]\rangle$.

Since $\mathfrak{p}^m\mathfrak{q}^n$ is principal in k_{pq}, $[\mathfrak{p}^m\mathfrak{q}^n]$ capitulates in $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, and $B_{10} = k_3k_4k_rk_{pq}$.

Since $\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}$ is principal in $\tilde{k}_{pq}$, $[\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}]$ capitulates in $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, and $B_7 = k_1k_2k_r\tilde{k}_{pq}$.

The remaining two $B_i > k_\mu$, $i \in \{1,2,3,4,7,10\}$, more precisely, $i \in \{1,7\}$ for $\mu = 1$, and $i \in \{2,7\}$ for $\mu = 2$, and $i \in \{3,10\}$ for $\mu = 3$, and $i \in \{4,10\}$ for $\mu = 4$, have norm class group $\langle[\mathfrak{p}\mathfrak{q}]\rangle$, respectively $\langle[\mathfrak{p}\mathfrak{q}^2]\rangle$, and *minimal* transfer kernel $\ker(T_{B_i/k_\mu}) \ge \langle[\mathfrak{p}^m\mathfrak{q}^n]\rangle$, respectively $\ker(T_{B_i/k_\mu}) \ge \langle[\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}]\rangle$. □

Proposition 12 and parts of its proof are now summarized in Table 18, with transposition in **boldface** font, based on Corollary 2. In this table, we give the norm class group (NCG) $N_{B_i/k_\mu}(\mathrm{Cl}_3(B_i))$ and the transfer kernel (TK) $\ker(T_{B_i/k_\mu})$, also in the symbolic form $\varkappa$ with place holders $1 \le x, \tilde{x}, y, \tilde{y}, z, \tilde{z}, w, \tilde{w} \le 4$, for each collection of four unramified cyclic cubic relative extensions B_j, $j = 1, \dots, 10$, of each base field k_μ, $\mu = 1, \dots, 4$, of the quartet.

TABLE 18. Norm class groups and minimal transfer kernels for Graph III.8

Base	k_1				k_2				k_3				k_4			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}^2$	$\mathfrak{p}\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}\mathfrak{q}^2$
TK	$\mathfrak{p}^m\mathfrak{q}^n$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}$	$\mathfrak{p}^m\mathfrak{q}^n$	$\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}^m\mathfrak{q}^n$	$\mathfrak{p}^{\tilde{m}}\mathfrak{q}^{\tilde{n}}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{p}^m\mathfrak{q}^n$
$\varkappa$	x	**3**	**2**	$\tilde{x}$	y	$\tilde{y}$	**4**	**3**	$\tilde{z}$	**3**	**2**	z	$\tilde{w}$	**3**	**2**	w

Theorem 18. ***(Second* 3*-class group for* III.8*.)*** *Let $(k_1, \dots, k_4)$ be the quartet of cyclic cubic number fields sharing the common conductor $c = pqr$, belonging to Graph 8 of Category* III *with combined cubic residue symbol $[p, q, r]_3 = \{r \to p \leftrightarrow q \leftarrow r\}$.*

*To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $1 \le \mu \le 4$, let the **principal factor** of k_{pq}, respectively $\tilde{k}_{pq}$, be $A(k_{pq}) = p^mq^n$, respectively $A(\tilde{k}_{pq}) = p^{\tilde{m}}q^{\tilde{n}}$, and additionally assume the **regular** situation where both $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3,3)$ are elementary bicyclic, whence $(m,n) = (\tilde{m}, \tilde{n})$.*

*Then there are several **minimal transfer kernel types** (mTKT) $\varkappa_0$ of k_μ, $1 \le \mu \le 4$, and the other possible capitulation types in ascending order $\varkappa_0 < \varkappa' < \varkappa''$, ending in the mandatory $\varkappa'' = (2100)$, type* b.10*:*
either $\varkappa_0 = (2111)$, type H.4*, $\varkappa' = (2110)$, type* d.19*, for $\mathcal{P} = 1$, or $\varkappa_0 = (2133)$, type* F.11*, $\varkappa' = (2130)$, type* d.23*, or* (2103)*, type* d.25*, for $\mathcal{P} = 2$, and the second 3-class group is $\mathfrak{M} \simeq$*

$$(8.24)\qquad \begin{cases} \langle 729, 34..39\rangle,\ \alpha = [111,111,21,21],\ \varkappa = (2100) & \text{if } \mathcal{N} = 2, \\ \langle 729, 41\rangle,\ \alpha = [111,111,22,21],\ \varkappa = (2110) & \text{if } \mathcal{P} = 1,\ \mathcal{N} = 3, \\ \langle 729, 42|43\rangle,\ \alpha = [111,111,22,21],\ \varkappa = (2130)|(2140) & \text{if } \mathcal{P} = 2,\ \mathcal{N} = 3, \\ \langle 2187, 65|67\rangle,\ \alpha = [111,111,22,22],\ \varkappa = (2111) & \text{if } \mathcal{P} = 1,\ \mathcal{N} = 4, \\ \langle 2187, 66|73\rangle,\ \alpha = [111,111,22,22],\ \varkappa = (2133) & \text{if } \mathcal{P} = 2,\ \mathcal{N} = 4, \end{cases}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j,\ I_j = 27\}$ and $\mathcal{P}$ is the number of prime divisors of p^mq^n. In any case, the 3-class field tower may have a group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu)$ bigger than $\mathfrak{M}$ although $d_2(\mathfrak{M}) \le 4$. Further, III.8 *is the unique graph where the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$ **cannot be of maximal class**.*

*Since the group order cannot be specified in the **(super-)singular** situation, only the capitulation type can be given. Additionally, the number $\tilde{\mathcal{P}}$ of prime divisors of $p^{\tilde{m}}q^{\tilde{n}}$ is used, and two cases are separated.*
If $(m,n) = (\tilde{m}, \tilde{n})$, then $\mathcal{P} = \tilde{\mathcal{P}}$, and all four types $\varkappa(k_\mu)$, $\mu = 1, \dots, 4$, coincide:

$$(8.25)\qquad \begin{cases} \varkappa = (2100),\ \text{b.10} & \text{if } \mathcal{N} = 2, \\ \varkappa = (2111),\ \text{H.4} & \text{if } \mathcal{P} = 1,\ \mathcal{N} = 4, \\ \varkappa = (2133),\ \text{F.11} & \text{if } \mathcal{P} = 2,\ \mathcal{N} = 4. \end{cases}$$

