# BCF-GROUPS WITH ELEVATED RANK DISTRIBUTION

DANIEL C. MAYER

Abstract. Infinitely many large Schur $\sigma$-groups $G$ with logarithmic order $\mathrm{lo}(G) = 19 + e$, non-elementary bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$, $e \ge 2$, elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$, punctured transfer kernel type $\varkappa(G) \sim (144; 4)$ and soluble length $\mathrm{sl}(G) = 3$ are constructed. Up to $e \le 4$, they are realized as 3-class field tower groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$. Their metabelianizations $M = G/G''$ are BCF-groups with $\mathrm{lo}(M) = 8+e$ and bicyclic third lower central factor $\gamma_3(M)/\gamma_4(M) \simeq C_3 \times C_3$.

## 1. Introduction

Let $G$ be a pro-3 group or finite 3-group with bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$ having one non-elementary factor with exponent $e \ge 2$. Then $G$ possesses four maximal self-conjugate subgroups $H_1, \dots, H_3; H_4$, and by the *rank distribution* of $G$ we understand the quartet

$$\varrho(G) := [\mathrm{rank}_3(H_1/H_1'), \dots, \mathrm{rank}_3(H_3/H_3'); \mathrm{rank}_3(H_4/H_4')] . \tag{1}$$

Let $(\gamma_j(G))_{j\ge1}$ be the lower central series of $G$. When the factors $\gamma_j(G)/\gamma_{j-1}(G) \simeq C_3$ are all cyclic, for $j \ge 2$, then $G$ is called a *CF-group*, according to Ascione et al. [4]. CF means cyclic factors. Otherwise, at least the factor $\gamma_3(G)/\gamma_4(G) \simeq C_3 \times C_3$ is bicyclic, and $G$ is called a *BCF-group*, according to Nebelung [28]. BCF means bicyclic or cyclic factors. Recall that, since $\gamma_2(G) = \langle s_2, \gamma_3(G)\rangle$, the factor $\gamma_2(G)/\gamma_3(G) \simeq C_3$ is always cyclic, generated by the main commutator $s_2 = [y, x]$ of the two-generated group $G = \langle x, y\rangle$. For a BCF-group $G$, we have $\gamma_3(G) = \langle s_3, t_3, \gamma_4(G)\rangle$ with higher non-trivial commutators $s_3 = [s_2, x]$ and $t_3 = [s_2, y]$.

In [27, § 2], we introduced the concept of *punctured transfer kernel types*

$$\varkappa(G) := [\ker(T_1), \dots, \ker(T_3); \ker(T_4)] \tag{2}$$

for 3-groups $G = \langle x, y\rangle$ with $G/G' \simeq C_{3^e} \times C_3$, $e \ge 2$. Here, $T_i : G/G' \to H_i/H_i'$ denotes the Artin transfer homomorphism from $G$ to $H_i$. It turned out that at least three kernels are two-dimensional, equal to the complete 3-elementary subgroup $\langle x^{e-1}, y, G'\rangle/G'$ of $G/G'$, when $G$ is a CF-group. Consequently, metabelian CF-groups can be realized arithmetically only by second 3-class groups $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, but not for imaginary quadratic fields with $d < 0$, where all kernels must be one-dimensional.

In [27, §§ 5 and 7], we investigated how BCF-groups $G$ with *moderate* rank distribution $\varrho(G) \in \{(2, 2, 2; 3), (2, 2, 3; 3)\}$ are populated by second 3-class groups $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ and 3-class field tower groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with non-elementary bicyclic 3-class groups $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$, $e \ge 2$.

In the present article we continue this research enterprise for BCF-groups $G$ with *elevated* rank distribution $\varrho(G) = (3, 3, 3; 3)$ and punctured transfer kernel type B.18, $\varkappa(G) \sim (144; 4)$. Their exo-genetic propagation has been clarified in [27, Thm. 17].

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## 2. Arithmetical realization

It is of the greatest importance to emphasize that the assumptions concerning the punctured transfer kernel type $\varkappa(G)$, and the *logarithmic abelian quotient invariants* of first order

$$\alpha_1(G) := [H_1/H_1', \ldots, H_3/H_3'; H_4/H_4'] \tag{3}$$

and of second order

$$\alpha_2(G) := \bigl(G/G'; [H_i/H_i'; (H_{i,j}/H_{i,j}')_{(H_i:H_{i,j})=3}]_{1\le i\le 4}\bigr) \tag{4}$$

of the Schur $\sigma$-groups in the following six main theorems are perfectly tailored for applications in algebraic number theory and class field theory. According to the Artin reciprocity law [2, 3], these invariants can be interpreted for an arbitrary algebraic number field $K$ as the punctured capitulation type $\varkappa(K) := [\ker(\tau_1), \ldots, \ker(\tau_3); \ker(\tau_4)]$ of the extension homomorphisms $\tau_i : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(L_i)$, $\mathfrak{a}\mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_{L_i})\mathcal{P}_{L_i}$, of 3-classes from $K$ to the four unramified cyclic cubic extensions $L_i$, the logarithmic abelian type invariants $\alpha_1(K) := [\mathrm{Cl}_3(L_1), \ldots, \mathrm{Cl}_3(L_3); \mathrm{Cl}_3(L_4)]$ of the 3-class groups of the fields $L_i$, and the logarithmic abelian type invariants

$$\alpha_2(K) := \bigl(\mathrm{Cl}_3(K); [\mathrm{Cl}_3(L_i); (\mathrm{Cl}_3(L_{i,j}))_{[L_{i,j}:L_i]=3}]_{1\le i\le 4}\bigr)$$

of all unramified (but not necessarily abelian) 3-extensions of degree at most nine of $K$. For details see [25]. In this article, we investigate applications to the simplest algebraic number fields, namely imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with negative fundamental discriminants $d < 0$.

## 3. Main theorems

The six *main theorems* are the crucial achievements of the present article. They show that Schur $\sigma$-groups [16, 1, 9] with *elevated* rank distribution also become *periodic* for sufficiently large exponents $e \ge 9$, similarly as Schur $\sigma$-groups with *moderate* rank distribution for $e \ge 5$, according to [27].

### 3.1. Schur $\sigma$-groups $G$.

In all theorems, the symbols $e^+ := e + 1$, $e^- := e - 1$ are used for abbreviation. Isomorphism classes of groups are identified in accordance with [6, 13, 17].

**Theorem 1.** *A total of* 54 *Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq (3^e, 3)$, punctured transfer kernel type* B.18, *$\varkappa(G) \sim (144; 4)$, elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$, first abelian quotient invariants $\alpha_1(G) \sim [e^+21, e11, e11; e^-21]$, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^+21; e2111, (e^+211)^3, (e^+2)^9],\\ &[e11; e2111, (e^+21)^3, (e21)^9],\\ &[e11; e2111, (e^+111)^3, (e^+2)^9];\\ &[e^-21; e2111, (e31)^3, (e21)^8, e^-22])\end{aligned} \tag{5}$$

*and (minimal) logarithmic order* $\mathrm{lo}(G) = 19 + e$ *is given for each $e \ge 9$ by the term*

$$G = W_a[-\#1; 1]^{e-9} - \#1; p - \#1; q - \#1; 1 \quad \text{with } p \in \{2, 3\} \text{ and } q \in \{1, 2, 3\}, \tag{6}$$

*where* 9 *distinct periodic roots with $1 \le a \le 9$, $\tilde{a} = 1$ for $a \le 3$, $\tilde{a} = 2$ for $a \ge 4$, are denoted by*

$$W_a := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \mathbf{24} - \#3; 14 - \#4; a - \#2; \tilde{a} - \#2; 1. \tag{7}$$

**Theorem 2.** *A total of* 162 *Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq (3^e, 3)$, punctured transfer kernel type* B.18, *$\varkappa(G) \sim (144; 4)$, elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$, first abelian quotient invariants $\alpha_1(G) \sim [e^+21, e11, e11; e^-21]$, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^+21; e2111, (e^+1111)^3, (e^+2)^9],\\ &[e11; e2111, (e^+21)^3, (e21)^9],\\ &[e11; e2111, (e^+21)^3, (e^+2)^9];\\ &[e^-21; e2111, (e31)^3, (e21)^8, e^-22])\end{aligned} \tag{8}$$

*and (minimal) logarithmic order* $\mathrm{lo}(G) = 19 + e$ *is given for each $e \ge 9$ by the term*

$$G = W_{a,b}[-\#1; 1]^{e-9} - \#1; p - \#1; q - \#1; 1 \quad \text{with } p \in \{2, 3\} \text{ and } 1 \le q \le N, \tag{9}$$

*where* 45 *distinct periodic roots with* $1 \le a \le 27$, $\tilde{a} = 1$, $1 \le b \le 3$, $N = 1$ *for* $a \in \{3, 4, 8, 12, 13, 17, 21, 22, 26\}$ *and* $\tilde{a} = 2$, $b = 1$, $N = 3$ *otherwise, are denoted by*

$$W_{a,b} := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \mathbf{26} - \#3; 14 - \#4; a - \#2; \tilde{a} - \#2; b. \tag{10}$$

**Theorem 3.** *A total of* 324 *Schur* $\sigma$*-groups* $G$ *with commutator quotient* $G/G' \simeq (3^e, 3)$*, punctured transfer kernel type* B.18, $\varkappa(G) \sim (144; 4)$*, elevated rank distribution* $\varrho(G) = (3, 3, 3; 3)$*, first abelian quotient invariants* $\alpha_1(G) \sim [e^{+}21, e11, e11; e^{-}21]$*, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^{+}21; e2111, (e^{+}211)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}21)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}111)^3, (e^{+}2)^9];\\ &[e^{-}21; e2111, (e31)^3, (e21)^8, e^{-}22])\end{aligned} \tag{11}$$

*and (minimal) logarithmic order* $\mathrm{lo}(G) = 19 + e$ *is given for each* $e \ge 9$ *by the term*

$$G = W_{\ell,k,a}[-\#1; 1]^{e-9} - \#1; p - \#1; q - \#1; r \text{ with } p \in \{2, 3\} \text{ and } q, r \in \{1, 2, 3\}, \tag{12}$$

*where* 18 *distinct periodic roots with* $(\ell, k) \in \{(\mathbf{28}, \mathbf{5}), (\mathbf{30}, \mathbf{2})\}$ *and* $1 \le a \le 9$ *are denoted by*

$$W_{\ell,k,a} := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \ell - \#\mathbf{3}; \mathbf{k} - \#4; a - \#2; 1 - \#2; 1. \tag{13}$$

**Theorem 4.** *A total of* 162 *Schur* $\sigma$*-groups* $G$ *with commutator quotient* $G/G' \simeq (3^e, 3)$*, punctured transfer kernel type* B.18, $\varkappa(G) \sim (144; 4)$*, elevated rank distribution* $\varrho(G) = (3, 3, 3; 3)$*, first abelian quotient invariants* $\alpha_1(G) \sim [e^{+}21, e11, e11; e^{-}21]$*, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^{+}21; e2111, (e^{+}211)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}21)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}21)^3, (e21)^9];\\ &[e^{-}21; e2111, (e211)^3, (e21)^8, e^{-}22])\end{aligned} \tag{14}$$

*and (minimal) logarithmic order* $\mathrm{lo}(G) = 19 + e$ *is given for each* $e \ge 9$ *by the term*

$$G = W_{a,b}[-\#1; 1]^{e-9} - \#1; p - \#1; q - \#1; 1 \text{ with } p \in \{2, 3\} \text{ and } 1 \le q \le N, \tag{15}$$

*where* 45 *distinct periodic roots with* $1 \le a \le 27$, $\tilde{a} = 1$ *for* $a \le 9$, $\tilde{a} = 2$ *for* $a \ge 10$, $1 \le b \le 3$, $N = 1$ *for* $a \in \{1, 5, 9, 11, 15, 16, 21, 22, 26\}$ *and* $b = 1$, $N = 3$ *otherwise, are denoted by*

$$W_{a,b} := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \mathbf{31} - \#3; 29 - \#4; a - \#2; \tilde{a} - \#2; b. \tag{16}$$

**Theorem 5.** *A total of* 162 *Schur* $\sigma$*-groups* $G$ *with commutator quotient* $G/G' \simeq (3^e, 3)$*, punctured transfer kernel type* B.18, $\varkappa(G) \sim (144; 4)$*, elevated rank distribution* $\varrho(G) = (3, 3, 3; 3)$*, first abelian quotient invariants* $\alpha_1(G) \sim [e^{+}21, e11, e11; e^{-}21]$*, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^{+}21; e2111, (e^{+}1111)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}21)^3, (e^{+}2)^9],\\ &[e11; e2111, (e^{+}21)^3, (e^{+}2)^9];\\ &[e^{-}21; e2111, (e31)^3, (e21)^8, e^{-}22])\end{aligned} \tag{17}$$

*and (minimal) logarithmic order* $\mathrm{lo}(G) = 19 + e$ *is given for each* $e \ge 9$ *by the term*

$$G = W_{a,b}[-\#1; 1]^{e-9} - \#1; p - \#1; q - \#1; 1 \text{ with } p \in \{2, 3\} \text{ and } q \in \{1, 2, 3\}, \tag{18}$$

*where* 27 *distinct periodic roots with* $1 \le a \le 9$ *and* $1 \le b \le 3$ *are denoted by*

$$W_{a,b} := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \mathbf{33} - \#3; 32 - \#4; a - \#2; 1 - \#2; b. \tag{19}$$

**Theorem 6.** *A total of* 162 *Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq (3^e,3)$, punctured transfer kernel type* B.18, *$\varkappa(G) \sim (144;4)$, elevated rank distribution $\varrho(G) = (3,3,3;3)$, first abelian quotient invariants $\alpha_1(G) \sim [e^+21, e11, e11; e^-21]$, second abelian quotient invariants*

$$\begin{aligned}\alpha_2(G) \sim (e1;&[e^+21; e2111, (e^+211)^3, (e^+2)^9],\\ &[e11; e2111, (e^+21)^3, (e^+2)^9],\\ &[e11; e2111, (e^+21)^3, (e^+2)^9];\\ &[e^-21; e2111, (e211)^3, (e21)^8, e^-22])\end{aligned} \tag{20}$$

*and (minimal) logarithmic order $\mathrm{lo}(G) = 19 + e$ is given for each $e \geq 9$ by the term*

$$G = W_{a,b}[-\#1;1]^{e-9} - \#1; p - \#1; q - \#1; 1 \text{ with } p \in \{2,3\} \text{ and } q \in \{1,2,3\}, \tag{21}$$

*where* 27 *periodic roots with $1 \leq a \leq 9$, $\tilde{a} = 1$ for $a \in \{2,6,7\}$, $\tilde{a} = 2$ otherwise, and $1 \leq b \leq 3$ are*

$$W_{a,b} := \langle 2187, 3\rangle - \#3; 2 - \#\mathbf{4}; \mathbf{37} - \#3; 32 - \#4; a - \#2; \tilde{a} - \#2; b. \tag{22}$$

**Remark 1.** The *periodic twig* $-\#1; p - \#1; q - \#1; r$ of the terms for the Schur $\sigma$-groups $G$ in the main theorems contains 6 *terminal leaves* on average. However, for Theorem 3 there are 18, and for Theorems 2 and 4 there are partially only 2.

