FIRST EXCITED STATE WITH MODERATE RANK DISTRIBUTION

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ABSTRACT. Evidence is provided for the existence of infinite periodic sequences of Schur σ-groups G with commutator quotient $G/G' \simeq C_{3^e} \times C_3$, $e \geq 7$, and logarithmic order $\mathrm{lo}(G) = 10 + e$. With respect to their maximal subgroups $H_1, \ldots, H_3; H_4$, they have *moderate* rank distribution $\varrho(G) = (\mathrm{rank}_3(H_i/H_i'))_{1 \leq i \leq 4} \sim (2, 2, 3; 3)$ and represent the *first excited state* of their punctured transfer kernel types $\varkappa(G)$, which is characterized by a polarized component of the abelian quotient invariants $\alpha_1(G) = (H_i/H_i')_{1 \leq i \leq 4}$ with $\mathrm{lo} = 6 + e$ in contrast to the ground state with $\mathrm{lo} = 4 + e$.

1. INTRODUCTION

This is the third (and last) of a series of three articles devoted to periodic sequences of Schur σ-groups G [13, 1, 9, 8] with bicyclic commutator quotients $G/G' \simeq C_{3^e} \times C_3$ having one non-elementary component with logarithmic exponent $e \geq 2$. The periodicity appears in the shape of an infinite chain of immediate p-descendants of finite 3-groups with variable $e \geq e_0$ bigger than a starting value. The Schur σ-groups arise as leaves of finite twigs with constant structure emanating from the vertices of the chain.

In the first article [22] of the trilogy, periodicity of pairs of *metabelian* Schur σ-groups sets in with $e_0 = 3$, and the *ground state* of non-metabelian Schur σ-groups G with moderate rank distribution $\varrho(G) \in \{(2, 2, 2; 3), (2, 2, 3; 3)\}$ begins to become periodic for $e_0 = 5$. The typical bifurcation between G and its metabelianization $M = G/G''$ *degenerates* to a simple p-descendant relation, already for $e \geq 4$, i.e., the siblings topology becomes a child topology [19].

The primary motivation for the investigations in the present article was the question what will happen with the more complicated fork topology between G and $M = G/G''$ for the *first excited state* of non-metabelian Schur σ-groups G with moderate rank distribution $\varrho(G) \in \{(2, 2, 2; 3), (2, 2, 3; 3)\}$, which was conjectured to become periodic for $e_0 = 7$ in the conclusion of [22, § 12]. We shall see that $e_0 = 7$ is confirmed, and the bifurcation degenerates to an iterated p-descendant relation, already for $e \geq 6$.

The most difficult situation of non-metabelian Schur σ-groups G with *elevated* rank distribution $\varrho(G) = (3, 3, 3; 3)$ was completely clarified in [23]. Periodicity sets in with $e_0 = 9$ and the extremely complicated fork topology freezes to a *common bifurcation of infinite order* for all values $e \geq 4$. An additional complication with decisive negative impact on experimental arithmetical realizations by 3-class tower groups $G \simeq \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of imaginary quadratic number fields K is the requirement of logarithmic abelian quotient invariants $\alpha_2(G)$ of second order, whereas in the present article and in [22] we have the immense benefit that invariants $\alpha_1(G)$ of first order suffice.

Date: Wednesday, 13 October 2021.

2010 *Mathematics Subject Classification.* Primary 20D15, 20E18, 20E22, 20F05, 20F12, 20F14; secondary 11R37, 11R32, 11R11, 11R20, 11R29, 11Y40 .

Key words and phrases. pro-3 groups, finite 3-groups, BCF-groups, generator rank, relation rank, Schur σ-groups, balanced presentation, extremal root path principle, low index normal subgroups, kernels of Artin transfers, abelian quotient invariants of first order, moderate rank distribution, p-group generation algorithm, descendant trees, antitony principle; three-stage Hilbert 3-class field towers, maximal unramified pro-3 extensions, unramified cyclic cubic extensions, imaginary quadratic fields, non-elementary bicyclic 3-class groups, Galois action, punctured capitulation types, minimal discriminants.