If $(m,n)\neq(\tilde{m},\tilde{n})$*, then the number of coinciding types must be indicated by formal exponents:*

$$(8.26)\quad \begin{cases} \varkappa=(2100)^4,\ \text{b.10} & \text{if } \mathcal{N}=2,\\ \varkappa=(2130)^2,\ \text{d.23},\ \varkappa=(2140)^2,\ \text{d.25} & \text{if } \mathcal{N}=3,\\ \varkappa=(2112)^4,\ \text{F.7} & \text{if } \mathcal{P}=\tilde{\mathcal{P}}=1,\ \mathcal{N}=4,\\ \varkappa=(2134)^2,\ \text{G.16},\ \varkappa=(2143)^2,\ \text{G.19} & \text{if } \mathcal{P}=\tilde{\mathcal{P}}=2,\ \mathcal{N}=4,\\ \varkappa=(2131)^2,\ \text{F.12},\ \varkappa=(2113)^2,\ \text{F.13} & \text{if } \mathcal{P}\neq\tilde{\mathcal{P}},\ \mathcal{N}=4. \end{cases}$$

Proof. We normalize the transpositions in Table 18 by the following convention $\varkappa=(21**)\sim(*32*)\sim(**43)$, taking the leading type of three equivalent types.

Since the transfer kernels $\ker(T_{B_i/k_\mu})$ for the *tame* extensions with $i\in\{5,6,8,9\}$ are *partial*, the corresponding indices $I_i=(U_i:V_i)=27$ of subfield units must be maximal, whence necessarily $\mathcal{N}\geq 2$. The associated 3-class numbers $h_3(B_i)=27$ are consistent with occurrence of *two elementary tricyclic* 3-class groups $\mathrm{Cl}_3(B_i)\simeq(3,3,3)$, connected with a *transposition* in the capitulation type $\varkappa(k_\mu)=(21**)$, according to Proposition 12.

In the proof of this proposition, it was also derived that, due to $r_i=2$, the 3-class numbers of the *wild* extensions are given by $h_3(B_i)=\frac{I_i}{3}h_3(k_{pq})\geq 27$, for $i\in\{1,2,10\}$, and $h_3(B_i)=\frac{I_i}{3}h_3(\tilde{k}_{pq})\geq 27$, for $i\in\{3,4,7\}$, where $I_i=(U_i:V_i)\in\{9,27\}$, and $9\mid h_3(k_{pq})$, $9\mid h_3(\tilde{k}_{pq})$.

It follows that maximal class $\mathrm{cc}(\mathfrak{M})=1$ is prohibited for two reasons, firstly by the Artin pattern $\varkappa\sim(21\varkappa_3\varkappa_4)$, $\alpha\sim[111,111,\alpha_3,\alpha_4]$, and secondly by the bi-polarization of order at least 27, which implies that α_3 and α_4 are bicyclic equal to (21) or bigger [16, pp. 289–292].

In fact, even $\mathrm{cc}(\mathfrak{M})=2$ is very restricted, because the candidates for $\mathfrak{M}$ must be descendants of the group $\langle 243,3\rangle$. The other two groups with two or three components (111) in the abelian type invariants α are discouraged, since $\varkappa\sim(1133)$, $\alpha\sim[21,111,21,111]$ for $\langle 243,7\rangle$ does not contain a transposition, and in $\varkappa\sim(2111)$, $\alpha\sim[111,21,111,111]$ for $\langle 243,4\rangle$, the transposition in $\varkappa$ is not associated with two elementary tricyclic components of α.

If $\mathcal{N}=2$, then the Artin pattern $\alpha=[111,111,21,21]$, $\varkappa=(2100)$ identifies one of the six groups $\langle 729,34..39\rangle$, $\alpha=[111,111,21,21]$, since $\langle 243,3\rangle$ is forbidden by Corollary 4.

If $\mathcal{N}=3$, then generally $\alpha=[111,111,22,21]$, with bipolarization consisting of copolarization (21), i.e. coclass 2, and polarization (22), i.e., class 4. Now, if $\mathcal{P}=1$, then the capitulation type $\varkappa=(2110)\sim(2120)$ contains a repetition, which identifies the group $\langle 729,41\rangle$. On the other hand, if $\mathcal{P}=2$, then the capitulation type $\varkappa=(2130)$ either contains a fixed point, which gives $\langle 729,42\rangle$, or $\varkappa=(2140)$ neither contains a repetition nor a fixed point, which gives $\langle 729,43\rangle$.

If $\mathcal{N}=4$, then generally $\alpha=[111,111,22,22]$, but a finer distinction is provided by $\mathcal{P}$. If $\mathcal{P}=1$, then the capitulation type $\varkappa=(2111)\sim(2122)$ contains two repetitions and becomes nearly constant, which identifies the groups $\langle 2187,65|67\rangle$ of coclass 3. However, if $\mathcal{P}=2$, then a fixed point and its repetition occurs in the capitulation type $\varkappa=(2133)\sim(2144)$, which leads to the groups $\langle 2187,66|73\rangle$, $\alpha=[111,111,22,22]$.