For each $e \geq 9$, all main theorems together yield 1026 Schur $\sigma$-groups $G$ with $\mathrm{lo}(G) = 19 + e$, which are descendants of 171 distinct periodic roots $W$ with fixed logarithmic order $\mathrm{lo}(W) = 25$.

Exemplarily we give a succinct proof for the last main theorem, namely Theorem 6.

*Proof.* (Proof of Theorem 6.) For a fixed step size $s \geq 1$, we denote by $N$ the number of all immediate descendants of a 3-group, and by $C$ the number of capable immediate descendants with positive nuclear rank $\nu \geq 1$. Generally, let $X := \langle 2187, 3\rangle - \#3; 2 - \#4; 37 - \#3; 32$. This is a non-metabelian 3-group of type $(729, 3)$. We consider a chain of exo-genetic propagations:

- $X$ has $N = C = 27$ for $s = \nu = 4$ but only the first 9 descendants are of type $(2187, 3)$.
- Each $X - \#4; a$ with $1 \leq a \leq 9$ has $N = C = 6$ for $s = \nu = 2$ but only the first, resp. second, descendant, indicated by $\tilde{a} \in \{1, 2\}$, is of type $(6561, 3)$.
- Each $X - \#4; a - \#2, \tilde{a}$ with $1 \leq a \leq 9$ has $N = C = 9$ for $s = \nu = 2$ but only the first 3 descendants are of type $(19683, 3)$.
- Each $W_{a,b} := X - \#4; a - \#2, \tilde{a} - \#2; b$ with $1 \leq a \leq 9$ and $1 \leq b \leq 3$ has 6 Schur $\sigma$-descendants $W_{a,b}[-\#1;1]^{e-9} - \#1; p - \#1; q - \#1; 1$ with $p \in \{2,3\}$ and $q \in \{1,2,3\}$, for each $e \geq 9$.

Together this census yields $9 \cdot 3 \cdot 6 = 162$ Schur $\sigma$-groups, for each $e \geq 9$. □

The following supplementary theorem provides a warranty for the fact that the information in the six main theorems is exhaustive and complete.

**Theorem 7.** ***(Exhaustion Theorem.)***
*Let $G$ be a Schur $\sigma$-group with non-elementary bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$, $e \geq 9$, punctured transfer kernel type* B.18, *$\varkappa(G) \sim (144;4)$, elevated rank distribution $\varrho(G) = (3,3,3;3)$, and first abelian quotient invariants $\alpha_1(G) \sim [(e+1)21, e11, e11; (e-1)21]$. Then*

- *if $G$ has logarithmic order $\mathrm{lo}(G) = 19 + e$, then $G$ is of one of the shapes in the six main Theorems 1 – 6 (inclusively the shape of $\alpha_2(G)$) and has soluble length $\mathrm{sl}(G) = 3$;*
- *if $G$ is not of one of the shapes in the six main Theorems 1 – 6, then $G$ has logarithmic order $\mathrm{lo}(G) > 19 + e$ and different second abelian quotient invariants $\alpha_2(G)$.*

3.2. **Second derived quotients $G/G''$.** Periodicity of metabelianizations with elevated rank distribution sets in earlier for $e \geq 5$ already.

**Corollary 1.** *The metabelianization $M = G/G''$ of a Schur $\sigma$-group $G$ with commutator quotient $G/G' \simeq (3^e, 3)$, $e \geq 5$, punctured transfer kernel type* B.18, *$\varkappa(G) \sim (144;4)$, logarithmic abelian quotient invariants of first order $\alpha_1(G) \sim [(e+1)21, e11, e11; (e-1)21]$, and logarithmic order $\mathrm{lo}(G) = 19 + e$ is given by one of the two candidates*

$$M \simeq \langle 2187, 3\rangle - \#3; 2 - \#2; 93[-\#1;1]^{e-5} - \#1; i \text{ with } i \in \{2,3\}. \tag{23}$$

*Their logarithmic order is* $\mathrm{lo}(M) = 8 + e$*, i.e. the second derived subgroup* $G''$ *is of constant logarithmic order* $\mathrm{lo}(G'') = 11$*, in fact, it is abelian of constant type* $G'' \simeq (332111)$*. A parametrized polycyclic power commutator presentation of the members* $\langle 2187, 3\rangle - \#3; 2 - \#2; 93[-\#1; 1]^{e-5}$ *of the infinite chain is given for* $e \ge 6$ *by*

$$\langle x, y \mid x^{3^e} = 1, y^3 = s_3 s_4^2, s_2^3 = s_4 t_4^2, s_3^3 = s_5, t_3^3 = s_5^2, [x^3, y] = s_4 t_4 s_5^2, [x^3, s_2] = s_5, t_5 = s_5\rangle \tag{24}$$

*in terms of the commutators* $s_2 = [y, x]$, $s_3 = [s_2, x]$, $t_3 = [s_2, y]$, $s_4 = [s_3, x]$, $t_4 = [t_3, y]$, $s_5 = [s_4, x]$, $t_5 = [t_4, y]$.

The justification of the periodicities in Theorems 1 – 6 and Corollary 1 will be developed in § 12.

## 4. Layout of the paper

Since the periodicity in the crucial Theorems 1 – 6 sets in with exponent $e = 9$, we devote §§ 6, 8 and 10 to the detailed discussion of the regular cases $2 \le e \le 4$. We do not go into the details of the irregular intermediate cases $5 \le e \le 8$, which are clarified sufficiently by Figure 4. In § 12 we illuminate the long and winding road to the actual verification of the periodicity of Schur $\sigma$-groups $G$ with elevated rank distribution $\varrho(G) = (3, 3, 3; 3)$ and commutator quotient $G/G' \simeq (3^e, 3)$, which was expected by ourselves for $e \ge 9$ in analogy to the periodicity for $e \ge 5$ in the case of moderate rank distribution [27]. Arithmetical applications to 3-class field tower groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d < 0$ and non-elementary 3-class groups $\mathrm{Cl}_3(K) \simeq (3^e, 3)$ are given in §§ 7, 9 and 11 for $2 \le e \le 4$. In §§ 13 and 14, where arithmetical realizations of $5 \le e \le 6$ are just possible (with CPU-time a week) it becomes clear that $e = 7$ (CPU-time several months) and $e = 8$ (CPU-time several years) are outside of a reasonable and realistic arithmetical enterprise, aggravated by internal Magma errors, due to huge absolute discriminants $|d|$. A conclusion concerning the general structure of the logarithmic abelian quotient invariants $\alpha_2$ of second order is eventually drawn in § 15. In §§ 6, 8 and 10, we also consider $\mathrm{lo}(G) > 19 + e$. The case $e = 2$ was also investigated in [26].

## 5. Root path to Schur $\sigma$-groups

In order to find $\sigma$-groups [24, Dfn. 3.1, p. 91], and in particular Schur $\sigma$-groups [16, 1, 9], $G$ with commutator quotient $G/G' \simeq (3^e, 3)$ and punctured transfer kernel type B.18, $\varkappa(G) \sim (144; 4)$, it is necessary to take into consideration the associated *scaffold type* b.31, $\varkappa \sim (044; 4)$, since the two-dimensional transfer kernel 0 of a parent can shrink to the one-dimensional transfer kernel 1 for a descendant. This is a consequence of the *antitony principle* for the Artin pattern $(\varkappa, \alpha)$ of parent descendant pairs. The situation is similar to [24, § 3.2.2 and Fig. 2, pp. 91–92] and [21, Fig. 1–2, pp. 24–25], both for elementary $G/G' \simeq (3, 3)$. Now we have non-elementary $G/G'$.

**Proposition 1.** *The root path of the* ***bifurcation*** $B := \langle 2187, 3\rangle - \#3; 2$ ***of infinite order****,*

$$1 \xleftarrow{s=2} \pi_p^3(B) = \langle 9, 2\rangle \xleftarrow{s=2} \pi_p^2(B) = \langle 81, 3\rangle \xleftarrow{s=3} \pi_p(B) = \langle 2187, 3\rangle \xleftarrow{s=3} B = \langle 2187, 3\rangle - \#3; 2, \tag{25}$$

*has step sizes* $(2, 2, 3, 3)$ *and contains two vertices with scaffold type* b.31, $\varkappa \sim (044; 4)$*, which give rise to Schur* $\sigma$*-groups* $G$ *with type* B.18, $\varkappa(G) \sim (144; 4)$*, and to their metabelianizations* $G/G''$*.*

*Proof.* There are only three groups $G$ with $G/G' \simeq (9, 3)$, i.e. $e = 2$, and order $\#G = 81$, namely the non-abelian groups $G \simeq \langle 81, 3\rangle$ with $\varkappa(G) \sim (000; 0)$, a.1, $G \simeq \langle 81, 4\rangle$ with $\varkappa(G) \sim (444; 4)$, A.20, and $G \simeq \langle 81, 6\rangle$ with $\varkappa(G) \sim (111; 1)$, A.1. According to the antitony principle for the Artin pattern $(\varkappa, \alpha)$, the latter two groups are discouraged as predecessors of descendants with $\varkappa \sim (044; 4)$ or $\varkappa \sim (144; 4)$. Moreover, they are not $\sigma$-groups. The unique remaining group $G = \langle 81, 3\rangle$ has the root path $G \xrightarrow{s=2} \pi_p(G) = \langle 9, 2\rangle = C_3 \times C_3 \xrightarrow{s=2} \pi_p^2(G) = \langle 1, 1\rangle = 1$. In order to stay at $e = 2$, the descendant $D = \langle 729, 10\rangle$ with scaffold type b.31, $\varkappa(D) \sim (044; 4)$, must be selected. The unique immediate $\sigma$-descendant $F = \langle 6561, 165\rangle$ of $D$ is already the fork between the desired Schur $\sigma$-group $S$ and its metabelianization $S/S'' \simeq F - \#2; 85$. See § 12, Figure 3. □

## 6. 3-groups with commutator quotient $(9,3)$

In Table 1, we list the second AQI $\alpha_2$ of the 30 non-metabelian step size-4 descendants $F-\#4;\ell$ with $1\le\ell\le 30$ of the metabelian fork $F=\langle 729,10\rangle-\#2;2$. The general structure of $\alpha_2$ is

$$\alpha_2(G)=[21;(\tau_0;22111,D_1),(211;22111,D_2),(211;22111,D_3);(211;22111,D_4)], \tag{26}$$

where each dodecuplet $D_i$, $1\le i\le 4$, consists of a triplet $T_i^3$ and a nonet $N_i^9$. The metabelianization $M=G/G''$ is given by the step size-2 descendant $F-\#2;m$ with $m\in\{82,83,84,85\}$ of $F$. The smallest logarithmic order, soluble length, of a Schur $\sigma$-descendant $S$ of $G$ is $\mathrm{lo}(S)$, $\mathrm{sl}(S)$.