Research supported by the Austrian Science Fund (FWF): projects J0497-PHY, P26008-N25, and by EUREA.

2. Main Theorems: periodic Schur σ-groups

In order to emphasize logical independence, we partition our main result into existence, uniqueness, explicit construction in virtue of periodicity, and structural invariants.

Theorem 1. ***(Existence Theorem.)*** *For each logarithmic exponent $e \ge 2$, and for each of three punctured transfer kernel types* (pTKT)*,*

$$\text{D.5},\ \varkappa(G) \sim (112;3),\quad \text{C.4},\ \varkappa(G) \sim (113;3),\quad \text{D.10},\ \varkappa(G) \sim (114;3), \tag{1}$$

there exists a unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order $\mathrm{lo}(G) = 10 + e$.

Observe that existence is warranted also in the pre-periodic range $2 \le e \le 6$.

Theorem 2. ***(Periodicity Theorem.)*** *For each logarithmic exponent $e \ge 7$, the unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 10+e$ *is given explicitly by the periodic sequence of descendants (notation according to* [5, 11]*)*

$$G \simeq W_\ell[-\#1;1]^{e-7} - \#1;i - \#1;1 - \#1;1, \quad i \in \{2,3\} \tag{2}$$

with periodic root $W_\ell = \mathrm{SmallGroup}(6561, 93) - \#2;1 - \#2;1 - \#2;\ell$*, where*

$$\begin{aligned} &\ell = 2 \text{ for type D.10},\ \varkappa(G) \sim (114;3),\\ &\ell = 4 \text{ for type C.4},\ \varkappa(G) \sim (113;3),\\ &\ell = 5 \text{ for type D.5},\ \varkappa(G) \sim (112;3).\end{aligned} \tag{3}$$

The metabelianization of G is given by

$$M = G/G'' \simeq W_\ell[-\#1;1]^{e-7} - \#1;i, \quad i \in \{2,3\}, \tag{4}$$

of log order $\mathrm{lo}(G) = 8 + e$*, in fact, G'' is cyclic of order 9 and is contained in the center of G.*

Theorem 3. ***(Structure Theorem.)*** *For each logarithmic exponent $e \ge 4$, the unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 10+e$ *possesses logarithmic abelian quotient invariants of first order*

$$\begin{aligned} &\alpha_1(G) = [(e+1)1, (e+1)1, e33; e11] \text{ for type D.10, and}\\ &\alpha_1(G) = [(e+1)1, (e+1)1, (e+1)32; e11] \text{ for type C.4 and D.5.}\end{aligned} \tag{5}$$

thus, in both cases, moderate rank distribution $\varrho(G) \sim (2,2,3;3)$, and soluble length $\mathrm{sl}(G) = 3$.

Observe that the periodic invariants partially also occur in the pre-periodic range $4 \le e \le 6$, but in different form for $2 \le e \le 3$. The proof will be developed in § 5, illustrated by Figure 1.

3. Pre-periodic Schur σ-groups

Since periodicity in Theorem 2 sets in with $e_0 = 7$, we must provide supplementary results for the pre-periodic range $2 \le e \le 6$. They are also justified in § 5 and illustrated by Figure 1.

Theorem 4. *The unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_9 \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 12$ *has first abelian quotient invariants* $\alpha_1(G) \sim (31, 31, 431; 211)$ *and is given by* $G \simeq \mathrm{SmallGroup}(2187, 168) - \#2;7 - \#1;4 - \#2;i$ *with metabelianization* $M \simeq \mathrm{SmallGroup}(2187, 168) - \#1;7 - \#1;4 - \#1;i$*, where $i \in \{2,9\}$ for type* D.5*, $i \in \{3,8\}$ for type* C.4*, and $i \in \{5,6\}$ for type* D.10*. Thus,* $F = \mathrm{SmallGroup}(2187, 168)$ *is fork between M and G.*