Concerning the **(super-)singular** situation, two cases are distinguished. If $(m,n)=(\tilde{m},\tilde{n})$, then $\mathcal{P}=1$, i.e., $(m,n)\in\{(0,1),(1,0)\}$, implies two identical repetitions in $\varkappa_0\sim(2111)\sim(2122)$, H.4; but $\mathcal{P}=2$, i.e., $(m,n)\in\{(1,1),(1,2),(2,1)\}$, produces a single fixed point in $\varkappa_0\sim(2133)\sim(2144)$, F.11. These two minimal transfer kernel types for $\mathcal{N}=4$ both expand to $\varkappa''\sim(2100)$ for $\mathcal{N}=2$. All three cases are uniform. If $\mathcal{N}=3$ were possible, then $\mathcal{P}=1$ would lead to type $\varkappa\sim(2110)\sim(2120)$, d.19, and $\mathcal{P}=2$ would either imply type $\varkappa\sim(2130)$, d.23, or $\varkappa\sim(2140)$, d.25. The latter case would be non-uniform, but $\mathcal{N}=3$ does not seem to occur at all.

If $(m,n)\neq(\tilde{m},\tilde{n})$, then nevertheless $\mathcal{P}=\tilde{\mathcal{P}}$ is possible, and then $\mathcal{P}=1$ implies two distinct repetitions in $\varkappa\sim(2112)\sim(2121)$, F.7, uniformly, whereas $\mathcal{P}=2$ leads to either two fixed points in $\varkappa\sim(2134)$, G.16 or a second transposition in $\varkappa\sim(2143)$, G.19. These permutation types would be non-uniform in two sub-doublets, but they are obviously forbidden, for an unknown reason. Finally, $\mathcal{P}\neq\tilde{\mathcal{P}}$ admits several distinct realizations with identical result: it always leads to a repetition, and additionally either to a fixed point in $\varkappa\sim(2131)$, F.12, or a non-fixed point in $\varkappa\sim(2131)$, F.13, non-uniformly in two sub-doublets. □

Corollary 12. ***(Non-uniformity of the quartet for* III.8.*)*** *If $(m,n)=(\tilde{m},\tilde{n})$, in particular always in the* ***regular*** *situation, the components of the quartet, all with 3-rank two,* ***uniformly*** *share a common capitulation type $\varkappa(k_\mu)$, common abelian type invariants $\alpha(k_\mu)$, and a common second 3-class group $\mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, for $1\le\mu\le 4$.* ***Otherwise****, the invariants may be* ***non-uniform****, divided in two sub-doublets.*

Proof. In the *regular* case, we must have $(m,n)=(\tilde{m},\tilde{n})$. All TKTs are either equivalent to F.11, with mandatory fixed point, if $\mathfrak{p}^m\mathfrak{q}^n\in\{\mathfrak{p}\mathfrak{q},\mathfrak{p}\mathfrak{q}^2\}$, or to H.4, if $\mathfrak{p}^m\mathfrak{q}^n\in\{\mathfrak{p},\mathfrak{q}\}$, according to Table 18. The potential non-uniformity was proved in Theorem 18. □

In Table 19, we summarize the prototypes of Graph III.8 in the same way as in Table 13. The group with multifurcation of order four is abbreviated by $P_7:=\langle 2187,64\rangle$. See the table and tree diagram [18, § 11, pp. 96–100, Tbl. 1, Fig. 5].

TABLE 19. Prototypes for Graph III.8

No.	c	$r\to p\leftrightarrow q\leftarrow r$	v^*	v	m,n	$\tilde{v}$	$\tilde{m},\tilde{n}$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	20 293	$13\to 7\leftrightarrow 223\leftarrow 13$	1	2	2, 1			b.10	$\langle 729,37..39\rangle$	≥ 2
2	41 509	$31\to 13\leftrightarrow 103\leftarrow 31$	1	2	1, 1			b.10	$\langle 729,37..39\rangle$	≥ 2
3	46 341	$19\to 9\leftrightarrow 271\leftarrow 19$	2	2	0, 1			b.10	$\langle 729,37..39\rangle$	≥ 2
5	52 497	$19\to 9\leftrightarrow 307\leftarrow 19$	1	2	2, 1			F.11	$\langle 2187,66\vert 73\rangle$	≥ 2
7	92 911	$13\to 7\leftrightarrow 1021\leftarrow 13$	1	2	1, 1			F.11	$\langle 2187,66\vert 73\rangle$	≥ 2
18	191 007	$19\to 9\leftrightarrow 1117\leftarrow 19$	2	2	1, 0			b.10	$\langle 729,37..39\rangle$	≥ 2
26	231 469	$43\to 7\leftrightarrow 769\leftarrow 43$	4	3	0, 1	3	0, 1	H.4	$P_7-\#2;34\vert 35$	≥ 3
40	387 729	$9\to 67\leftrightarrow 643\leftarrow 9$	3	3	2, 1	3	2, 1	F.11	$P_7-\#2;36\vert 38$	≥ 2
92	756 499	$43\to 73\leftrightarrow 241\leftarrow 43$	3	3	0, 1	3	1, 1	F.12	$P_7-\#2;43\vert 46\vert 51\vert 53$	≥ 2
								F.13	$P_7-\#2;41\vert 47\vert 50\vert 52$	≥ 3
93	758 233	$7\to 19\leftrightarrow 5701\leftarrow 7$	3	3	1, 1	3	1, 1	F.11	$P_7-\#2;36\vert 38$	≥ 2
101	806 869	$7\to 73\leftrightarrow 1579\leftarrow 7$	5	3	1, 1	5	1, 1	F.11		≥ 2
105	831 001	$67\to 79\leftrightarrow 157\leftarrow 67$	4	3	1, 1	3	2, 1	d.23	$\langle 6561,678\rangle$	≥ 3
								d.25	$\langle 6561,679\vert 680\rangle$	≥ 3
102	945 117	$19\to 9\leftrightarrow 5527\leftarrow 19$	3	3	1, 0	3	1, 1	F.12	$P_7-\#2;43\vert 46\vert 51\vert 53$	≥ 2
								F.13	$P_7-\#2;41\vert 47\vert 50\vert 52$	≥ 3
162	1 301 287	$31\to 13\leftrightarrow 3229\leftarrow 31$	4	4	0, 1	3	2, 1	d.23		≥ 2
								d.25		≥ 2
164	1 305 937	$31\to 103\leftrightarrow 409\leftarrow 31$	6	4	1, 1	5	1, 1	F.11		≥ 2
183	1 463 917	$13\to 7\leftrightarrow 16087\leftarrow 13$	3	3	0, 1	3	1, 0	F.7	$P_7-\#2;55\vert 56\vert 58$	≥ 2
185	1 483 767	$19\to 9\leftrightarrow 8677\leftarrow 19$	3	3	2, 1	3	0, 1	F.12	$P_7-\#2;43\vert 46\vert 51\vert 53$	≥ 2
								F.13	$P_7-\#2;41\vert 47\vert 50\vert 52$	≥ 2
253	2 068 587	$19\to 9\leftrightarrow 12097\leftarrow 19$	5	3	1, 1	5	0, 1	F.12		≥ 2
								F.13		≥ 2
385	2 991 987	$19\to 9\leftrightarrow 17497\leftarrow 19$	3	3	1, 0	3	2, 1	F.12	$P_7-\#2;43\vert 46\vert 51\vert 53$	≥ 2
								F.13	$P_7-\#2;41\vert 47\vert 50\vert 52$	≥ 2
468	3 556 699	$97\to 37\leftrightarrow 991\leftarrow 97$	6	5	1, 1	3	2, 1	d.23		≥ 2
								d.25		≥ 2
651	4 686 019	$109\to 13\leftrightarrow 3307\leftarrow 109$	4	4	0, 1	3	1, 0	F.7	$P_7-\#2;55\vert 56\vert 58$	≥ 2