Table 1. Invariants of $G=\langle 729,10\rangle-\#2;2-\#4;\ell$ with $1\le\ell\le 30$

| $\ell$ | $\tau_0$ | $T_1$ | $N_1$ | $T_2$ | $N_2$ | $T_3$ | $N_3$ | $T_4$ | $N_4$ | $m$ | $\mathrm{lo}(S)$ | $\mathrm{sl}(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 222 | 22111 | 221 | 2211 | 221 | 2211 | 221 | 3111 | 32 | 82 | $\infty$ | $\infty$ |
| 2 | 222 | 22111 | 221 | 321 | 32 | 321 | 32 | 321 | 32 | 83 | 21 | 3 |
| 7 | 222 | 22111 | 221 | 321 | 32 | 321 | 32 | 321 | 32 | 82 | 21 | 3 |
| 3 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 3111 | 32 | 82 | 25 | 4 |
| 4 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 2211 | 221 | 83 | 21 | 3 |
| 6 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 2211 | 221 | 83 | 21 | 3 |
| 5 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 3111 | 32 | 82 | 25 | 3 |
| 8 | 222 | 22111 | 221 | 321 | 221 | 321 | 221 | 321 | 32 | 83 | 24 | 4 |
| 9 | 222 | 22111 | 221 | 3111 | 32 | 3111 | 32 | 3111 | 32 | 83 | $\infty$ | $\infty$ |
| 10 | 222 | 3211 | 221 | 321 | 32 | 321 | 221 | 3111 | 32 | 82 | 21 | 3 |
| 12 | 222 | 3211 | 221 | 321 | 32 | 321 | 221 | 3111 | 32 | 83 | 21 | 3 |
| 13 | 222 | 3211 | 221 | 321 | 32 | 321 | 221 | 3111 | 32 | 82 | 21 | 3 |
| 15 | 222 | 3211 | 221 | 321 | 32 | 321 | 221 | 3111 | 32 | 83 | 21 | 3 |
| 11 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 3111 | 32 | 83 | 21 | 3 |
| 14 | 222 | 3211 | 221 | 321 | 32 | 321 | 32 | 3111 | 32 | 83 | 21 | 3 |
| 16 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 3111 | 32 | 84 | 25 | 4 |
| 17 | 321 | 3211 | 32 | 321 | 32 | 321 | 221 | 3111 | 32 | 85 | 21 | 3 |
| 24 | 321 | 3211 | 32 | 321 | 32 | 321 | 221 | 3111 | 32 | 85 | 21 | 3 |
| 18 | 321 | 3211 | 32 | 321 | 221 | 321 | 221 | 3111 | 32 | 84 | 28 | 4 |
| 19 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 3111 | 32 | 85 | 21 | 3 |
| 23 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 3111 | 32 | 85 | 21 | 3 |
| 25 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 3111 | 32 | 84 | 21 | 3 |
| 27 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 3111 | 32 | 85 | 21 | 3 |
| 20 | 321 | 31111 | 32 | 3111 | 32 | 3111 | 32 | 3111 | 32 | 84 | $\infty$ | $\infty$ |
| 21 | 321 | 31111 | 32 | 321 | 32 | 321 | 32 | 321 | 221 | 85 | 21 | 3 |
| 28 | 321 | 31111 | 32 | 321 | 32 | 321 | 32 | 321 | 221 | 84 | 21 | 3 |
| 22 | 321 | 3211 | 32 | 321 | 32 | 321 | 32 | 2211 | 221 | 84 | 21 | 3 |
| 26 | 321 | 3211 | 32 | 321 | 32 | 321 | 221 | 2211 | 221 | 85 | 24 | 4 |
| 29 | 321 | 31111 | 32 | 321 | 32 | 321 | 32 | 321 | 32 | 85 | 21 | 3 |
| 30 | 321 | 31111 | 32 | 3111 | 32 | 3111 | 32 | 2211 | 221 | 85 | $\infty$ | $\infty$ |

**Theorem 8.** *The Schur $\sigma$-groups $S$ with commutator quotient $S/S'\simeq(9,3)$, punctured transfer kernel type* B.18*, $\varkappa(S)\sim(144;4)$, and first AQI $\alpha_1(S)\sim(\tau_0,211,211;211)$ are descendants of* 30 *non-metabelian 3-groups $G=\langle 729,10\rangle-\#2;2-\#4;\ell$ whose invariants are listed in Table 1. In the case of finite order $\mathrm{lo}(S)<\infty$, their invariants usually coincide with those of the predecessor $G$. For $\mathrm{lo}(S)=21$ they have three stages, $\mathrm{sl}(S)=3$, for $\mathrm{lo}(S)\in\{24,28\}$ four stages, $\mathrm{sl}(S)=4$, and for $\mathrm{lo}(S)=25$ they have $3\le\mathrm{sl}(S)\le 4$. Their metabelianization $S/S''\simeq G/G''$ is $M=\langle 729,10\rangle-\#2;2-\#2;m$, where $m\in\{82,83\}$, $\tau_0=222$ for $1\le\ell\le 15$, and $m\in\{84,85\}$, $\tau_0=321$ for $16\le\ell\le 30$.*

The minimum $\mathrm{lo}(S)=21$ occurs for 20 values $\ell$, 24 for 2, 25 for 3, 28 for 1, and $\infty$ for 4.

## 7. Imaginary quadratic fields $K$ with $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$

The 875 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-1\,000\,000 < d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$ were computed by means of Magma [17] in 7 782 seconds of CPU time, that is more than two hours. In Table 2, the first nineteen cases with punctured capitulation type B.18, $\varkappa(K) \sim (144; 4)$, are listed. The abelian quotient invariants $\alpha_1(K)$ of first order of only eleven of them are *uni-polarized* and in the *ground state*. For details see [26].

Table 2. Nineteen fields $K = \mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$ and $\varkappa(K) \sim (144; 4)$

| No. | $d$ | factors | $\alpha_1(K)$ | remark |
|---|---|---|---|---|
| 45 | $-89\,923$ | prime | $(222, 211, 211; 321)$ | bi-polarized |
| 87 | $-150\,319$ | $13, 31, 373$ | $(321, 211, 211; 211)$ | |
| 124 | $-194\,703$ | $3, 64\,901$ | $(321, 211, 211; 211)$ | |
| 161 | $-242\,255$ | $5, 13, 3\,727$ | $(222, 211, 211; 321)$ | bi-polarized |
| 203 | $-294\,983$ | $13, 22\,691$ | $(222, 211, 211; 211)$ | |
| 304 | $-389\,371$ | $401, 971$ | $(431, 211, 211; 211)$ | first excited state |
| 305 | $-389\,435$ | $5, 71, 1\,097$ | $(222, 211, 211; 211)$ | |
| 330 | $-409\,380$ | $2, 3, 5, 6\,823$ | $(222, 211, 211; 321)$ | bi-polarized |
| 397 | $-481\,567$ | $271, 1\,777$ | $(222, 211, 211; 321)$ | bi-polarized |
| 413 | $-494\,771$ | $61, 8\,111$ | $(321, 211, 211; 211)$ | |
| 418 | $-497\,859$ | $3, 263, 631$ | $(321, 211, 211; 321)$ | bi-polarized |
| 438 | $-518\,835$ | $3, 5, 34\,589$ | $(222, 211, 211; 211)$ | |
| 470 | $-553\,807$ | $433, 1\,279$ | $(222, 211, 211; 211)$ | |
| 482 | $-566\,168$ | $2, 17, 23, 181$ | $(321, 211, 211; 211)$ | |
| 635 | $-761\,855$ | $5, 17, 8\,963$ | $(222, 211, 211; 211)$ | |
| 637 | $-763\,972$ | $2, 11, 97, 179$ | $(222, 211, 211; 211)$ | |
| 661 | $-793\,992$ | $2, 3, 33\,083$ | $(321, 211, 211; 321)$ | bi-polarized |
| 729 | $-857\,743$ | prime | $(431, 211, 211; 321)$ | highly bi-polarized |
| 743 | $-876\,948$ | $2, 3, 73\,079$ | $(222, 211, 211; 211)$ | |

In Table 3, we give the abelian quotient invariants $\alpha_2(K)$ of second order of the eleven fields in the uni-polarized ground state contained in Table 2. The general structure of $\alpha_2(K)$ is the following

$$\alpha_2(K) = [21; (\tau_0; 22111, D_1), (211; 22111, D_2), (211; 22111, D_3); (211; 22111, D_4)] \tag{27}$$

where $\tau_0 \in \{222, 321\}$, and each dodecuplet $D_i$, $1 \le i \le 4$, consists of a triplet and a nonet.

Table 3. Details for eleven fields $K = \mathbb{Q}(\sqrt{d})$ in Table 2

| No. | $\tau_0$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | remark |
|---|---|---|---|---|---|---|
| 87 | 321 | $(41111)^3(32)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | ref. 29 |
| 124 | 321 | $(3211)^3(32)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | $(3111)^3(32)^9$ | ref. 16,19,23,25,27 |
| 203 | 222 | $(32111)^3(221)^9$ | $(3211)^3(32)^9$ | $(3211)^3(221)^9$ | $(2221)^3(221)^9$ | extreme |
| 305 | 222 | $(32111)^3(221)^9$ | $(3211)^3(32)^9$ | $(3211)^3(221)^9$ | $(2221)^3(221)^9$ | extreme |
| 413 | 321 | $(3211)^3(32)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | $(3111)^3(32)^9$ | ref. 16,19,23,25,27 |
| 438 | 222 | $(3211)^3(221)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | $(2211)^3(221)^9$ | ref. 4,6 |
| 470 | 222 | $(3211)^3(221)^9$ | $(321)^3(32)^9$ | $(321)^3(221)^9$ | $(3111)^3(32)^9$ | ref 10,12,13,15 |
| 482 | 321 | $(32211)^3(32)^9$ | $(3221)^3(32)^9$ | $(3221)^3(32)^9$ | $(3221)^3(221)^9$ | extreme |
| 635 | 222 | $(3211)^3(221)^9$ | $(321)^3(32)^9$ | $(321)^3(32)^9$ | $(3111)^3(32)^9$ | ref 3,5,11,14 |
| 637 | 222 | $(3211)^3(221)^9$ | $(321)^3(32)^9$ | $(321)^3(221)^9$ | $(3111)^3(32)^9$ | ref 10,12,13,15 |
| 743 | 222 | $(3211)^3(221)^9$ | $(321)^3(32)^9$ | $(321)^3(221)^9$ | $(3111)^3(32)^9$ | ref 10,12,13,15 |

The following theorem provides evidence of a new class of algebraic number fields with 3-class group of type $\mathrm{Cl}_3(K) \simeq (9,3)$ whose 3-class field tower consists of exactly three stages.

**Theorem 9.** *An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with non-elementary 3-class group $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$ of rank two, punctured capitulation type B.18, $\varkappa(K) \sim (144;4)$, and abelian type invariants $\alpha_2(K)$ of second order of the shape in Formula (27) with either*

$$\tau_0 = 2^3,\ D_1 = (32111)^3(221)^9,\ D_2 = (321)^3(32)^9,\ D_3 = (321)^3(32)^9,\ D_4 = (321)^3(32)^9 \tag{28}$$

*or*

$$\tau_0 = 2^3,\ D_1 = (3211)^3(221)^9,\ D_2 = (321)^3(32)^9,\ D_3 = (321)^3(32)^9,\ D_4 = (2211)^3(221)^9 \tag{29}$$

*or*

$$\tau_0 = 2^3,\ D_1 = (3211)^3(221)^9,\ D_2 = (321)^3(32)^9,\ D_3 = (321)^3(221)^9,\ D_4 = (3111)^3(32)^9 \tag{30}$$

*or*

$$\tau_0 = 321,\ D_1 = (41111)^3(32)^9,\ D_2 = (321)^3(32)^9,\ D_3 = (321)^3(32)^9,\ D_4 = (321)^3(32)^9 \tag{31}$$

*possesses a finite 3-class field tower*

$$K = \mathrm{F}_3^0(K) < \mathrm{F}_3^1(K) < \mathrm{F}_3^2(K) < \mathrm{F}_3^3(K) = \mathrm{F}_3^\infty(K)$$

*with precise length $\ell_3(K) = 3$.*

In the following corollary, Theorem 9 is supplemented by information on the Galois group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ and its metabelianization $M = G/G'' \simeq \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$.

**Corollary 2.** *Let $K$ be a field with properties as in the assumptions of Theorem 9. Then the automorphism group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of the full 3-class field tower of $K$ is a non-metabelian Schur $\sigma$-group with soluble length $\mathrm{sl}(G) = 3$, order $\#G = 3^{21}$ and nilpotency class $\mathrm{cl}(G) = 9$. The second 3-class group $M = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of $K$ is a metabelian $\sigma$-group of order $\#M = 3^{10}$ and nilpotency class $\mathrm{cl}(M) = 5$.*

*Proof.* Formula (28) leads to either $\ell = 2$, $m = 83$ [26, Lem. 10] or $\ell = 7$, $m = 82$.
Formula (29) leads to $\ell \in \{4,6\}$, $m = 83$ and $108 = 81 + 27$ candidates for $G$ [26, Lem. 6].
Formula (30) leads to either $\ell \in \{10,13\}$, $m = 82$ or $\ell \in \{12,15\}$, $m = 83$ [26, Lem. 8].
Formula (31) leads to $\ell = 29$, $m = 85$ and 27 candidates for $G$ [26, Lem. 10].
Let $B := \langle 6561, 165\rangle = \langle 729, 10\rangle - \#2; 2$ in the notation of [6, 13] be the common fork of the root paths of all finite 3-groups $G$ with non-elementary bicyclic commutator quotient $G/G' \simeq C_9 \times C_3$, punctured transfer kernel type B.18, $\varkappa(G) \sim (144;4)$, and logarithmic abelian quotient invariants of first order $\alpha_1(G) = \bigl(21; (\tau_0, 211, 211; 211)\bigr)$ with $\tau_0 \in \{222, 321\}$. Then the candidates for $G$ are given in the shape $B - \#4; \ell - \#2; k - \#4; j - \#1; i - \#2; h$ with $1 \le \ell \le 72$, $1 \le k \le 41$, $1 \le j \le 27$, $1 \le i \le 5$, where $k$ is determined uniquely as a function $k = k(\ell)$ of $\ell$, $j$ runs through all possible values, $i$ is determined uniquely as a function $i = i(j)$ of $j$, and $1 \le h \le 3$ [26]. □

**Example 1.** The quadratic fields $K$ with fundamental discriminants
$d = -518\,835$ and $\ell \in \{4,6\}$,
respectively $d \in \{-553\,807, -763\,972, -876\,948\}$ and $\ell \in \{10,12,13,15\}$,
respectively $d = -150\,319$ and $\ell = 29$,
have punctured capitulation type $\varkappa(K) \sim (144;4)$ and are examples of field possessing a 3-class field tower with exactly three stages, $\ell_3(K) = 3$, of relative degrees

$$[\mathrm{F}_3^3(K) : \mathrm{F}_3^2(K)] = 3^{11}, \quad [\mathrm{F}_3^2(K) : \mathrm{F}_3^1(K)] = 3^7, \quad [\mathrm{F}_3^1(K) : \mathrm{F}_3^0(K)] = 3^3,$$

and Galois group $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of order $3^{21}$.