Theorem 5. *The unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{27} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 13$ *has first abelian quotient invariants* $\alpha_1(G) \sim (41, 41, 432; 311)$ *and is given by* $G \simeq \mathrm{SmallGroup}(6561, 98) - \#2;1 - \#1;1 - \#2;i$ *with metabelianization* $M \simeq \mathrm{SmallGroup}(6561, 98) - \#1;3 - \#1;1 - \#1;i$*, where $i \in \{2,3\}$ for type* D.10*, $i \in \{5,9\}$ for type* C.4*, and $i \in \{6,8\}$ for type* D.5*. Thus,* $F = \mathrm{SmallGroup}(6561, 98)$ *is the fork between M and G.*

Theorem 6. *The unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{81} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 14$ *has first abelian quotient invariants* $\alpha_1(G) \sim (51, 51, 532; 411)$ *or* $\alpha_1(G) \sim (51, 51, 433; 411)$ *and is given by* $G \simeq F - \#2; 1 - \#2; i$ *with metabelianization* $M \simeq F - \#1; 2 - \#1; i$*, in terms of the fork* $F = \mathrm{SmallGroup}(6561, 93) - \#2; 6$*, where* $i \in \{2, 3\}$ *for type* D.10*,* $i \in \{5, 9\}$ *for type* C.4*, and* $i \in \{6, 8\}$ *for type* D.5.

Theorem 7. *The unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{243} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 15$ *has first abelian quotient invariants* $\alpha_1(G) \sim (61, 61, 632; 511)$ *or* $\alpha_1(G) \sim (61, 61, 533; 511)$ *and is given by* $G \simeq F - \#2; i - \#1; 1$ *with metabelianization* $M \simeq F - \#1; (i + 1)$*, in terms of the fork* $F = \mathrm{SmallGroup}(6561, 93) - \#2; 1 - \#2; 6$*, where* $i \in \{2, 3\}$ *for type* D.10*,* $i \in \{5, 9\}$ *for type* C.4*, and* $i \in \{6, 8\}$ *for type* D.5.

Theorem 8. *The unique pair of Schur σ-groups G with commutator quotient $G/G' \simeq C_{729} \times C_3$ and logarithmic order* $\mathrm{lo}(G) = 16$ *has first abelian quotient invariants* $\alpha_1(G) \sim (71, 71, 732; 611)$ *or* $\alpha_1(G) \sim (71, 71, 633; 611)$ *and is given by* $G \simeq M - \#1; 1 - \#1; 1$ *as an iterated p-descendant of the metabelianization* $M \simeq \mathrm{SmallGroup}(6561, 93) - \#2; 1 - \#2; 1 - \#2; i$*, where* $i \in \{7, 8\}$ *for type* D.10*,* $i \in \{10, 14\}$ *for type* C.4*, and* $i \in \{11, 13\}$ *for type* D.5.

4. Arithmetical realizations

The automorphism group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of the maximal unramified pro-3 extension $\mathrm{F}_3^\infty(K)$ of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with negative fundamental discriminant $d < 0$ must be a Schur σ-group [13, 9, 8].

With the aid of the computational algebra system Magma [6, 7, 14], we conducted a search for fundamental discriminants $-10^9 < d < 0$ such that the 3-class group $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$ is non-elementary bicyclic with $2 \le e \le 8$, and the capitulation in the four unramified cyclic cubic extensions of K is the first excited state of one of the three types C.4, D.5, D.10 under investigation or of the closely related type D.6. The class field routines by Fieker [10] were employed.

In order to enable comparison with analogous cases of imaginary quadratic number field K with elementary bicyclic 3-class group $\mathrm{Cl}_3(K) \simeq C_3 \times C_3$, we begin with a recall of arithmetical information in [17, Fig. 1–2, pp. 24–25] concerning the first excited state of capitulation types in section E which are characterized by the polarization 43 of $\alpha_1(K)$. The rank distribution is either $\varrho(K) \sim (2, 3, 2, 2)$ for the former two types or $\varrho(K) = (2, 2, 2, 2)$ for the latter two types.