Example 8. Since III.8 is the graph with most sparse population by far, Ayadi [2, pp. 89–90] was unable to give any examples. We found many, but not all, prototypes. These are the minimal conductors for each scenario in Theorem 18. In the **regular** case, they have been found for $\mathcal{N}\in\{2,4\}$, but not for $\mathcal{N}=3$. There are some **regular** prototypes: $c=20\,293$ with symbol $\{13\to 7\leftrightarrow 223\leftarrow 13\}$, $v=1$, and $\mathfrak{M}=\langle 729,37..39\rangle$; $c=46\,341$ with symbol $\{19\to 9\leftrightarrow 271\leftarrow 19\}$, $v=2$, and $\mathfrak{M}=\langle 729,37..39\rangle$; $c=52\,497$ with symbol $\{19\to 9\leftrightarrow 307\leftarrow 19\}$, $v=1$, and $\mathfrak{M}=\langle 2187,66|73\rangle$. Furthermore, there is a **super-singular** prototype $c=231\,469$ with symbol $\{43\to 7\leftrightarrow 769\leftarrow 43\}$, type H.4, and $\mathfrak{M}=\langle 2187,64\rangle-\#2;i,\ i\in\{34,35\}$, with $d_2(\mathfrak{M})=5$, outside of the library [5], not treated by Theorem 18.

8.5. **Category III, Graph 9.** Let $(k_1,\ldots,k_4)$ be a quartet of cyclic cubic number fields sharing the common conductor $c=pqr$, belonging to Graph 9 of Category III with combined cubic residue symbol $[p,q,r]_3=\{r\leftarrow p\leftrightarrow q\leftarrow r\}$.

Proposition 13. ***(Quartet with* 3*-rank two for* III.9.*)*** *For fixed $\mu\in\{1,2,3,4\}$, let $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ be the prime ideals of k_μ over p,q,r, that is, $p\mathcal{O}_{k_\mu}=\mathfrak{p}^3$, $q\mathcal{O}_{k_\mu}=\mathfrak{q}^3$, $r\mathcal{O}_{k_\mu}=\mathfrak{r}^3$, then the* ***principal factor*** *of k_μ is $A(k_\mu)=p$, and the 3-class group of k_μ is*

$$\mathrm{Cl}_3(k_\mu)=\langle[\mathfrak{q}],[\mathfrak{r}]\rangle\simeq(3,3). \tag{8.27}$$

In terms of n and $\tilde{n}$ in $A(k_{pq}) = p^m q^n$ and $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, the ranks of the ***wild*** *extensions are*

(8.28) $\quad r_1 = r_2 = r_{10} = 3$ *iff* $n \neq 0$ *iff* $q \mid A(k_{pq})$ *and* $r_3 = r_4 = r_7 = 3$ *iff* $\tilde{n} \neq 0$ *iff* $q \mid A(\tilde{k}_{pq})$.

Proof. By Proposition 3, principal factors are $A(k_{pr}) = A(\tilde{k}_{pr}) = p$, since $r \leftarrow p$, and $A(k_{qr}) = A(\tilde{k}_{qr}) = r$, since $q \leftarrow r$. Further, by Proposition 4, $A(k_\mu) = p$, for all $1 \le \mu \le 4$, since p is universally repelling $r \leftarrow p \rightarrow q$. Since $\mathfrak{p} = \alpha\mathcal{O}_{k_\mu}$ is a principal ideal, its class $[\mathfrak{p}] = 1$ is trivial, whereas the classes $[\mathfrak{q}], [\mathfrak{r}]$ are non-trivial. By Corollary 3,
since $\mathfrak{q}$ is principal ideal in k_q, the class $[\mathfrak{q}]$ capitulates in $B_6 = k_1k_4k_q\tilde{k}_{pr}$ and $B_9 = k_2k_3k_qk_{pr}$;
since $\mathfrak{r}$ is principal ideal in k_r, the class $[\mathfrak{r}]$ capitulates in $B_7 = k_1k_2k_r\tilde{k}_{pq}$ and $B_{10} = k_3k_4k_rk_{pq}$.
However, since $\mathfrak{r}$ is principal ideal in k_{qr} and $\tilde{k}_{qr}$, the class $[\mathfrak{r}]$ also capitulates in $B_1 = k_1k_{pq}k_{pr}k_{qr}$, $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$, $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, $B_5 = k_1k_3k_p\tilde{k}_{qr}$, and $B_8 = k_2k_4k_pk_{qr}$.