**Remark 2.** The quadratic fields $K$ with fundamental discriminants
$d = -761\,855$ and $\ell \in \{3,5,11,14\}$,
respectively $d_K \in \{-194\,703, -494\,771\}$ and $\ell \in \{16,19,23,25,27\}$,
have punctured capitulation type $\varkappa(K) \sim (144;4)$ and $3 \le \ell_3(K) \le 4$.

The quadratic fields $K$ with fundamental discriminants $d \in \{-294\,983, -389\,435\}$ and punctured capitulation type $\varkappa(K) \sim (144;4)$ have an infinite 3-class field tower.

## 8. 3-groups with commutator quotient $(27, 3)$

In Table 4, we list the second AQI $\alpha_2$ of the 30 non-metabelian step size-4 descendants $F - \#4;\ell$ with $43 \le \ell \le 72$ of the metabelian fork $F = \langle 2187, 3\rangle - \#2;10$. The general structure of $\alpha_2$ is

$$\alpha_2(G) = [31; (421; 32111, D_1), (311; 32111, D_2), (311; 32111, D_3); (221; 32111, D_4)], \tag{32}$$

where each dodecuplet $D_i$, $1 \le i \le 3$, consists of a triplet $T_i^3$ and a nonet $N_i^9$. However, $D_4$ consists of a triplet $T_4^3$ and either a nonet $N_4^9$ or an octet $O_4^8$ and a singlet $S_4$. The metabelianization $M = G/G''$ is given by the step size-2 descendant $F - \#2;m$ with $m \in \{88, 90\}$ of the fork $F$. The smallest logarithmic order, soluble length, of a Schur $\sigma$-descendant $S$ of $G$ is $\mathrm{lo}(S)$, $\mathrm{sl}(S)$.

Table 4. Invariants of $G = \langle 2187, 3\rangle - \#2;10 - \#4;\ell$ with $43 \le \ell \le 72$

| $\ell$ | $T_1$ | $N_1$ | $T_2$ | $N_2$ | $T_3$ | $N_3$ | $T_4$ | $N_4$ | $O_4$ | $S_4$ | $m$ | $\mathrm{lo}(S)$ | $\mathrm{sl}(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 43 | 4211 | 42 | 421 | 42 | 421 | 42 | 3211 | 321 | | | 88 | 37 | 4 |
| 58 | 4211 | 42 | 421 | 42 | 421 | 42 | 3211 | 321 | | | 90 | 37 | 4 |
| 44 | 4211 | 42 | 421 | 321 | 4111 | 42 | 331 | | 321 | 222 | 88 | 22 | 3 |
| 59 | 4211 | 42 | 421 | 321 | 4111 | 42 | 331 | | 321 | 222 | 90 | 22 | 3 |
| 45 | 4211 | 42 | 421 | 321 | 421 | 321 | 3211 | 321 | | | 88 | 34 | 4 |
| 62 | 4211 | 42 | 421 | 321 | 421 | 321 | 3211 | 321 | | | 90 | 34 | 4 |
| 46 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | | 321 | 222 | 88 | 22 | 3 |
| 50 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | | 321 | 222 | 88 | 22 | 3 |
| 63 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | | 321 | 222 | 90 | 22 | 3 |
| 65 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | | 321 | 222 | 90 | 22 | 3 |
| 47 | 41111 | 42 | 4111 | 42 | 4111 | 42 | 3211 | 321 | | | 88 | $\infty$ | $\infty$ |
| 60 | 41111 | 42 | 4111 | 42 | 4111 | 42 | 3211 | 321 | | | 90 | $\infty$ | $\infty$ |
| 48 | 41111 | 42 | 421 | 42 | 421 | 321 | 331 | | 321 | 222 | 88 | 22 | 3 |
| 61 | 41111 | 42 | 421 | 42 | 421 | 321 | 331 | | 321 | 222 | 90 | 22 | 3 |
| 49 | 4211 | 42 | 421 | 42 | 3211 | 321 | 331 | 321 | | | 88 | 25 | 3 |
| 64 | 4211 | 42 | 421 | 42 | 3211 | 321 | 331 | 321 | | | 90 | 25 | 3 |
| 51 | 4211 | 42 | 421 | 42 | 421 | 321 | 3211 | | 321 | 222 | 88 | 22 | 3 |
| 66 | 4211 | 42 | 421 | 42 | 421 | 321 | 3211 | | 321 | 222 | 90 | 22 | 3 |
| 52 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | 321 | | | 88 | 25 | 3 |
| 70 | 4211 | 42 | 421 | 42 | 4111 | 42 | 331 | 321 | | | 90 | 25 | 3 |
| 53 | 4211 | 42 | 421 | 321 | 3211 | 321 | 331 | | 321 | 222 | 88 | 28 | 4 |
| 54 | 4211 | 42 | 421 | 42 | 421 | 42 | 3211 | | 321 | 222 | 88 | 22 | 3 |
| 72 | 4211 | 42 | 421 | 42 | 421 | 42 | 3211 | | 321 | 222 | 90 | 22 | 3 |
| 55 | 41111 | 42 | 421 | 42 | 421 | 321 | 331 | 321 | | | 88 | 25 | 3 |
| 67 | 41111 | 42 | 421 | 42 | 421 | 321 | 331 | 321 | | | 90 | 25 | 3 |
| 56 | 41111 | 42 | 421 | 42 | 421 | 42 | 331 | | 321 | 222 | 88 | 22 | 3 |
| 68 | 41111 | 42 | 421 | 42 | 421 | 42 | 331 | | 321 | 222 | 90 | 22 | 3 |
| 57 | 41111 | 42 | 4111 | 42 | 3211 | 321 | 3211 | | 321 | 222 | 88 | $\infty$ | $\infty$ |
| 69 | 41111 | 42 | 4111 | 42 | 3211 | 321 | 3211 | | 321 | 222 | 90 | $\infty$ | $\infty$ |
| 71 | 4211 | 42 | 421 | 321 | 3211 | 321 | 331 | | 321 | 222 | 90 | 25 | 3 |

**Theorem 10.** *The Schur $\sigma$-groups $S$ with commutator quotient $S/S' \simeq (27, 3)$, punctured transfer kernel type* B.18, *$\varkappa(S) \sim (144; 4)$, and first AQI $\alpha_1(S) \sim (421, 311, 311; 221)$ are descendants of the* 30 *non-metabelian 3-groups $G = \langle 2187, 3\rangle - \#2;10 - \#4;\ell$ whose invariants are listed in Table 4. In the case of finite order $\mathrm{lo}(S) < \infty$, their invariants coincide with those of the predecessor $G$. For $22 \le \mathrm{lo}(S) \le 25$ they have three stages $\mathrm{sl}(S) = 3$, and for $28 \le \mathrm{lo}(S) < \infty$ they have four stages $\mathrm{sl}(S) = 4$. For $43 \le \ell \le 57$ their metabelianization $S/S'' \simeq G/G''$ is $M = \langle 2187, 3\rangle - \#2;10 - \#2;88$, and for $58 \le \ell \le 72$ it is $M = \langle 2187, 3\rangle - \#2;10 - \#2;90$.*

The minimum $\mathrm{lo}(S) = 22$ occurs for 14 values $\ell$, 25 for 7, 28 for 1, 34 for 2, 37 for 2, and $\infty$ for 4.

## 9. Imaginary quadratic fields $K$ with $\mathrm{Cl}_3(K) \simeq C_{27} \times C_3$

The 930 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-3\,000\,000 < d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_{27} \times C_3$ were computed together with their punctured capitulation types $\varkappa(K)$ and first abelian type invariants $\alpha_1(K)$ by means of the computational algebra system Magma [17] in 19 132 seconds of CPU time, that is more than 5 hours. In Table 5, the first 16 cases with punctured capitulation type B.18, $\varkappa(K) \sim (144; 4)$, and *uni-polarized* abelian type invariants $\alpha_1(K)$ of first order in the *ground state* are listed. Bi-polarized cases and excited states are excluded.

Table 5. Sixteen fields $K = \mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K) \simeq C_{27} \times C_3$ and $\varkappa(K) \sim (144; 4)$

| No. | $d$ | factors | $\alpha_1(K)$ |
|---|---|---|---|
| 15 | $-163\,736$ | $2, 97, 211$ | $(421, 311, 311; 221)$ |
| 25 | $-218\,123$ | $59, 3\,697$ | $(421, 311, 311; 221)$ |
| 75 | $-428\,935$ | $5, 13, 6\,599$ | $(421, 311, 311; 221)$ |
| 121 | $-615\,467$ | $19, 29, 1\,117$ | $(421, 311, 311; 221)$ |
| 202 | $-892\,459$ | $31, 28\,789$ | $(421, 311, 311; 221)$ |
| 234 | $-985\,727$ | $463, 2\,129$ | $(421, 311, 311; 221)$ |
| 304 | $-1\,216\,407$ | $3, 47, 8\,627$ | $(421, 311, 311; 221)$ |
| 322 | $-1\,263\,279$ | $3, 421\,093$ | $(421, 311, 311; 221)$ |
| 328 | $-1\,283\,531$ | $701, 1\,831$ | $(421, 311, 311; 221)$ |
| 357 | $-1\,358\,087$ | prime | $(421, 311, 311; 221)$ |
| 407 | $-1\,502\,187$ | $3, 500\,729$ | $(421, 311, 311; 221)$ |
| 425 | $-1\,561\,043$ | $11, 191, 743$ | $(421, 311, 311; 221)$ |
| 433 | $-1\,588\,196$ | $2, 23, 61, 283$ | $(421, 311, 311; 221)$ |
| 475 | $-1\,752\,787$ | $67, 26\,161$ | $(421, 311, 311; 221)$ |
| 508 | $-1\,853\,828$ | $2, 463\,457$ | $(421, 311, 311; 221)$ |
| 590 | $-2\,052\,195$ | $3, 5, 136\,813$ | $(421, 311, 311; 221)$ |

In Table 6, we give the abelian type invariants $\alpha_2(K)$ of second order of the 16 fields in the uni-polarized ground state contained in Table 5. They were computed with the aid of Magma [17] in 74 318 seconds of CPU time, that is nearly 21 hours. The general structure of $\alpha_2(K)$ is

$$\alpha_2(K) = [31; (421; 32111, D_1), (311; 32111, D_2), (311; 32111, D_3); (221; 32111, D_4)] \quad (33)$$

where each dodecuplet $D_i$, $1 \le i \le 3$, consists of a triplet and a nonet, and $D_4$ consists of a triplet, an octet and a singlet. A reference to Table 4 is added. It usually admits the determination of the length $\ell_3(K)$ of the 3-class field tower of $K$.

**Example 2.** According to Tables 5 and 6 together with Theorem 11, we get the following 5 examples of 3-class field towers with precisely three stages, $\ell_3(K) = \mathrm{sl}(S) = 3$:

- $d \in \{-218\,123, -1\,358\,087\}$ both with $\ell \in \{51, 66\}$,
- $d = -892\,459$ with $\ell \in \{44, 59\}$,
- $d = -1\,263\,279$ with $\ell \in \{52, 70\}$ and $\mathrm{lo}(S) = 25$,
- $d = -1\,752\,787$ with $\ell \in \{46, 50, 63, 65\}$.

In contrast, the 3-class field tower is infinite for $d \in \{-163\,736, -428\,935, -985\,727, -1\,561\,043\}$. No statement is possible for $d \in \{-1\,216\,407, -1\,283\,531, -1\,502\,187, -1\,853\,828, -2\,052\,195\}$, since the associated Schur $\sigma$-groups $S$ are unknown.

**Example 3.** As a particular highlight we point out the unique example of a 3-class field **tower with precisely four stages**, $\ell_3(K) = \mathrm{sl}(S) = 4$, for $d = -1\,588\,196$ with $\ell \in \{43, 58\}$ and $\mathrm{lo}(S) = 37$. As opposed, the precise length is unknown for $d = -615\,467$ with $\ell \in \{53, 71\}$ and $3 \le \ell_3(K) = \mathrm{sl}(S) \le 4$.