Example 1. For $e = 1$, the hits with absolutely minimal discriminants are
$d = -262\,744$ for type E.14, $\varkappa(K) \sim (3122)$, $\alpha_1(K) \sim (43, 111, 21, 21)$,
$d = -268\,040$ for type E.6, $\varkappa(K) \sim (1122)$, $\alpha_1(K) \sim (43, 111, 21, 21)$,
$d = -297\,079$ for type E.9, $\varkappa(K) \sim (2334)$, $\alpha_1(K) \sim (21, 43, 21, 21)$,
$d = -370\,740$ for type E.8, $\varkappa(K) \sim (2234)$, $\alpha_1(K) \sim (21, 43, 21, 21)$.

Now we come to examples for non-elementary bicyclic 3-class groups $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$, $e \ge 2$. For $e = 2$, the polarization of $\alpha_1(K)$ is irregular but uniform for all types. It is given by $(e + 2)31$ instead of $(e + 1)32$.

Example 2. For $e = 2$, the hits with absolutely minimal discriminants are
$d = -210\,164$ for type D.6, $\varkappa(K) \sim (123; 1)$, $\alpha_1(K) \sim (31, 31, 31; 431)$,
$d = -320\,968$ for type C.4, $\varkappa(K) \sim (113; 3)$, $\alpha_1(K) \sim (31, 31, 431; 211)$,
$d = -354\,232$ for type D.10, $\varkappa(K) \sim (114; 3)$, $\alpha_1(K) \sim (31, 31, 431; 211)$,
$d = -776\,747$ for type D.5, $\varkappa(K) \sim (112; 3)$, $\alpha_1(K) \sim (31, 31, 431; 211)$.

For $e = 3$, the polarization of $\alpha_1(K)$ is uniform for all types. It is given by $(e + 1)32$.

Example 3. For $e = 3$, the hits with absolutely minimal discriminants are
$d = -642\,491$ for type D.5, $\varkappa(K) \sim (112; 3)$, $\alpha_1(K) \sim (41, 41, 432; 311)$,
$d = -1\,021\,523$ for type D.6, $\varkappa(K) \sim (123; 1)$, $\alpha_1(K) \sim (41, 41, 41; 432)$,
$d = -1\,052\,072$ for type C.4, $\varkappa(K) \sim (113; 3)$, $\alpha_1(K) \sim (41, 41, 432; 311)$,
$d = -1\,265\,747$ for type D.10, $\varkappa(K) \sim (114; 3)$, $\alpha_1(K) \sim (41, 41, 432; 311)$.

For all $e \geq 4$, the polarization of $\alpha_1(K)$ depends on the type. For C.4 and D.5 it is $(e+1)32$, whereas for D.10 and D.6 we have the variant $e33$.

Example 4. For $e = 4$, the hits with absolutely minimal discriminants are
$d = -2\,249\,263$ for type D.6, $\varkappa(K) \sim (123;1)$, $\alpha_1(K) \sim (51,51,51;433)$,
$d = -2\,959\,235$ for type C.4, $\varkappa(K) \sim (113;3)$, $\alpha_1(K) \sim (51,51,532;411)$,
$d = -4\,076\,823$ for type D.10, $\varkappa(K) \sim (114;3)$, $\alpha_1(K) \sim (51,51,433;411)$,
$d = -5\,231\,284$ for type D.5, $\varkappa(K) \sim (112;3)$, $\alpha_1(K) \sim (51,51,532;411)$.

Example 5. For $e = 5$, the hits with absolutely minimal discriminants are
$d = -5\,593\,787$ for type D.10, $\varkappa(K) \sim (114;3)$, $\alpha_1(K) \sim (61,61,533;511)$,
$d = -14\,885\,751$ for type C.4, $\varkappa(K) \sim (113;3)$, $\alpha_1(K) \sim (61,61,632;511)$,
$d = -18\,597\,255$ for type D.5, $\varkappa(K) \sim (112;3)$, $\alpha_1(K) \sim (61,61,632;511)$,
$d = -18\,731\,096$ for type D.6, $\varkappa(K) \sim (123;1)$, $\alpha_1(K) \sim (61,61,61;533)$.