For the wild bicyclic bicubic fields B_j, $j \in \{1,2,3,4,7,10\}$, the rank r_j is calculated with row operations on the associated principal factor matrices M_j:

$$M_1 = M_2 = \begin{pmatrix} 1 & 0 & 0 \\ m & n & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ m & n & 0 \end{pmatrix},\ M_3 = M_4 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{m} & \tilde{n} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \tilde{m} & \tilde{n} & 0 \end{pmatrix}.$$

For $B_1 = k_1k_{pq}k_{pr}k_{qr}$ and $B_2 = k_2k_{pq}\tilde{k}_{pr}\tilde{k}_{qr}$, $M_1 = M_2$ leads to the decisive pivot element n in the middle column, for $B_{10} = k_3k_4k_rk_{pq}$, M_{10} also leads to n. So rank $r_1 = r_2 = r_{10} = 3$ iff $n \neq 0$.

For $B_3 = k_3\tilde{k}_{pq}\tilde{k}_{pr}k_{qr}$ and $B_4 = k_4\tilde{k}_{pq}k_{pr}\tilde{k}_{qr}$, $M_3 = M_4$ leads to the decisive pivot element $\tilde{n}$ in the middle column, for $B_7 = k_1k_2k_r\tilde{k}_{pq}$, M_7 also leads to $\tilde{n}$. So rank $r_3 = r_4 = r_7 = 3$ iff $\tilde{n} \neq 0$. □

In terms of capitulation targets in Corollary 2, Proposition 13 and parts of its proof are now summarized in Table 20 with transpositions in **bold** font.

TABLE 20. Norm class groups and minimal transfer kernels for Graph III.9

Base	k_1				k_2				k_3				k_4			
Ext	B_1	B_5	B_6	B_7	B_2	B_7	B_8	B_9	B_3	B_5	B_9	B_{10}	B_4	B_6	B_8	B_{10}
NCG	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}^2$	$\mathfrak{qr}^2$	$\mathfrak{qr}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}^2$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{qr}$	$\mathfrak{qr}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{qr}^2$
TK	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{r}$
$\varkappa$	3	**3**	**2**	3	4	4	**4**	**3**	3	**3**	**2**	3	2	**3**	**2**	2

Theorem 19. ***(Second 3-class group for*** **III.9*.)*** *To identify the second 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, $1 \le \mu \le 4$, let the* ***principal factor*** *of k_{pq}, respectively $\tilde{k}_{pq}$, be $A(k_{pq}) = p^m q^n$, respectively $A(\tilde{k}_{pq}) = p^{\tilde{m}} q^{\tilde{n}}$, and additionally assume the* ***regular*** *situation where both $\mathrm{Cl}_3(k_{pq}) \simeq \mathrm{Cl}_3(\tilde{k}_{pq}) \simeq (3,3)$ are elementary bicyclic, whence $(m,n) = (\tilde{m},\tilde{n})$.*

Then the ***minimal transfer kernel type*** *(mTKT) $\varkappa_0$ of k_μ, $1 \le \mu \le 4$, and other possible capitulation types in ascending order $\varkappa_0 < \varkappa' < \varkappa'' < \varkappa'''$, ending in the mandatory $\varkappa''' = (2000)$, type* a.3**, are $\varkappa_0 = (2111)$, type* H.4*, $\varkappa' = (2110)$, type* d.19*, $\varkappa'' = (2100)$, type* b.10*, and the second 3-class group is $\mathfrak{M} \simeq$*

$$(8.29)\qquad \begin{cases} \langle 81,7\rangle,\ \alpha = [111,11,11,11],\ \varkappa = (2000) & \text{if } n \neq 0,\ \mathcal{N} = 1, \\ \langle 729,34..39\rangle,\ \alpha = [111,111,21,21],\ \varkappa = (2100) & \text{if } n = 0,\ \mathcal{N} = 2, \\ \langle 729,41\rangle,\ \alpha = [111,111,22,21],\ \varkappa = (2110) & \text{if } n = 0,\ \mathcal{N} = 3, \\ \langle 2187,65|67\rangle,\ \alpha = [111,111,22,22],\ \varkappa = (2111) & \text{if } n = 0,\ \mathcal{N} = 4, \end{cases}$$

where $\mathcal{N} := \#\{1 \le j \le 10 \mid k_\mu < B_j,\ I_j = 27\}$. Only in the first case, the 3-class field tower has certainly the group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(k_\mu)/k_\mu) \simeq \mathfrak{M}$ and length $\ell_3(k_\mu) = 2$, otherwise $\ell_3(k_\mu) \ge 3$ cannot be excluded, even if $d_2(\mathfrak{M}) \le 4$.

Proof. The essence of the proof is a systematic evaluation of the facts proved in Proposition 13 and illustrated by Table 20, ordered by increasing indices $I_j := (U_j : V_j)$ of subfield units and, accordingly, by Lemma 2, shrinking transfer kernels $\ker(T_{B_j/k_\mu})$, with $1 \le j \le 10$, $1 \le \mu \le 4$.

(1) For the maximal TKT, $\varkappa''' \sim (2000)$, called a.3* in conjunction with ATI $[111, 11, 11, 11]$, we must have $n \ne 0$, $\tilde{n} \ne 0$ and by (8.28) *wild* ranks $r_j = 3$ and indices $I_j = 3$ for all $j = 1, 2, 3, 4, 7, 10$, causing eight (because 7 is used twice over k_1 and k_2 and 10 is used twice over k_3 and k_4) minimal 3-class numbers $h_3(B_j) = h_3(k_{pq}) = h_3(\tilde{k}_{pq}) = 9$, by (8.3), and ATI $\mathrm{Cl}_3(B_j) \simeq (11)$, characterisic for a group of coclass $\mathrm{cc}(\mathfrak{M}) = 1$, i.e. maximal class, namely $\mathfrak{M} \simeq \langle 81, 7\rangle$. However, the elementary tricyclic component (111) of the ATI requires *tame* indices $I_j = 27$ for $j = 5, 8$, and thus $\mathcal{N} = 1$ for each $1 \le \mu \le 4$ (because 5 is used twice over k_1 and k_3 and 8 is used twice over k_2 and k_4).