Table 6. Details for the fields $K = \mathbb{Q}(\sqrt{d})$ in Table 5

| No. | $D_1$ | $D_2$ | $D_3$ | $D_4$ | reference | $\ell_3(K)$ |
|---|---|---|---|---|---|---|
| 15 | $(42111)^3(42)^9$ | $(4211)^3(42)^9$ | $(3221)^3(321)^9$ | $(3311)^3(321)^8(222)$ | $57,69$ var. | $\infty$ |
| 25 | $(4211)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(3211)^3(321)^8(222)$ | $51,66$ | 3 |
| 75 | $(42111)^3(42)^9$ | $(4211)^3(42)^9$ | $(4211)^3(321)^9$ | $(3221)^3(321)^8(222)$ | $57,69$ var. | $\infty$ |
| 121 | $(4211)^3(42)^9$ | $(421)^3(321)^9$ | $(3211)^3(321)^9$ | $(331)^3(321)^8(222)$ | $53,71$ | 4 or 3 |
| 202 | $(4211)^3(42)^9$ | $(4111)^3(42)^9$ | $(421)^3(321)^9$ | $(331)^3(321)^8(222)$ | $44,59$ | 3 |
| 234 | $(42111)^3(42)^9$ | $(4211)^3(42)^9$ | $(4211)^3(321)^9$ | $(3311)^3(321)^8(222)$ | $57,69$ var. | $\infty$ |
| 304 | $(51111)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(331)^3(321)^8(222)$ | $48,61$ var. | ? |
| 322 | $(4211)^3(42)^9$ | $(421)^3(42)^9$ | $(4111)^3(42)^9$ | $(331)^3(321)^9$ | $52,70$ | 3 |
| 328 | $(51111)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(331)^3(321)^9$ | $55,67$ var. | ? |
| 357 | $(4211)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(3211)^3(321)^8(222)$ | $51,66$ | 3 |
| 407 | $(51111)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(331)^3(321)^8(222)$ | $48,61$ var. | ? |
| 425 | $(42111)^3(42)^9$ | $(4211)^3(42)^9$ | $(4211)^3(321)^9$ | $(3311)^3(321)^8(222)$ | $57,69$ var. | $\infty$ |
| 433 | $(4211)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(42)^9$ | $(3211)^3(321)^8(222)$ | $43,58$ | 4 |
| 475 | $(4211)^3(42)^9$ | $(421)^3(42)^9$ | $(4111)^3(42)^9$ | $(331)^3(321)^8(222)$ | $46,50,63,65$ | 3 |
| 508 | $(51111)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(321)^9$ | $(331)^3(321)^9$ | $55,67$ var. | ? |
| 590 | $(51111)^3(42)^9$ | $(421)^3(42)^9$ | $(421)^3(42)^9$ | $(331)^3(321)^8(222)$ | $56,68$ var. | ? |

**Theorem 11.** *For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with 3-class group $\mathrm{Cl}_3(K) \simeq (27,3)$ and punctured capitulation type* B.18, $\varkappa \sim (144;4)$, *the 3-class field tower consists of precisely three stages with Schur $\sigma$-group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of order $\#G = 3^{22}$ and nilpotency class $\mathrm{cl}(G) = 9$, if the following conditions for the abelian type invariants $\alpha_2(K)$ of second order in Formula* (33) *are satisfied. In the notation of the SmallGroups database* [6] *and the ANUPQ package* [13], *the 3-class field tower group is given by*

$$G \simeq \langle 2187,3\rangle - \#2;10 - \#4;\ell - \#2;k(\ell) - \#4;j - \#1;i(j) - \#2;h, \tag{34}$$

*where $43 \le \ell \le 72$ is determined by the second AQI $\alpha_2$, $1 \le k \le 41$ is determined by $\ell$, $1 \le j \le 27$ is arbitrary, $1 \le i \le 2$ is determined by $j$, and $1 \le h \le N$ is arbitrary below an upper bound $N \in \{1,3\}$ determined by $\ell$.*

- *$\ell \in \{51,66\}$, $N = 3$, i.e. 162 candidates for $G$,*
  *if $D_1 = (4211)^3(42)^9$, $D_2 = (421)^3(42)^9$, $D_3 = (421)^3(321)^9$, $D_4 = (3211)^3(321)^8(222)$;*
- *$\ell \in \{44,59\}$, $N = 1$, i.e. 54 candidates for $G$,*
  *if $D_1 = (4211)^3(42)^9$, $D_2 = (421)^3(321)^9$, $D_3 = (4111)^3(42)^9$, $D_4 = (331)^3(321)^8(222)$.*

*The metabelianization $M = G/G'' \simeq \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$, which is isomorphic to the second 3-class group of $K$, has order $\#M = 3^{11}$, nilpotency class $\mathrm{cl}(M) = 5$ and is given by*

$$M \simeq \langle 2187,3\rangle - \#2;10 - \#2;m, \tag{35}$$

*where $m = 88$ if $\ell \le 57$, and $m = 90$ if $\ell \ge 58$.*

*Proof.* Among the 14 descendants $\langle 2187,3\rangle - \#2;10 - \#4;\ell$ which give rise to Schur $\sigma$-groups of minimal order $3^{22}$, that is $\ell \in \{44,46,48,50,51,54,56,59,61,63,65,66,68,72\}$, the second AQI in the statements are unique. It remains to check the other 16 values of $43 \le \ell \le 72$ with Tbl. 4. □

## 10. 3-groups with commutator quotient $(81,3)$

In Table 7, we list the second AQI $\alpha_2$ of the 30 non-metabelian step size-4 descendants $B - \#4;\ell$ with $80 \le \ell \le 109$ of the metabelian fork $B = \langle 2187,3\rangle - \#3;2$. The general structure of $\alpha_2$ is

$$\alpha_2(G) = [41;(521;42111,D_1),(411;42111,D_2),(411;42111,D_3);(321;42111,D_4)] \tag{36}$$

where each dodecuplet $D_i$, $1 \le i \le 3$, consists of a triplet $T_i^3$ and a nonet $N_i^9$, and $D_4$ usually consists of a triplet $T_4^3$, an octet $O_4^8$ and a singlet $S_4$. The metabelianization $M = G/G''$ is given by the step size-2 descendant $B - \#2; m$ with $m \in \{100, 102\}$ of the fork $B$. The smallest logarithmic order, soluble length, of a Schur $\sigma$-descendant $S$ of $G$ is $\mathrm{lo}(S)$, $\mathrm{sl}(S)$.

TABLE 7. Invariants of $G = \langle 2187, 3\rangle - \#3; 2 - \#4; \ell$ with $80 \le \ell \le 109$

| $\ell$ | $T_1$ | $N_1$ | $T_2$ | $N_2$ | $T_3$ | $N_3$ | $T_4$ | $D_4$ | $O_4$ | $S_4$ | $m$ | $\mathrm{lo}(S)$ | $\mathrm{sl}(S)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 80 | 5211 | 52 | 521 | 52 | 521 | 52 | 3311 | $(331)^6(322)^3$ | | | 100 | 46 | 4 |
| 95 | 5211 | 52 | 521 | 52 | 521 | 52 | 3311 | $(331)^6(322)^3$ | | | 102 | 46 | 4 |
| 81 | 5211 | 52 | 521 | 421 | 5111 | 52 | 431 | | 421 | 322 | 100 | 23 | 3 |
| 96 | 5211 | 52 | 521 | 421 | 5111 | 52 | 431 | | 421 | 322 | 102 | 23 | 3 |
| 82 | 5211 | 52 | 521 | 421 | 521 | 421 | 3311 | $(331)^6(322)^3$ | | | 100 | 40 | 4 |
| 99 | 5211 | 52 | 521 | 421 | 521 | 421 | 3311 | $(331)^6(322)^3$ | | | 102 | 40 | 4 |
| 83 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | | 421 | 322 | 100 | 23 | 3 |
| 87 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | | 421 | 322 | 100 | 23 | 3 |
| 100 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | | 421 | 322 | 102 | 23 | 3 |
| 102 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | | 421 | 322 | 102 | 23 | 3 |
| 84 | 51111 | 52 | 5111 | 52 | 5111 | 52 | 3311 | $(331)^6(322)^3$ | | | 100 | $\infty$ | $\infty$ |
| 97 | 51111 | 52 | 5111 | 52 | 5111 | 52 | 3311 | $(331)^6(322)^3$ | | | 102 | $\infty$ | $\infty$ |
| 85 | 51111 | 52 | 521 | 52 | 521 | 421 | 431 | | 421 | 322 | 100 | 23 | 3 |
| 98 | 51111 | 52 | 521 | 52 | 521 | 421 | 431 | | 421 | 322 | 102 | 23 | 3 |
| 86 | 5211 | 52 | 521 | 52 | 4211 | 421 | 431 | $(331)^6(322)^3$ | | | 100 | 29 | 3 |
| 101 | 5211 | 52 | 521 | 52 | 4211 | 421 | 431 | $(331)^6(322)^3$ | | | 102 | 29 | 3 |
| 88 | 5211 | 52 | 521 | 52 | 521 | 421 | 4211 | | 421 | 322 | 100 | 23 | 3 |
| 103 | 5211 | 52 | 521 | 52 | 521 | 421 | 4211 | | 421 | 322 | 102 | 23 | 3 |
| 89 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | $(331)^6(322)^3$ | | | 100 | 29 | 3 |
| 107 | 5211 | 52 | 521 | 52 | 5111 | 52 | 431 | $(331)^6(322)^3$ | | | 102 | 29 | 3 |
| 90 | 5211 | 52 | 521 | 421 | 4211 | 421 | 431 | | 421 | 322 | 100 | 29 | 4 |
| 91 | 5211 | 52 | 521 | 52 | 521 | 52 | 4211 | | 421 | 322 | 100 | 23 | 3 |
| 109 | 5211 | 52 | 521 | 52 | 521 | 52 | 4211 | | 421 | 322 | 102 | 23 | 3 |
| 92 | 51111 | 52 | 521 | 52 | 521 | 421 | 431 | $(331)^6(322)^3$ | | | 100 | 29 | 3 |
| 104 | 51111 | 52 | 521 | 52 | 521 | 421 | 431 | $(331)^6(322)^3$ | | | 102 | 29 | 3 |
| 93 | 51111 | 52 | 521 | 52 | 521 | 52 | 431 | | 421 | 322 | 100 | 23 | 3 |
| 105 | 51111 | 52 | 521 | 52 | 521 | 52 | 431 | | 421 | 322 | 102 | 23 | 3 |
| 94 | 51111 | 52 | 5111 | 52 | 4211 | 421 | 4211 | | 421 | 322 | 100 | $\infty$ | $\infty$ |
| 106 | 51111 | 52 | 5111 | 52 | 4211 | 421 | 4211 | | 421 | 322 | 102 | $\infty$ | $\infty$ |
| 108 | 5211 | 52 | 521 | 421 | 4211 | 421 | 431 | | 421 | 322 | 102 | 26 | 3 |

**Theorem 12.** *The Schur $\sigma$-groups $S$ with commutator quotient $S/S' \simeq (81, 3)$, punctured transfer kernel type* B.18*, $\varkappa(S) \sim (144; 4)$, and first AQI $\alpha_1(S) \sim (521, 411, 411; 321)$ are descendants of the* 30 *non-metabelian* 3*-groups $G = \langle 2187, 3\rangle - \#3; 2 - \#4; \ell$ whose invariants are listed in Table 7. In the case of finite order $\mathrm{lo}(S) < \infty$, their invariants coincide with those of the predecessor $G$. For $23 \le \mathrm{lo}(S) \le 26$ they have three stages $\mathrm{sl}(S) = 3$, for $40 \le \mathrm{lo}(S) < \infty$ four stages $\mathrm{sl}(S) = 4$ and for $\mathrm{lo}(S) \le 29$ they have $3 \le \mathrm{sl}(S) \le 4$. For $80 \le \ell \le 94$ their metabelianization $S/S'' \simeq G/G''$ is $M = \langle 2187, 3\rangle - \#3; 2 - \#2; 100$, and for $95 \le \ell \le 109$ it is $M = \langle 2187, 3\rangle - \#3; 2 - \#2; 102$.*

The minimum $\mathrm{lo}(S) = 23$ occurs for 14 values $\ell$, 26 for 1, 29 for 7, 40 for 2, 46 for 2, and $\infty$ for 4.

**Remark 3.** Table 4 and Theorem 10 were completed on 24 August 2021. After the discovery of the fork $B = \langle 2187, 3\rangle - \#3; 2$ on 26 August 2021, Table 7 could be computed immediately: For all $1 \le e \le 4$, exemplary representatives of multiplets of Schur $\sigma$-groups $S$ with commutator quotient $S/S' \simeq (3^e, 3)$ can be found according to the *principle of extremal root paths* (see Figure 3),

$$S \xrightarrow{s=2} \pi(S) \xrightarrow{s=1} \pi^2(S) \xrightarrow{s=4} \pi^3(S) \xrightarrow{s=2} \pi^4(S) \xrightarrow{s=4} \pi^5(S) = B.$$

This is not possible any longer for $e \ge 5$, due to the beginning discrepancy between parents $\pi(G)$ and $p$-parents $\pi_p(G)$ of finite 3-groups $G$.