Example 6. For $e = 6$, the hits with absolutely minimal discriminants are
$d = -11\,591\,183$ for type D.5, $\varkappa(K) \sim (112;3)$, $\alpha_1(K) \sim (71,71,732;611)$,
$d = -17\,740\,111$ for type C.4, $\varkappa(K) \sim (113;3)$, $\alpha_1(K) \sim (71,71,732;611)$,
$d = -33\,942\,367$ for type D.6, $\varkappa(K) \sim (123;1)$, $\alpha_1(K) \sim (71,71,71;633)$.

Example 7. For $e = 7$, the hits with absolutely minimal discriminants are
$d = -111\,733\,415$ for type D.10, $\varkappa(K) \sim (114;3)$, $\alpha_1(K) \sim (81,81,733;711)$,
$d = -116\,407\,871$ for type D.5, $\varkappa(K) \sim (112;3)$, $\alpha_1(K) \sim (81,81,832;711)$.

Example 8. For $e = 8$, the hits with absolutely minimal discriminants are
$d = -98\,311\,919$ for type D.5, $\varkappa(K) \sim (112;3)$, $\alpha_1(K) \sim (91,91,932;811)$.

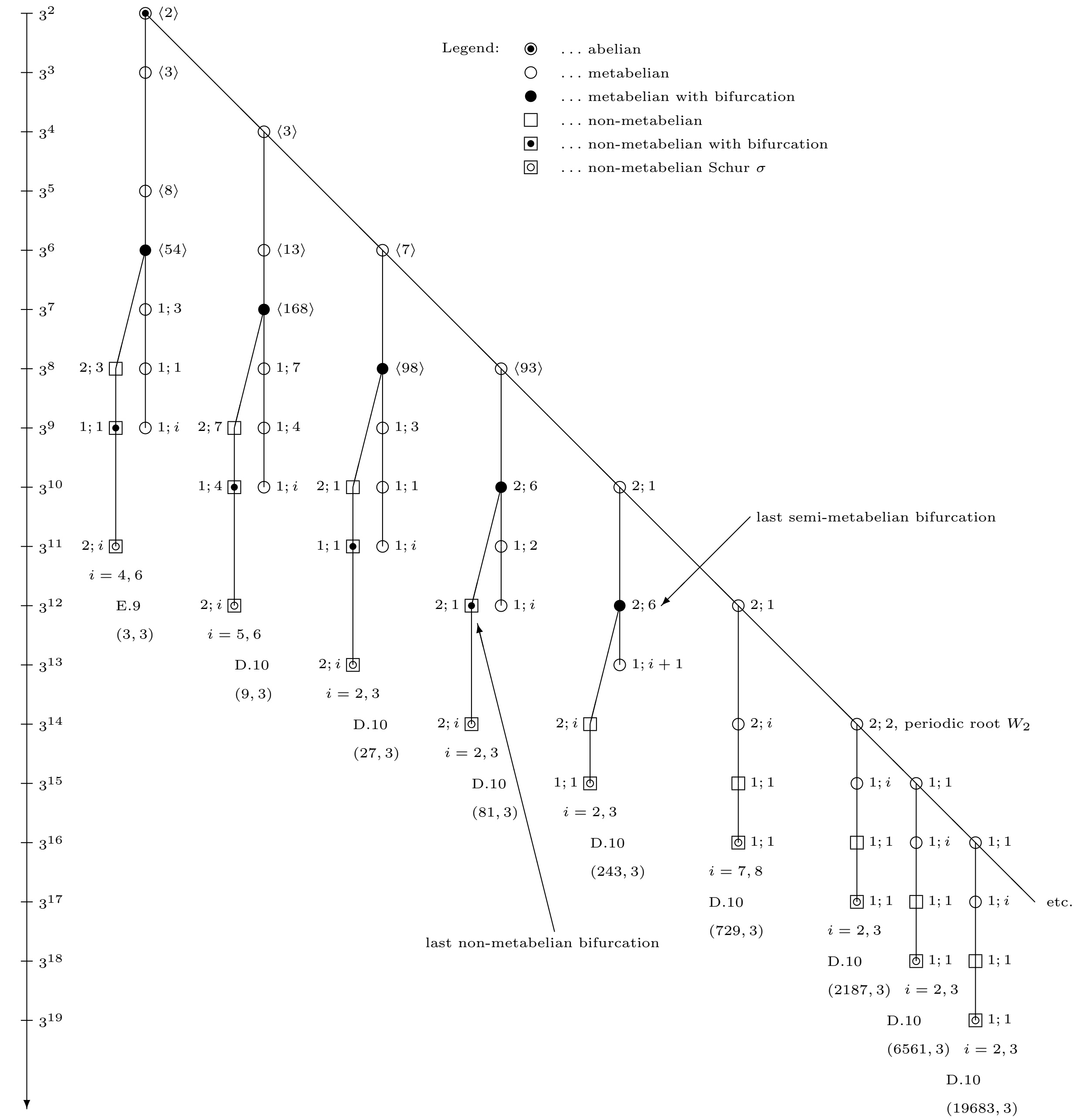
Theorem 9. ***(Three Stage Tower Theorem.)*** *Any imaginary quadratic number field* $K = \mathbb{Q}(\sqrt{d})$*,* $d < 0$*, with non-elementary bicyclic* 3*-class group* $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$*,* $e \geq 2$*, and Artin pattern* $\mathrm{AP}(K) = (\varkappa(K), \alpha_1(K))$ *given by Formulas* (1) *for* $\varkappa(K)$ *and* (5) *for* $\alpha_1(K)$ *has a finite* 3*-class field tower* $\mathrm{F}_3^\infty(K)$ *with precisely three stages,* $\ell_3(K) = 3$*. For* $e \geq 7$*, the* 3*-class tower group* $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ *is given by Formula* (2)*, and the second* 3*-class group* $M = G/G'' \simeq \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ *by Formula* (4)*, both in dependence on Formula* (3)*.*