(2) Next, one of the total TK shrinks to a transposition, $\varkappa'' \sim (2100)$, b.10, which requires a group of coclass $\mathrm{cc}(\mathfrak{M}) \ge 2$, implying, firstly, *tame* indices $I_j = 27$ also for $j = 6, 9$, and thus $\mathcal{N} = 2$ for each $1 \le \mu \le 4$ (because 6 is used twice over k_1 and k_4 and 9 is used twice over k_2 and k_3), and, secondly, (from now on) necessarily both $n = \tilde{n} = 0$, implying *wild* ranks $r_j = 2$ indices $I_j \in \{9, 27\}$ for all $j = 1, 2, 3, 4, 7, 10$, here $I_j = 9$, 3-class numbers $h_3(B_j) = 3 \cdot h_3(k_{pq}) = 3 \cdot h_3(\tilde{k}_{pq}) = 3 \cdot 9 = 27$, and thus ATI $\alpha \sim [111, 111, 21, 21]$, leading to $\mathfrak{M} \simeq \langle 729, 34..39\rangle$, in view of Corollary 4.

(3) Now another total TK shrinks to a repetition, $\varkappa' \sim (2110)$, d.19, the first three wild indices for $j = 1, 2, 10$ become maximal $I_j = 27$, causing $\mathcal{N} = 3$ and four (because 10 is used twice over k_3 and k_4) maximal new 3-class numbers $h_3(B_j) = 9 \cdot h_3(\tilde{k}_{pq}) = 9 \cdot 9 = 81$, and thus ATI $\alpha \sim [111, 111, 22, 21]$, uniquely identifying $\mathfrak{M} \simeq \langle 729, 41\rangle$.

(4) Finally, for the minimal TKT, $\varkappa_0 \sim (2111)$, H.4, the remaining three wild indices for $j = 3, 4, 7$ become maximal $I_j = 27$, causing $\mathcal{N} = 4$ and four (because 7 is used twice over k_1 and k_2) maximal new 3-class numbers $h_3(B_j) = 9 \cdot h_3(\tilde{k}_{pq}) = 9 \cdot 9 = 81$, and thus ATI $\alpha \sim [111, 111, 22, 22]$, enforcing a group of coclass $\mathrm{cc}(\mathfrak{M}) = 3$ namely $\mathfrak{M} \simeq \langle 2187, 65|67\rangle$.

□

Corollary 13. ***(Uniformity of the quartet for* III.9*.)*** *The components of the quartet, all with 3-rank two, share a common capitulation type $\varkappa(k_\mu)$, common abelian type invariants $\alpha(k_\mu)$, and a common second 3-class group $\mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$, for $1 \le \mu \le 4$.*

Proof. This follows immediately from Theorem 19. □

Example 9. Prototypes for Graph III.9 are the minimal conductors for each scenario in Theorem 19. They have been found for all $\mathcal{N} \in \{1, 2, 3, 4\}$.

There are **regular** cases: $c = 16\,471$ with symbol $\{13 \leftarrow 181 \leftrightarrow 7 \leftarrow 13\}$, $v^* = 1$, and $\mathfrak{G} = \mathfrak{M} = \langle 81, 7\rangle$; $c = 89\,487$ with symbol $\{9 \leftarrow 163 \leftrightarrow 61 \leftarrow 9\}$, $v^* = 2$, and $\mathfrak{M} = \langle 729, 41\rangle$; $c = 109\,291$ with symbol $\{7 \leftarrow 13 \leftrightarrow 1201 \leftarrow 7\}$, $v^* = 2$, and $\mathfrak{M} = \langle 729, 34..36\rangle$; $c = 193\,921$ with symbol $\{7 \leftarrow 13 \leftrightarrow 2131 \leftarrow 7\}$, $v^* = 2$, and $\mathfrak{M} = \langle 729, 37..39\rangle$; and, with **extreme statistic delay**, $c = 707\,517$ with ordinal number 145, symbol $\{9 \leftarrow 127 \leftrightarrow 619 \leftarrow 9\}$, $v^* = 2$, and $\mathfrak{M} = \langle 2187, 65|67\rangle$ with $d_2(\mathfrak{M}) = 5$.

Only one **super-singular** case for $c < 2 \cdot 10^5$: It is $c = 197\,239$ with symbol $7 \leftarrow 1483 \leftrightarrow 19 \leftarrow 7$, $v^* = 4$, and $\mathfrak{M} = \langle 729, 37..39\rangle$. Astonishingly, no bigger order and coclass of $\mathfrak{M}$, due to $n \ne 0$.

In Table 21, we summarize the prototypes of Graph III.9 in the same way as in Table 13.

TABLE 21. Prototypes for Graph III.9

No.	c	$r \leftarrow p \leftrightarrow q \leftarrow r$	v^*	v	m, n	$\tilde{v}$	$\tilde{m}, \tilde{n}$	capitulation type	$\mathfrak{M}$	$\ell_3(k)$
1	16 471	$13 \leftarrow 181 \leftrightarrow 7 \leftarrow 13$	1	2	1, 1	2	1, 1	a.3*	$\langle 81, 7\rangle$	$= 2$
15	89 487	$9 \leftarrow 163 \leftrightarrow 61 \leftarrow 9$	2	2	1, 0	2	1, 0	d.19	$\langle 729, 41\rangle$	≥ 2
19	109 291	$7 \leftarrow 13 \leftrightarrow 1201 \leftarrow 7$	2	2	1, 0	2	1, 0	b.10	$\langle 729, 34..36\rangle$	≥ 2
28	193 921	$7 \leftarrow 13 \leftrightarrow 2131 \leftarrow 7$	2	2	1, 0	2	1, 0	b.10	$\langle 729, 37..39\rangle$	≥ 2
31	197 239	$7 \leftarrow 1483 \leftrightarrow 19 \leftarrow 7$	4	3	0, 1	3	0, 1	b.10	$\langle 729, 37..39\rangle$	≥ 2
145	707 517	$9 \leftarrow 127 \leftrightarrow 619 \leftarrow 9$	2	2	1, 0	2	1, 0	H.4	$\langle 2187, 65\|67\rangle$	≥ 3