## 11. Imaginary quadratic fields $K$ with $\mathrm{Cl}_3(K) \simeq C_{81} \times C_3$

The 2174 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-20\,000\,000 < d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_{81} \times C_3$ were computed by means of the computational algebra system Magma [17] in 141 586 seconds of CPU time, that is nearly two days. In Table 8, the first eight cases with punctured capitulation type B.18, $\varkappa(K) \sim (144;4)$, are listed. The abelian quotient invariants $\alpha_1(K)$ of first order of only six of them are *uni-polarized* and in the *ground state.*

Table 8. Eight fields $K = \mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K) \simeq C_{81} \times C_3$ and $\varkappa(K) \sim (144;4)$

| No. | $d$ | factors | $\alpha_1(K)$ | remark |
|---|---|---|---|---|
| 31 | $-936\,311$ | prime | $(521, 411, 411; 321)$ | |
| 49 | $-1\,240\,879$ | $107, 11\,597$ | $(521, 411, 411; 321)$ | |
| 57 | $-1\,437\,179$ | $19, 75\,641$ | $(521, 411, 411; 321)$ | |
| 78 | $-1\,723\,864$ | $2, 215\,483$ | $(521, 411, 411; 321)$ | |
| 80 | $-1\,749\,655$ | $5, 349\,931$ | $(521, 411, 411; 321)$ | |
| 86 | $-1\,818\,223$ | $11, 165\,293$ | $(521, 411, 411; 321)$ | |
| 96 | $-1\,854\,319$ | $281, 6\,599$ | $(532, 411, 411; 321)$ | first excited state |
| 109 | $-2\,003\,179$ | $61, 32\,839$ | $(521, 411, 411; 332)$ | bi-polarized |

In Table 9, we give the abelian quotient invariants $\alpha_2(K)$ of second order of the six fields in the uni-polarized ground state contained in Table 8. The general structure of $\alpha_2(K)$ is the following

$$\alpha_2(K) = [41; (521; 42111, D_1), (411; 42111, D_2), (411; 42111, D_3); (321; 42111, D_4)] \tag{37}$$

where each dodecuplet $D_i$, $1 \le i \le 3$, consists of a triplet and a nonet, and $D_4$ usually consists of a triplet, an octet and a singlet. But the constitution of $D_4$ may occasionally be irregular.

Table 9. Details for six fields $K = \mathbb{Q}(\sqrt{d})$ in Table 8

| No. | $D_1$ | $D_2$ | $D_3$ | $D_4$ | remark |
|---|---|---|---|---|---|
| 31 | $(5211)^3(52)^9$ | $(5111)^3(52)^9$ | $(521)^3(421)^9$ | $(431)^3(421)^8(322)$ | ref. 81, 96 |
| 49 | $(5211)^3(52)^9$ | $(5111)^3(52)^9$ | $(521)^3(52)^9$ | $(431)^3(421)^8(322)$ | ref. 83, 87, 100, 102 |
| 57 | $(5211)^3(52)^9$ | $(521)^3(52)^9$ | $(521)^3(421)^9$ | $(4211)^3(421)^8(322)$ | ref. 88, 103 |
| 78 | | | | $(431)^3(421)^8(322)$ | Magma int. err. |
| 80 | $(61111)^3(52)^9$ | $(521)^3(52)^9$ | $(521)^3(421)^9$ | $(431)^3(331)^6(322)^3$ | irregular |
| 86 | $(5211)^3(52)^9$ | $(5111)^3(52)^9$ | $(521)^3(52)^9$ | $(431)^3(421)^8(322)$ | ref. 83, 87, 100, 102 |

**Theorem 13.** *For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with 3-class group $\mathrm{Cl}_3(K) \simeq (81,3)$ and punctured capitulation type B.18, $\varkappa \sim (144;4)$, the 3-class field tower consists of precisely three stages with Schur $\sigma$-group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of order $\#G = 3^{23}$ and nilpotency class $\mathrm{cl}(G) = 9$, if the following conditions for the abelian quotient invariants $\alpha_2(K)$ of second order in Formula (37) are satisfied. In the notation of the SmallGroups database [6] and the ANUPQ package [13], the 3-class field tower group is given by*

$$G \simeq \langle 2187, 3\rangle - \#3; 2 - \#4; \ell - \#2; k(\ell) - \#4; j - \#1; i(j) - \#2; h, \tag{38}$$

*where $80 \le \ell \le 109$ is determined by the second AQI $\alpha_2$, $1 \le k \le 41$ is determined by $\ell$, $1 \le j \le 27$ is arbitrary, $1 \le i \le 2$ is determined by $j$, and $1 \le h \le N$ is arbitrary below an upper bound $N \in \{1,3\}$ determined by $\ell$.*

- *$\ell \in \{81, 96\}$, $N = 1$, i.e. 54 candidates for $G$, if $D_1 = (5211)^3(52)^9$, $D_2 = (521)^3(421)^9$, $D_3 = (5111)^3(52)^9$, $D_4 = (431)^3(421)^8(322)$;*

- $\ell \in \{83, 87, 100, 102\}$, $N = 3$, *i.e.* 324 *candidates for* $G$,
  *if* $D_1 = (5211)^3(52)^9$, $D_2 = (521)^3(52)^9$, $D_3 = (5111)^3(52)^9$, $D_4 = (431)^3(421)^8(322)$;
- $\ell \in \{88, 103\}$, $N = 3$, *i.e.* 162 *candidates for* $G$,
  *if* $D_1 = (5211)^3(52)^9$, $D_2 = (521)^3(52)^9$, $D_3 = (521)^3(421)^9$, $D_4 = (4211)^3(421)^8(322)$.

*The metabelianization* $M = G/G'' \simeq \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$, *which is isomorphic to the second* 3*-class group of* $K$, *has order* $\#M = 3^{12}$, *nilpotency class* $\mathrm{cl}(M) = 5$ *and is given by*

$$M \simeq \langle 2187, 3\rangle - \#3; 2 - \#2; m, \tag{39}$$

*where* $m = 100$ *if* $\ell \le 93$, *and* $m = 102$ *if* $\ell \ge 96$.

*Proof.* The root $\langle 2187, 3\rangle - \#3; 2$ can be viewed as usual descendant of $\langle 6561, 216\rangle$ with step size $s = 2$. Among the 14 descendants $\langle 2187, 3\rangle - \#3; 2 - \#4; \ell$ which give rise to Schur $\sigma$-groups of minimal order $3^{23}$, that is $\ell \in \{81, 83, 85, 87, 88, 91, 93, 96, 98, 100, 102, 103, 105, 109\}$, the second AQI in the statements are unique. It remains to check the other 16 values of $80 \le \ell \le 109$. □

**Example 4.** According to Tables 8 and 9 together with Theorem 13, we get the following 4 examples of 3-class field towers with precisely three stages, $\ell_3(K) = \mathrm{sl}(S) = 3$:

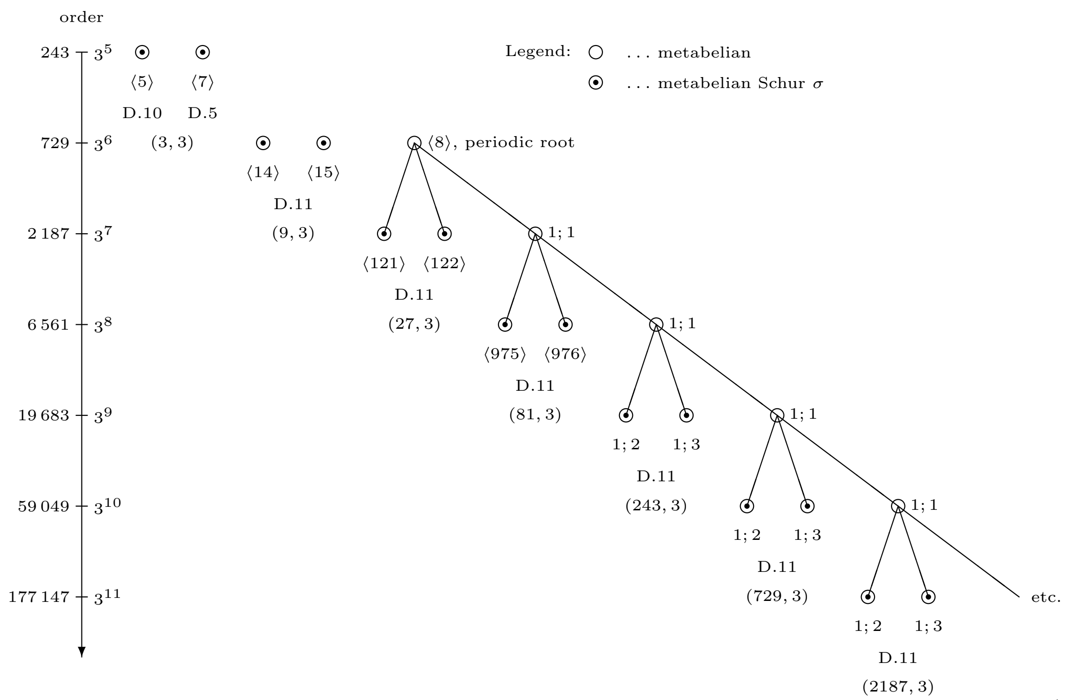
- $d = -936\,311$ with $\ell \in \{81, 96\}$,
- $d = -1\,240\,879$ and $d = -1\,818\,223$ both with $\ell \in \{83, 87, 100, 102\}$,
- $d = -1\,437\,179$ with $\ell \in \{88, 103\}$.

In contrast, no statement is possible for $d = -1\,749\,655$.

## 12. Motivation for seeking the new periodicities of Schur $\sigma$-groups

In our previous work [27, § 7, Thm. 4 and Thm. 7], we found a periodicity of pairs of *metabelian* Schur $\sigma$-groups $G$ with $G/G' \simeq (3^e, 3)$, $e \ge 3$, and type D.11, $\varkappa \sim (124; 1)$, which is illustrated by Figure 1.

Figure 1. Periodic metabelian Schur $\sigma$-groups $G$ with $G/G' \simeq (3^e, 3)$, $e \ge 3$




In the Figures 1 – 4, all directed edges lead from descendants $D$ to $p$-parents $\pi_p(D) = D/P_{c_p-1}(D)$, rather than to parents $\pi(D) = D/\gamma_c(D)$. The figures admit actual descendant construction.

In the main theorem [27, § 9, Thm. 12] of the previous work, we provided evidence of another periodicity of pairs of *non-metabelian* Schur $\sigma$-groups $G$ with $G/G' \simeq (3^e, 3)$, $e \geq 5$, and four types D.5, $\varkappa \sim (211;3)$, C.4, $\varkappa \sim (311;3)$, D.10, $\varkappa \sim (411;3)$, and D.6, $\varkappa \sim (123;1)$, which is illustrated for one member of the pair of type D.10 by Figure 2.

FIGURE 2. Schur $\sigma$-groups $G$ with $\varrho(G) \sim (2,2,3;3)$, $G/G' \simeq (3^e, 3)$, $2 \leq e \leq 7$