Proof. According to the Galois correspondence of field theory and the Artin reciprocity law of class field theory [2], the maximal self-conjugate subgroups $H_1, \ldots, H_3; H_4$ of the automorphism group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of the maximal unramified pro-3 extension of an algebraic number field K with bicyclic 3-class group $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$ correspond to the unramified cyclic cubic extensions $L_1, \ldots, L_3; L_4$ of K, and the abelian quotient invariants $\alpha_1(G) = [G/G'; (H_i/H_i')_{1 \leq i \leq 4}]$ coincide with the abelian type invariants of 3-class groups $\alpha_1(K) = [\mathrm{Cl}_3(K); (\mathrm{Cl}_3(L_i))_{1 \leq i \leq 4}]$. According to Artin's theory of the transfer [3], the Schur transfer homomorphisms $T_i : G/G' \to H_i/H_i'$ correspond to the extension homomorphisms $\tau_i : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(L_i)$ of 3-ideal classes, and the punctured transfer kernel type $\varkappa(G) = (\ker(T_i))_{1 \leq i \leq 4}$ coincides with the punctured capitulation type $\varkappa(K) = (\ker(\tau_i))_{1 \leq i \leq 4}$. This was discussed in more detail in [15] and is the foundation of the strategy of *pattern recognition via Artin transfers* [21], which is due to the coincidence of Artin patterns $\mathrm{AP}(G) = (\varkappa(G), \alpha_1(G)) = (\varkappa(K), \alpha_1(K)) = \mathrm{AP}(K)$. Finally, the soluble length $\mathrm{sl}(G)$ is equal to the length $\ell_3(K)$ of the 3-class field tower of K. For an imaginary quadratic field K, the 3-class tower group G must be a Schur σ-group [26, 13]. □