9. Conclusions

In this work, we have seen that order and structure of the **second** 3-class group $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k)/k)$ of a cyclic cubic number field k with conductor $c = pqr$ divisible by three prime(power)s p, q, r and elementary bicyclic 3-class group $\mathrm{Cl}_3(k) \simeq (3,3)$ depends on arithmetical invariants of other cyclic cubic **auxiliary fields**, associated with k. The field k is component of a quartet $(k_1, \ldots, k_4)$ of cyclic cubic fields sharing the common conductor c. The graph $[p, q, r]_3$ which is combined by the cubic residue symbols $\left(\frac{p}{q}\right)_3, \left(\frac{q}{p}\right)_3, \left(\frac{p}{r}\right)_3, \left(\frac{r}{p}\right)_3, \left(\frac{q}{r}\right)_3, \left(\frac{r}{q}\right)_3$ decides whether one, or two, or no, component(s) of the quartet have a 3-class group of rank $\varrho(k_\mu) = 3$, and accordingly the conductor $c = pqr$ is called of Category I, or II, or III. For Category I, the order of the 3-class group of the unique component with $\varrho(k_{\mu_0}) = 3$ is crucial. For Category II, the orders of both 3-class groups of the two components with $\varrho(k_{\mu_1}) = \varrho(k_{\mu_2}) = 3$ exert an impact. For Category III, the behavior is uniform with abelian $\mathfrak{M} \simeq (3,3)$, if $[p, q, r]_3$ does not contain mutual cubic residues (Graphs 1–4), otherwise there is exactly one pair $p \leftrightarrow q$ of mutual cubic residues (Graphs 5–9), and the auxiliary fields with decisive 3-class groups are the two subfields k_{pq} and $\tilde{k}_{pq}$ of the absolute genus field k^* of k, having the partial conductor pq. In each case, the **principal factors** (norms of ambiguous principal ideals) determine the fine structure in form of uniform or non-uniform second 3-class groups $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$. Explicit numerical investigations indicate that there is no upper bound for the orders of the 3-class groups $\mathrm{Cl}_3(k_{\mu_0})$, respectively $\mathrm{Cl}_3(k_{\mu_1})$ and $\mathrm{Cl}_3(k_{\mu_2})$, respectively $\mathrm{Cl}_3(k_{pq})$ and $\mathrm{Cl}_3(\tilde{k}_{pq})$. In the regular situation, these orders are 27, respectively 9, in the singular situation, they are 81, respectively 27, but in the super-singular situation, they are at least 243, respectively 27, and the **orders may increase unboundedly**. Concrete numerical examples are known with orders up to 729.

Bicyclic bicubic fields B_j, $j = 1, \ldots, 10$, constitute the **capitulation targets** of the cyclic cubic fields k_μ, $\mu = 1, \ldots, 4$. The introduction of important new concepts, the **minimal and maximal capitulation type** (mTKT), $\varkappa_0$ and $\varkappa_\infty$, permitted recognition of common patterns for several Graphs, partially in distinct Categories.

The four Graphs II.1, II.2, III.7, III.9 share the same ordered sequence of TKTs, $\varkappa_0 \sim (2111) < (2110) < (2100) < (2000) \sim \varkappa_\infty$, called H.4, d.19, b.10, a.3*, although the proofs and details are quite different. In terms of splitting prime ideals $\mathfrak{q}\mathcal{O}_{B_j} = \mathfrak{Q}_1\mathfrak{Q}_2\mathfrak{Q}_3$, $\mathfrak{r}\mathcal{O}_{B_\ell} = \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$, all these TKTs contain a crucial **transposition**, due to elementary tricyclic 3-class groups $\mathrm{Cl}_3(B_j) = \langle[\mathfrak{Q}_1,][\mathfrak{Q}_2,][\mathfrak{Q}_3]\rangle$, $\mathrm{Cl}_3(B_\ell) = \langle[\mathfrak{R}_1,][\mathfrak{R}_2,][\mathfrak{R}_3]\rangle$, and twisted capitulation kernels $\ker(T_{B_j/k_\mu}) = \langle[\mathfrak{r}]\rangle$, $\ker(T_{B_\ell/k_\mu}) = \langle[\mathfrak{q}]\rangle$, which restricts the group $\mathfrak{M}$ to descendants of $\langle 243, 3\rangle$ (except $\langle 81, 7\rangle$, where a total transfer kernel hides the transposition).

Similarly, the two graphs I.1, I.2 admit another characteristic ordered sequence of TKTs, $\varkappa_0 \sim (4231) < (0231) < (0200), (0001) < (0000) \sim \varkappa_\infty$, called G.16, c.21, a.2, a.3, a.1, with **two fixed points**, which restrict the group $\mathfrak{M}$ to descendants of $\langle 243, 8\rangle$ (except $\langle 81, 8\rangle$, $\langle 81, 10\rangle$, $\langle 243, 25\rangle$, $\langle 243, 27\rangle$, where total transfer kernels partially or completely hide the fixed points).

A remarkable outsider is Graph III.8 with a veritable wealth of exotic capitulation types, but restricted to the unusual maximal TKT $\varkappa_\infty \sim (2100)$, b.10, forced by mandatory transposition.

Due to the lack of cubic residue conditions between the prime divisors of the conductor $c = pqr$, two Graphs I.1, III.5 admit the absolute maximum of all TKTs $\varkappa = (0000)$ (non-abelian!).

It might be worth one's while to point out that a glance at α_2 in Tables 2 and 3 reveals that the commutator subgroup of all encountered second 3-class groups $\mathfrak{M}$, respectively 3-class tower groups $\mathfrak{G}$, has order $\#(\mathfrak{M}') \geq 9$, respectively $\#(\mathfrak{G}') \geq 9$, which means that the class number of the Hilbert 3-class field $\mathrm{F}_3^1(k)$ is divisible by 9, for all cyclic cubic fields k, with the exception of $t = 1$, the regular cases for $t = 2$, and the Graphs $1, \ldots, 4$ of Category III for $t = 3$.