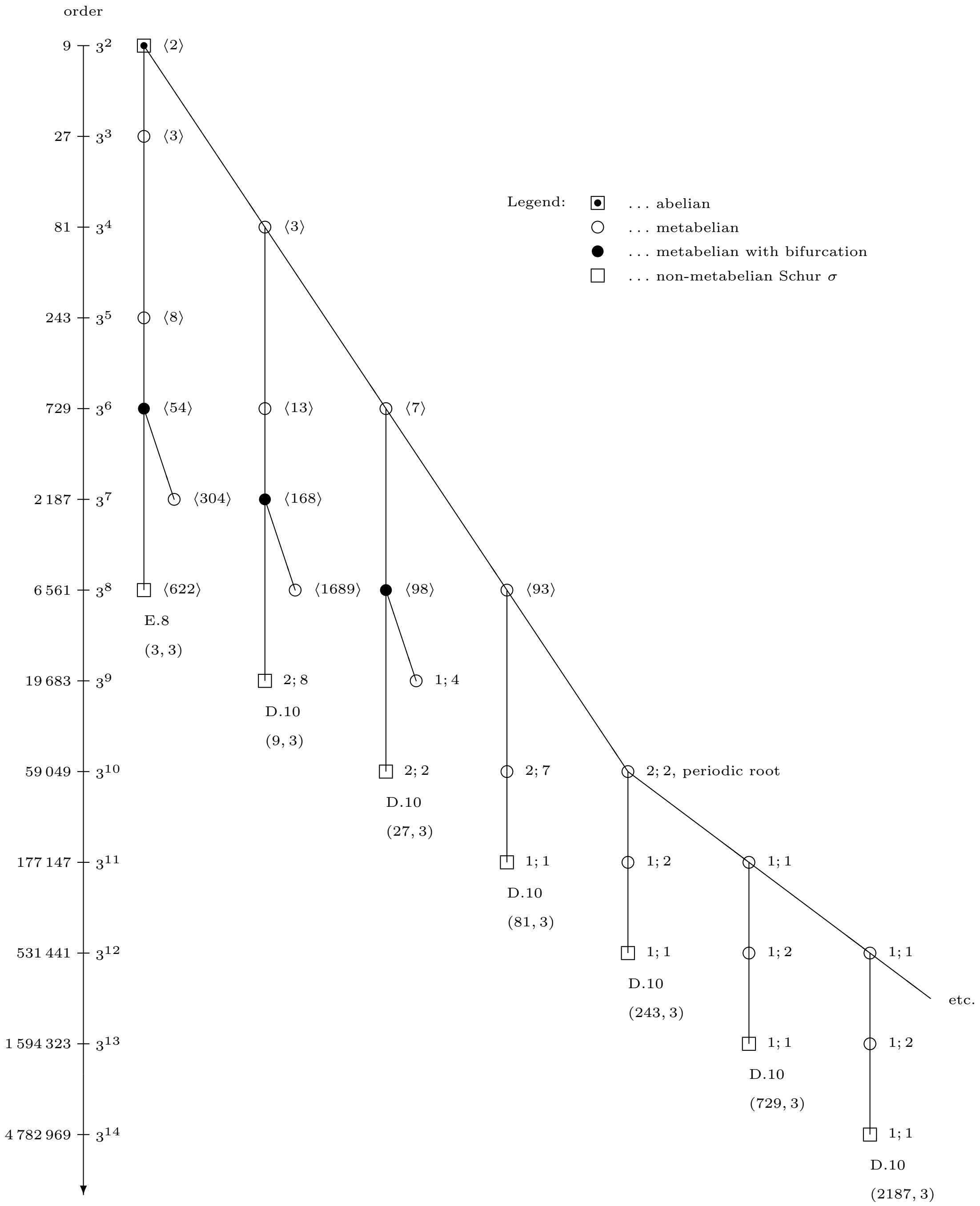
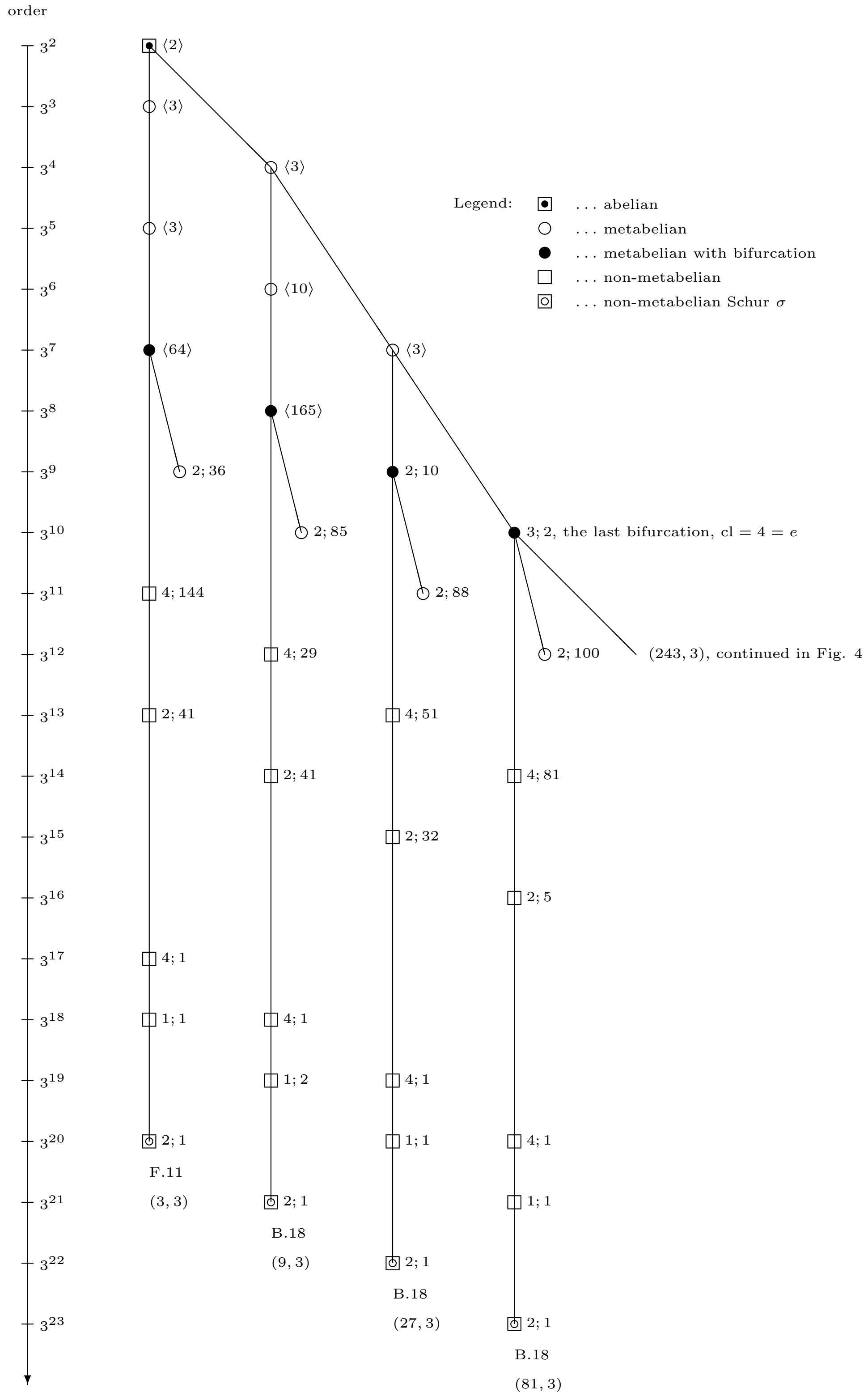



Figure 2 shows that the construction process for the eight non-metabelian Schur $\sigma$-groups $G$ with order $\#G = 3^{7+e}$ and punctured transfer kernel types D.10, C.4, D.5, and D.6, becomes increasingly difficult for the commutator quotients $G/G' \simeq (27,3)$, $(81,3)$, $(243,3)$. For the commutator quotient $G/G' \simeq (729,3)$, however, an **unexpected tranquilization** occurs, and the construction process becomes settled with a **simple step size one periodicity**.

FIGURE 3. Schur $\sigma$-groups $G$ with $\varrho(G) \sim (3,3,3;3)$, $G/G' \simeq (3^e,3)$, $2 \leq e \leq 4$



The investigation of periodic Schur $\sigma$-groups $G$ with *moderate* rank distribution $\varrho(G) \sim (2,2,2;3)$ or $\varrho(G) \sim (2,2,3;3)$ was completed in [27]. Although we were conscious that the difficulties will increase significantly, the tree diagram in Figure 2 inspired us to look at cases with *elevated* rank distribution on 21 August 2021. In Figure 3, we see how large Schur $\sigma$-groups $G$ with logarithmic order $\mathrm{lo}(G) = 19 + e$ and commutator quotient $G/G' \simeq (3^e, 3)$, $1 \le e \le 4$, can be constructed with the aid of the $p$-group generation algorithm [29, 30], which is implemented in the ANUPQ package [13] of the computational algebra system Magma [17]. In these four cases, the exponent $e$ is not bigger than the nilpotency class $\mathrm{cl}(F) = 4$ of the metabelian fork $F$ with bifurcation to non-metabelian vertices

$$G \overset{s=2}{\longrightarrow} \pi(G) \overset{s=1}{\longrightarrow} \pi^2(G) \overset{s=4}{\longrightarrow} \pi^3(G) \overset{s=2}{\longrightarrow} \pi^4(G) \overset{s=4}{\longrightarrow} \pi^5(G) = F.$$

They form the *extremal root path* of the Schur $\sigma$-group $G$, which is weighted by the maximal step sizes $s = \nu$ equal to the *nuclear rank* of the parent. In this region, parents and $p$-parents coincide.

Figure 3 for $2 \le e \le 4$, which is continued by Figure 4 for $4 \le e \le 13$, documents the stagnating state of our research enterprise on 31 August 2021, due to group theoretic problems. The initial cases were still in the region where parents and $p$-parents coincide,

$$\text{for } e = 3: \quad \langle 2187, 3\rangle \overset{s=2}{\longleftarrow} \langle 2187, 3\rangle - \#2;10 \overset{s=4}{\longleftarrow} \langle 2187, 3\rangle - \#2;10 - \#4;51 \longleftarrow \qquad \text{etc.}$$

$$\text{for } e = 4: \quad \langle 2187, 3\rangle - \#3;2 \overset{s=4}{\longleftarrow} \langle 2187, 3\rangle - \#3;2 - \#4;81 \longleftarrow \qquad \text{etc.}$$

However, the case $e = 5$ was outside of our reach already. We tried to look at the descendant $\langle 2187, 3\rangle - \#3;2 - \#5;1$, which has $G/G' \simeq (243, 3)$, but we got too big AQI of first order, namely $(622, 511, 511; 421)$ instead of $(621, 511, 511; 421)$.

At the commutator quotient $(81, 3) = (3^e, 3)$ with $e = 4$ the exponent $e$ overtakes the nilpotency class of the bifurcation $\mathrm{cl}(F) = 4$. It was not clear if the bifurcation will vanish for $(243, 3)$, but eventually it turned out that $B = \langle 2187, 3\rangle - \#3;2$ is simultaneous bifurcation for all commutator quotients $(3^e, 3)$ with $e \ge 4$. It can thus be called a *bifurcation of infinite order.*

After a lot of trial and error we succeeded in the construction of the desired Schur $\sigma$-groups G with logarithmic order 24, $G/G' \simeq (243, 3)$, i.e. $e = 5$, type B.18, $\varkappa \sim (144; 4)$, AQI $\alpha_1 \sim (621, 511, 511; 421)$, and $\mathrm{sl} = 3$. The mystery was solved on 06 September 2021 in the following way, which finally lead to Figure 4 on 13 September 2021. Let $B := \langle 2187, 3\rangle - \#3;2$.

In a first step, we looked for the metabelianization $M = G/G''$, and we got two unique solutions: $M = B - \#2;93 - \#1;i$ with $i \in \{2, 3\}$.

In a second step, we sought the non-metabelian Schur $\sigma$-group G. There are 15 possible starting points, $B - \#4;k$ with $23 \le k \le 37$, but only $k \in \{24, 26, 28, 30, 31, 33, 37\}$ leads to Schur $\sigma$-groups with minimal $\mathrm{lo}(G) = 19 + e$. (The other values of $k$ lead to $M = B - \#2;92 - \#1;i$ with $i \in \{2, 3\}$.) Exemplarily we take $k = 37$ in Figure 4. The **classical root path** with respect to the usual lower central **becomes disconnected**. The first non-metabelian vertex is irregular, $B - \#4;37 - \#1;i$ with $i \in \{2, 3\}$. It is isolated, since it has nuclear rank zero, and thus is useless for the construction. The remaining four non-metabelian vertices are regularly connected, beginning at $B - \#4;37 - \#3;j$ with $j \in \{73, 114\}$. We take $j = 73$ in Figure 4, which therefore illustrates a particular instance of the main Theorem 6. The structure of the relevant tree diagrams for the other five main Theorems 1 – 5 is the same as in Figure 4.

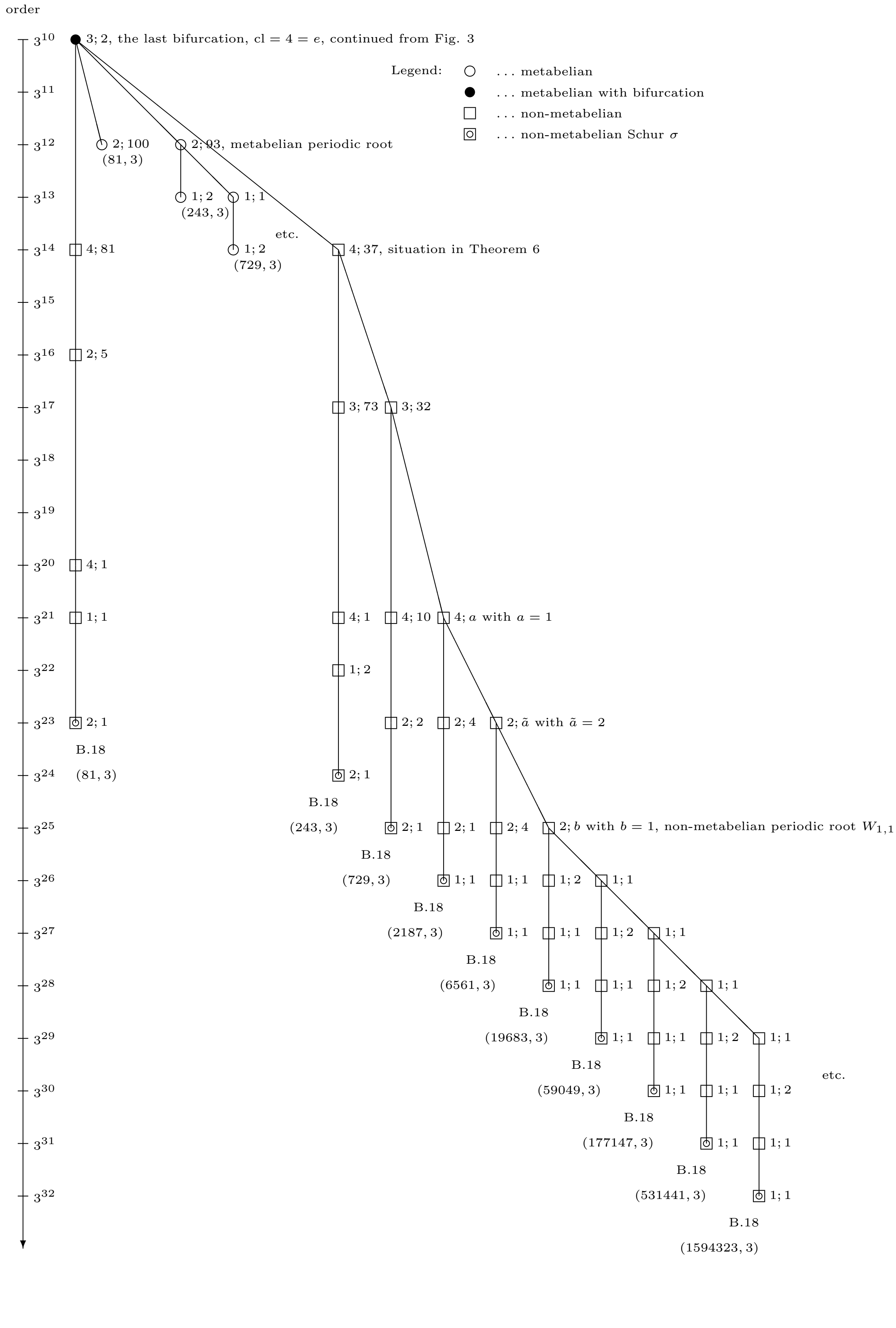
Concerning the bifurcations, we have the following information:

**Theorem 14.** *The bifurcations possess nearly identical pc-presentations: there are in fact only three bifurcations,* $B = \langle 6561, 165\rangle$ *for* $(9, 3)$*,* $B = \langle 2187, 3\rangle - \#2;10$ *for* $(27, 3)$*, and the* ***bifurcation of infinite order*** $B = \langle 2187, 3\rangle - \#3;2$ *for any* $(3^e, 3)$ *with* $e \ge 4$*. Denote some crucial commutators by* $s_2 = [y, x]$*,* $s_3 = [s_2, x]$*,* $t_3 = [s_2, y]$*,* $s_4 = [s_3, x]$*,* $t_4 = [t_3, y]$*,* $s_5 = [s_4, x]$*,* $t_5 = [t_4, y]$*. Then the polycyclic pc-presentation is given by*

$$B = \langle x, y \mid x^{3^e} = 1, y^3 = s_3 s_4^2, s_2^3 = s_4 t_4^2, [x^3, y] = s_4 t_4 \rangle \tag{40}$$

*with* $e = 2$*, respectively* $e = 3$*, respectively* $e = 4$*.*

FIGURE 4. Schur $\sigma$-groups $G$ with $\varrho(G) \sim (3,3,3;3)$, $G/G' \simeq (3^e,3)$, $4 \le e \le 13$



## 13. Imaginary quadratic fields $K$ with $\mathrm{Cl}_3(K) \simeq C_{243} \times C_3$

The 1784 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-60\,000\,000 < d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_{243} \times C_3$ were computed by means of the computational algebra system Magma [17] in 451 227 seconds of CPU time, that is nearly a full week. In Table 10, the first seven cases with punctured capitulation type B.18, $\varkappa(K) \sim (144; 4)$, are listed. The abelian quotient invariants $\alpha_1(K)$ of first order of only six of them are *uni-polarized* and in the *ground state*.

Table 10. Seven fields $K = \mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K) \simeq C_{243} \times C_3$ and $\varkappa(K) \sim (144; 4)$

| No. | $d$ | factors | $\alpha_1(K)$ | remark |
|---|---|---|---|---|
| 60 | $-5\,629\,151$ | $11, 631, 811$ | $(621, 511, 511; 421)$ | |
| 65 | $-5\,702\,003$ | prime | $(621, 511, 511; 421)$ | |
| 73 | $-6\,124\,411$ | prime | $(621, 511, 511; 421)$ | |
| 77 | $-6\,219\,188$ | $2, 1\,554\,797$ | $(621, 511, 511; 421)$ | |
| 116 | $-8\,513\,951$ | prime | $(621, 511, 511; 432)$ | bi-polarized |
| 149 | $-10\,401\,044$ | $2, 41, 63\,421$ | $(621, 511, 511; 421)$ | |
| 155 | $-10\,607\,215$ | $5, 2\,121\,443$ | $(621, 511, 511; 421)$ | |

In Table 11, we give the abelian quotient invariants $\alpha_2(K)$ of second order of the six fields in the uni-polarized ground state contained in Table 10. The general structure of $\alpha_2(K)$ is the following

$$\alpha_2(K) = [51; (621; 52111, D_1), (511; 52111, D_2), (511; 52111, D_3); (421; 52111, D_4)] \tag{41}$$

where each dodecuplet $D_i$ usually consists of a triplet and a nonet. Only the constitution of $D_4$ is occasionally irregular.

Table 11. Details for six fields $K = \mathbb{Q}(\sqrt{d})$ in Table 10

| No. | $D_1$ | $D_2$ | $D_3$ | $D_4$ | remark |
|---|---|---|---|---|---|
| 60 | $(6211)^3(62)^9$ | $(621)^3(62)^9$ | $(5211)^3(521)^9$ | $(531)^3(431)^6(422)^2(332)$ | irregular |
| 65 | | | | | Magma int. err. |
| 73 | $(6211)^3(62)^9$ | $(6111)^3(62)^9$ | | $(531)^3(521)^8(422)$ | Magma int. err. |
| 77 | | $(621)^3(62)^9$ | | $(531)^3(521)^8(422)$ | Magma int. err. |
| 149 | | | | | Magma int. err. |
| 155 | | | | | Magma int. err. |

## 14. Imaginary quadratic fields $K$ with $\mathrm{Cl}_3(K) \simeq C_{729} \times C_3$

The 263 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-60\,000\,000 < d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_{729} \times C_3$ were computed by means of the computational algebra system Magma [17] in 411 074 seconds of CPU time, that is nearly a full week. In Table 12, the eleven cases with punctured capitulation type B.18, $\varkappa(K) \sim (144; 4)$, are listed. The abelian quotient invariants $\alpha_1(K)$ of first order of only eight of them are *uni-polarized* and in the *ground state*.
In Table 13, we give the abelian quotient invariants $\alpha_2(K)$ of second order of the eight fields in the uni-polarized ground state contained in Table 12. The general structure of $\alpha_2(K)$ is the following

$$\alpha_2(K) = [61; (721; 62111, D_1), (611; 62111, D_2), (611; 62111, D_3); (521; 62111, D_4)] \tag{42}$$

where each dodecuplet $D_i$ usually consists of a triplet and a nonet. Only the constitution of $D_4$ is frequently (or even always) irregular.

Table 12. Eleven fields $K=\mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K)\simeq C_{729}\times C_3$ and $\varkappa(K)\sim(144;4)$

| No. | $d$ | factors | $\alpha_1(K)$ | remark |
|---|---|---|---|---|
| 9 | $-8\,716\,319$ | $2\,111, 4\,129$ | $(721,611,611;521)$ | |
| 17 | $-11\,598\,911$ | $19, 610\,469$ | $(721,611,611;521)$ | |
| 28 | $-17\,054\,671$ | prime | $(732,611,611;521)$ | first excited state |
| 94 | $-32\,670\,951$ | $3, 10\,890\,317$ | $(721,611,611;521)$ | |
| 133 | $-38\,393\,396$ | $2, 9\,598\,349$ | $(721,611,611;521)$ | |
| 141 | $-39\,551\,231$ | $17, 283, 8\,221$ | $(721,611,611;543)$ | highly bi-polarized |
| 144 | $-39\,948\,359$ | $11, 719, 5\,051$ | $(721,611,611;521)$ | |
| 197 | $-50\,631\,279$ | $3, 293, 57\,601$ | $(721,611,611;521)$ | |
| 198 | $-50\,963\,071$ | $439, 116\,089$ | $(721,611,611;521)$ | |
| 242 | $-57\,507\,455$ | $5, 11\,501\,491$ | $(721,611,611;521)$ | |
| 247 | $-58\,142\,996$ | $2, 14\,535\,749$ | $(721,611,611;532)$ | bi-polarized |

Table 13. Details for eight fields $K=\mathbb{Q}(\sqrt{d})$ in Table 12

| No. | $D_1$ | $D_2$ | $D_3$ | $D_4$ | remark |
|---|---|---|---|---|---|
| 9 | $(7211)^3(72)^9$ | $(721)^3(621)^9$ | $(721)^3(621)^9$ | $(5311)^3(531)^6(522)^2(432)$ | irregular |
| 17 | $(7211)^3(72)^9$ | $(6211)^3(621)^9$ | $(721)^3(72)^9$ | $(631)^3(531)^6(522)^2(432)$ | irregular |
| 94 | | | | | Magma int. err. |
| 133 | | | | | Magma int. err. |
| 144 | | | | | Magma int. err. |
| 197 | | | | $(631)^3(531)^6(522)^2(432)$ | irregular |
| 198 | | | | $(5321)^3(531)^6(522)^2(432)$ | irregular |
| 242 | $(7211)^3(72)^9$ | $(7111)^3(72)^9$ | | $(631)^3(531)^6(522)^2(432)$ | irregular |

**Remark 4.** In § 4 we have mentioned that it is rather hopeless to continue the search for imaginary quadratic fields $K=\mathbb{Q}(\sqrt{d})$ with bigger 3-class groups $\mathrm{Cl}_3(K)\simeq C_{3^e}\times C_3$ for $e\ge 7$. Firstly because of the immense amount of required CPU-time, and secondly in view of Magma internal errors which occur with increasing frequency during the computation of abelian type invariants $\alpha_2(K)$ of the second order for nonic relative extensions $L/K$ with absolute degree 18.

Nevertheless, we mention some interesting observations in experiments with $e=7$ and $e=8$. Concerning $e=7$, we found six ground states of type B.18 with $\alpha_1(K)\sim(821,711,711;621)$ for $d\in\{-37\,648\,463,-42\,705\,359,-122\,519\,927,-138\,616\,719,-154\,511\,167,-193\,538\,383\}$, and a bipolarization with $\alpha_1(K)\sim(821,711,711;632)$ for $d=-206\,130\,371$. Five of these seven discriminants are prime. The first two minimal hits of the desired 3-class group are the primes
$d=-32\,681\,951$ with $\alpha_1(K)\sim(81,81,821;711)$ and type D.5, $\varkappa(K)\sim(112;3)$,
$d=-35\,574\,431$ with $\alpha_1(K)\sim(81,81,711;711)$ and type D.11, $\varkappa(K)\sim(124;1)$.

Concerning $e=8$, we were at least able to discover three minimal hits of the desired 3-class group, though not of type B.18, $\varkappa(K)\sim(144;4)$. All discriminants are prime:
$d=-98\,311\,919$ with $\alpha_1(K)\sim(91,91,932;811)$ a first excited state of type D.5, $\varkappa(K)\sim(112;3)$,
$d=-201\,210\,239$ with $\alpha_1(K)\sim(91,91,811;811)$ and type D.11, $\varkappa(K)\sim(124;1)$, and
$d=-209\,606\,759$ with $\alpha_1(K)\sim(91,91,91;822)$ and type D.6, $\varkappa(K)\sim(123;1)$.

## 15. A general theorem

The previous sections with concrete results for various fixed values of the exponent $2\le e\le 20$ in the non-elementary bicyclic commutator quotient $G/G'\simeq(3^e,3)\hat{=}(e1)$ suggest the following generalization with upper bound $B:=20$.

**Theorem 15.** *In dependence on the exponent* $3 \le e \le B$*, the abelian quotient invariants* $\alpha_2(G)$ *of second order of finite Schur* $\sigma$*-groups* $G$ *with commutator quotient* $G/G' \simeq (e1)$*, punctured transfer kernel type* B.18*,* $\varkappa(K) \sim (144;4)$*, and logarithmic order* $\mathrm{lo}(G) = 19 + e$ *are given by*

$$\alpha_2(G) = [e1;((e+1)21;e2111,D_1),(e11;e2111,D_2),(e11;e2111,D_3);((e-1)21;e2111,D_4)], \tag{43}$$

*where each dodecuplet* $D_i$ *consists of a triplet* $T_i^3$ *and a nonet* $N_i^9$*, except for* $i = 4$*, where the nonet* $N_4^9$ *is replaced by an octet* $O_4^8$ *and a singlet* $S_4$*:*

$$\begin{aligned} &T_1 \in \{(e+1)211,(e+1)1111\}, \quad N_1 = (e+1)2, \\ &T_i \in \{(e+1)21,(e+1)111\}, \quad N_i \in \{(e+1)2,e21\}, \quad \text{for } 2 \le i \le 3, \\ &T_4 \in \{e31,e211\}, \quad O_4 = e21, \quad S_4 = (e-1)22. \end{aligned} \tag{44}$$

**Conjecture 1.** Theorem 15 remains true for any upper bound $B \ge 21$.

**Remark 5.** $T_i = e211$ for $2 \le i \le 3$ can also occur but it leads to bigger logarithmic order $\mathrm{lo}(G) > 19 + e$. The same is true for a nonet $N_4^9$ with $N_4 = e21$ in the fourth dodecuplet $D_4$. Theorem 15 was stated on 31 August 2021.

## 16. Conclusion

In our invited key note [25] at the 3rd International Conference on Mathematics and its Applications (ICMA) Casablanca, 28 February 2020, we offered supervision of a Ph.D. thesis about 3-groups with non-elementary bicyclic commutator quotient to the young researchers who listened to our talk and presentation with vigilance. That was eighteen months ago, immediately before the breakout of the worldwide Corona crisis, which prohibited any further scientific collaboration with personal contact. In the present article and its predecessor [27] we actually wrote this "thesis" ourselves, thereby discovering several groundbreaking and totally unexpected simple periodicities.

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Naglergasse 53, 8010 Graz, Austria

*E-mail address*: algebraic.number.theory@algebra.at

*URL*: http://www.algebra.at