5. Proof and tree diagram

The proof of the preperiodic Theorems 4 – 8 and finally of the periodic Main Theorems 1 – 3 can be developed in accordance with the tree diagram in Figure 1. All directed edges of this tree lead from descendants D to p-parents $\pi_p(D) = D/P_{c_p-1}(D)$, $c_p = \mathrm{cl}_p(D)$, rather than to parents $\pi(D) = D/\gamma_c(D)$, $c = \mathrm{cl}(D)$. Consequently, the figure admits actual descendant construction.

FIGURE 1. Schur σ-groups G with $\varrho(G) \sim (2,2,3;3)$, $G/G' \simeq (3^e,3)$, $2 \le e \le 9$



The p-group generation algorithm [12] by Newman [24] and O'Brien [25] is implemented in the ANUPQ package [11] of the computational algebra system Magma [14, 7, 6]. This algorithm is used to construct all immediate p-descendants of an assigned finite p-group. Repeated recursive applications of the algorithm, guided by the strategy of pattern recognition via Artin transfers [21], eventually produce Figure 1. In each step, only σ-descendants are allowed to pass the filter. The figure shows an *infinite main trunk* and *finite twigs* emanating from the vertices of the trunk. Propagation along the trunk is exo-genetic with increasing commutator quotient $(3^e,3) \mapsto (3^{e+1},3)$, whereas all vertices of a twig share a common abelianization and the propagation is endo-genetic. The leftmost twig is included in order to point out analogy to the *elementary* commutator quotient $(3,3)$. It was computed in [20, Fig. 6, p. 110], where historical information is provided for type E.9, $\varkappa \sim (2231) \sim (3231)$, in [20, § 4, pp. 107–111].

All the other twigs concern *non-elementary* commutator quotients $(3^e, 3)$, $2 \le e \le 9$, restricted to the particular punctured transfer kernel type D.10, $\varkappa \sim (114;3)$. Figure 1 remains unchanged for the other two types C.4 and D.5 when the parameter i is selected according to the preperiodic Theorems 4 – 8 for $2 \le e \le 6$, and the *periodic root* W_ℓ for $e \ge 7$ is replaced according to Formula (3) in the periodic Main Theorem 2. More changes are required for type D.6, $\varkappa \sim (123;1)$, which is only included in the number theoretic § 4 but not in the group theoretic §§ 2 and 3. For instance, the coclass tree with root $\langle 729, 13\rangle$ and bifucation at $\langle 2187, 168\rangle$ [22, Fig. 5], which is responsible for three types D.10, C.4 and D.5, must be replaced by the coclass tree with root $\langle 729, 21\rangle$ and bifucation at $\langle 2187, 191\rangle$ [22, Fig. 6] or with root $\langle 729, 18\rangle$ and bifucation at $\langle 2187, 181\rangle$. Both of the latter coclass trees give rise to the single relevant type D.6. The initialization of the construction process at $e = 2$ is described in [23, § 5], but now the *scaffold type* b.31, $\varkappa \sim (044;4)$, must be replaced by type d.10, $\varkappa \sim (110;3)$, associated with types D.10, C.4 and D.5. The search immediately leads to the root $\langle 729, 13\rangle$ and the fork $\langle 2187, 168\rangle$. Capable descendants of both step sizes $s \in \{1, 2\}$ have relative identifiers $\{1, 4, 7\}$, but only 7 is relevant for the desired types.

In the leftmost three twigs with $1 \le e \le 3$, the fork topologies [19], which begin at the bifurcation, are isomorphic as directed graphs. However, in the next two twigs with $4 \le e \le 5$, the fork topologies shrink gradually, and for all $e \ge 6$, the bifurcation vanishes leaving a descendant topology. The vertices on twigs are BCF-groups [23].