For Category I and II, we expect a rather rigid impact of the groups $\mathfrak{M} = \mathrm{Gal}(\mathrm{F}_3^2(k_\mu)/k_\mu)$ for $\mathrm{Cl}_3(k_\mu) \simeq (3,3)$ on the groups $\mathrm{Gal}(\mathrm{F}_3^2(k_\nu)/k_\nu)$ for $\mathrm{Cl}_3(k_\nu) \simeq (3,3,3)$, as suggested by the numerous tables in [20]. This research line will be pursued further in a forthcoming paper.

10. Acknowledgements

Both authors gratefully acknowledge computational aid by Bill Allombert, who used the algorithm of Aurel Page [21] to determine the 3-class groups of Hilbert 3-class fields and thus α_2 in Corollary 4. The second author acknowledges that his research was supported by the Austrian Science Fund (FWF): projects J0497-PHY and P26008-N25, and by the Research Executive Agency of the European Union (EUREA): project Horizon Europe 2021–2027.

References

[1] J. A. Ascione, G. Havas, and C. R. Leedham-Green, *A computer aided classification of certain groups of prime power order*, Bull. Austral. Math. Soc. **17** (1977), 257–274, microfiche supplement p. 320.

[2] M. Ayadi, *Sur la capitulation des 3-classes d'idéaux d'un corps cubique cyclique*, Thèse de doctorat, Université Laval, Québec, 1995.

[3] M. Ayadi, A. Azizi, and M. C. Ismaili, *The capitulation problem for certain number fields*, Advanced Studies in Pure Mathemetics **30** (2001), Class Field Theory – Its Centenary and Prospect, pp. 467–482.

[4] P. Barrucand and H. Cohn, Remarks on principal factors in a relative cubic field, *J. Number Theory* **3** (1971), No. 2, 226–239.

[5] H. U. Besche, B. Eick, and E. A. O'Brien, *The SmallGroups Library – a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in Magma.

[6] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[7] W. Bosma, J. Cannon, C. Fieker, and A. Steel (eds.), *Handbook of Magma functions*, Ed. 2.28, Sydney, 2023.

[8] A. Derhem, *Sur les corps cubiques cycliques de conducteur divisible par deux premiers*, Manuscript, Casablanca, 2002.

[9] C. Fieker, *Computing class fields via the Artin map*, Math. Comp. **70** (2001), No. 235, 1293–1303.

[10] G. Gamble, W. Nickel, and E. A. O'Brien, *ANU p-Quotient — p-Quotient and p-Group Generation Algorithms*, 2006, an accepted GAP package, available also in MAGMA.

[11] G. Gras, *Sur les ℓ-classes d'idéaux dans les extensions cycliques relatives de degré premier ℓ*, Ann. Inst. Fourier, Grenoble **23** (1973), No. 4, 1–44.

[12] Magma Developer Group, *Magma Computational Algebra System*, Version 2.28-5, Sydney, 2023, available from `http://magma.maths.usyd.edu.au`.

[13] Magma Developer Group, *Magma, Data for groups of order* 3^8, `data3to8.tar.gz`, Sydney, 2012, available from `http://magma.maths.usyd.edu.au`.

[14] D. C. Mayer, *Multiplicities of dihedral discriminants*, Math. Comp. **58** (1992), No. 198, 831–847 and S55–S58.

[15] D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), No. 3–4, 467–495.

[16] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015) No. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015.

[17] D. C. Mayer, *New number fields with known p-class tower*, Tatra Mt. Math. Pub. **64** (2015), 21–57, DOI 10.1515/tmmp-2015-0040, Special Issue on Number Theory and Cryptology '15.

[18] D. C. Mayer, *Co-periodicity isomorphisms between forests of finite p-groups*, Adv. Pure Math. **8** (2018) No. 1, 77–140, DOI 10.4236/apm.2018.81008, Special Issue on Group Theory, January 2018.

[19] D. C. Mayer, *Pattern recognition via Artin transfers applied to class field towers*, 3rd International Conference on Mathematics and its Applications (ICMA) 2020, Faculté des Sciences d' Ain Chock Casablanca (FSAC), Université Hassan II, Casablanca, Morocco, invited keynote 28 February 2020, available from `http://www.algebra.at/DCM@ICMA2020Casablanca.pdf`.

[20] D. C. Mayer, *Theoretical and experimental approach to p-class field towers of cyclic cubic number fields*, Sixièmes Journées d'Algèbre, Théorie des Nombres et leurs Applications (JATNA) 2022, Faculté des Sciences, Université Mohammed Premier, Oujda, Morocco, four invited keynotes 25–26 November 2022, available from `http://www.algebra.at/CyclicCubicTheoryAndExperiment.pdf`.

[21] A. Page, *abelianbnf*, Pari/gp script, 2020, Institute de Mathématiques de Bordeaux (IMB), Lithe and fast algorithmic number theory (LFANT), available from `https://inria.hal.science/hal-02961482`.

[22] C. J. Parry, *Bicyclic Bicubic Fields*, Canad. J. Math. **42** (1990), No. 3, 491–507.

[23] A. Scholz und O. Taussky, *Die Hauptideale der kubischen Klassenkörper imaginär quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluß auf den Klassenkörperturm*, J. Reine Angew. Math. **171** (1934), 19–41.

[24] I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

[25] O. Taussky, *Über eine Verschärfung des Hauptidealsatzes für algebraische Zahlkörper*, J. Reine Angew. Math. **168** (1932), 193–210.

[26] O. Taussky-Todd, *Arnold Scholz zum Gedächtnis*, Math. Nachrichten **7** (1952), 379–386.

Algebraic theories and applications research team (ATA). Ecole Normale Supèrieure of Moulay Ismail University (ENS-UMI). ADDRESS: ENS, B.P. 3104, TOULAL, MEKNÈS, MOROCCO
E-mail address: s.aouissi@umi.ac.ma

NAGLERGASSE 53, 8010 GRAZ, AUSTRIA
E-mail address: algebraic.number.theory@algebra.at
URL: http://www.algebra.at