The vertices of the main trunk are CF-groups [4] with type a.1, $\varkappa = (000;0)$, and $\varrho \sim (2,2,3;3)$. For $2 \le e \le 5$, the first vertex of the twig has scaffold type d.10, $\varkappa \sim (110;3)$, but for $e \ge 6$, the twigs entirely consist of vertices sharing a common type, D.10, C.4 or D.5.

Up to $e \le 6$, the construction process is straightforward, but for $e = 7$ considerable difficulties arise, because it is hard to determine the next vertex on the main trunk.

Since an attempt with hypothetical next main trunk root $\langle 6561, 93\rangle - \#2;1 - \#2;1 - \#2;1$ and 1709 descendants D up to p-class $\mathrm{cl}_p(D) \le 11$ only led to three pairs of Schur σ-groups with commutator quotient $(2187, 3)$ and $\mathrm{lo} = 20$ in the *second* excited state $\alpha_1 \sim (81, 81, 744; 711)$ for type D.10, respectively $\alpha_1 \sim (81, 81, 843; 711)$ for type C.4 and D.5, we returned to a tour de force computation starting at the previous vertex $\langle 6561, 93\rangle - \#2;1 - \#2;1$ on the main trunk.

Lemma 1. *Among the* 1708 *descendants D up to p-class* $\mathrm{cl}_p(D) \le 10$ *of the main trunk vertex* $\langle 6561, 93\rangle - \#2;1 - \#2;1$*, there occur the following Schur σ-groups S:*

(1) *the expected three pairs with* $S/S' \simeq (729, 3)$ *and* $\mathrm{lo}(S) = 16$ *in the first excited state* $\alpha_1 \sim (71, 71, 633; 611)$ *for type* D.10*, respectively* $\alpha_1 \sim (71, 71, 732; 611)$ *for type* C.4*,* D.5*,*
(2) *the desired three pairs with* $S/S' \simeq (2187, 3)$ *and* $\mathrm{lo}(S) = 17$ *in the first excited state* $\alpha_1 \sim (81, 81, 733; 711)$ *for type* D.10*, respectively* $\alpha_1 \sim (81, 81, 832; 711)$ *for type* C.4*,* D.5*,*
(3) *three unexpected pairs with* $S/S' \simeq (729, 3)$ *and* $\mathrm{lo}(S) = 19$ *in the second excited state* $\alpha_1 \sim (71, 71, 644; 611)$ *for type* D.10*, respectively* $\alpha_1 \sim (71, 71, 743; 611)$ *for type* C.4*,* D.5*,*
(4) *and (as a superfluous byproduct) a quartet of type* B.2*,* $\varkappa(S) \sim (111;2)$*, with* $S/S' \simeq (729, 3)$ *and* $\mathrm{lo}(S) = 19$ *in the first excited state* $\alpha_1 \sim (71, 71, 732; 611)$*.*

A back track search, starting from item (2) in Lemma 1, finally reveals three suitable periodic roots W_ℓ given in Formula (3) of Theorem 2.

6. Conclusion

In the present article, our hypothesis in [22, § 12] that periodicity of Schur σ-groups G with $G/G' \simeq C_{3^e} \times C_3$ and one of the punctured transfer kernel types D.10, C.4 and D.5 in the *first excited state* will set in with $e \ge e_0 = 7$ was verified.

The question concerning the evolution of the fork topology between G and its metabelianization $M = G/G''$, for which we were really unable to make any prediction, was answered by a gradual shrinking of the twigs in Figure 1 for $4 \le e \le 7$, coming along with a last non-metabelian (second) bifurcation for $e = 4$ and a last semi-metabelian (first) bifurcation for $e = 5$. The fork topology completely degenerates to a descendant topology for all $e \ge 6$ and periodicity of the shortest possible twigs sets in with $e \ge e_0 = 7$.

7. Acknowledgements

The author acknowledges support by the Austrian Science Fund (FWF): Projects J0497-PHY and P26008-N25, and by the Research Executive Agency of the European Union (EUREA).

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