DEEP TRANSFERS OF p-CLASS TOWER GROUPS

DANIEL C. MAYER

Abstract. Let p be a prime. For any finite p-group G, the deep transfers $T_{H,G'} : H/H' \to G'/G''$ from the maximal subgroups H of index $(G : H) = p$ in G to the derived subgroup G' are introduced as an innovative tool for identifying G uniquely by means of the family of kernels $\varkappa_d(G) = (\ker(T_{H,G'}))_{(G:H)=p}$. For all finite 3-groups G of coclass $\mathrm{cc}(G) = 1$, the family $\varkappa_d(G)$ is determined explicitly. The results are applied to the Galois groups $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ of the Hilbert 3-class towers of all real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d > 1$, 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and total 3-principalization in each of their four unramified cyclic cubic extensions E/F. A systematic statistical evaluation is given for the complete range $1 < d < 10^7$, and a few exceptional cases are pointed out for $1 < d < 10^8$.

1. Introduction

The layout of this paper is the following. Deep transfers of finite p-groups G, with an assigned prime number p, are introduced as an innovative supplement to the (usual) shallow transfers [13] in § 2. The family $\varkappa_d(G) = (\ker(T_{H,G'}))_{(G:H)=p}$ of the kernels of all deep transfers of G is called the *deep transfer kernel type* of G and will play a crucial role in this paper. For all finite 3-groups G of coclass $\mathrm{cc}(G) = 1$, the deep transfer kernel type $\varkappa_d(G) = (\ker(T_{H_i,G'}))_{1\le i\le 4}$ is determined explicitly with the aid of commutator calculus in § 3 using a parametrized polycyclic power-commutator presentation of G [6, 27, 28]. In the concluding § 4, the orders of the deep transfer kernels are sufficient for identifying the Galois group $G_3^{\infty}F := \mathrm{Gal}(F_3^{(\infty)}/F)$ of the maximal unramified pro-3 extension of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and total 3-principalization in each of their four unramified cyclic cubic extensions $E_1, \dots, E_4$.

2. Shallow and deep transfer of p-groups

With an assigned prime number $p \ge 2$, let G be a finite p-group. Since our focus in this paper will be on the simplest possible non-trivial situation, we assume that the abelianization G/G' of G is of elementary type (p, p) with rank two. For applications in number theory, concerning p-class towers, the Artin pattern has proved to be a decisive collection of information on G.

Definition 2.1. The *Artin pattern* $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ of G consists of two families

$$\tau(G) := (H_i/H_i')_{1\le i\le p+1} \text{ and } \varkappa(G) := (\ker(T_{G,H_i}))_{1\le i\le p+1} \tag{2.1}$$

containing the targets and kernels of the Artin transfer homomorphisms $T_{G,H_i} : G/G' \to H_i/H_i'$ [20] from G to its $p+1$ maximal subgroups H_i with $i \in \{1, \dots, p+1\}$. Since the maximal subgroups form the shallow layer $\mathrm{Lyr}_1(G)$ of subgroups of index $(G : H_i) = p$ of G, we shall call the T_{G,H_i} the *shallow transfers* of G, and $\varkappa_s(G) := \varkappa(G)$ the *shallow transfer kernel type* (sTKT) of G.

Date: August 28, 2017.
2000 *Mathematics Subject Classification.* Primary 11R37, 11R29, 11R11, 11R20, 11Y40; Secondary 20D15, 20E18, 20E22, 20F05, 20F12, 20F14, 20–04.
Key words and phrases. Hilbert p-class field towers, p-class groups, p-principalization, quadratic fields, dihedral fields of degree $2p$; finite p-groups, two-step centralizers, polarization principle, descendant trees, p-group generation algorithm, p-multiplicator rank, relation rank, generator rank, deep transfers, shallow transfers, partial order and monotony principle of Artin patterns, parametrized polycyclic pc-presentations, commutator calculus.
Research supported by the Austrian Science Fund (FWF): P 26008-N25.

We recall [13] that the sTKT is usually simplified by a family of non-negative integers, in the following way. For $1 \le i \le p+1$,

$$\varkappa_s(G)_i := \begin{cases} j & \text{if } \ker(T_{G,H_i}) = H_j/G' \text{ for some } j \in \{1,\ldots,p+1\}, \\ 0 & \text{if } \ker(T_{G,H_i}) = G/G'. \end{cases} \tag{2.2}$$

The progressive innovation in this paper, however, is the introduction of the deep Artin transfer.

Definition 2.2. By the *deep transfers* we understand the Artin transfer homomorphisms $T_{H_i,G'} : H_i/H_i' \to G'/G''$ [20] from the maximal subgroups $H_1,\ldots,H_{p+1}$ to the commutator subgroup G' of G, which forms the deep layer $\mathrm{Lyr}_2(G)$ of the (unique) subgroup of index $(G:G') = p^2$ of G with abelian quotient G/G'. Accordingly, we call the family

$$\varkappa_d(G) = (\#\ker(T_{H_i,G'}))_{1\le i\le p+1} \tag{2.3}$$

the *deep transfer kernel type* (dTKT) of G.

We point out that, as opposed to the sTKT, the members of the dTKT are only *cardinalities*, since this will suffice for reaching our intended goals in this paper. This preliminary coarse definition is open to further refinement in subsequent publications. (See the proof of Theorem 3.1.)

3. Identification of 3-groups by deep transfers

The drawback of the sTKT is the fact that occasionally several non-isomorphic p-groups G share a common Artin pattern $\mathrm{AP}(G) := (\tau(G), \varkappa_s(G))$ [26, Thm. 7.2, p. 158]. The benefit of the dTKT is its ability to distinguish the members of such batches of p-groups which have been inseparable up to now. After the general introduction of the dTKT for arbitrary p-groups in § 2, we are now going to demonstrate its advantages in the particular situation of the prime $p = 3$ and finite 3-groups G of coclass $\mathrm{cc}(G) = 1$, which are necessarily metabelian with second derived subgroup $G'' = 1$ and abelianization $G/G' \simeq C_3 \times C_3$, according to Blackburn [5].

For the statement of our main theorem, we need a precise ordering of the four maximal subgroups $H_1,\ldots,H_4$ of the group $G = \langle x,y\rangle$, which can be generated by two elements x,y, according to the Burnside basis theorem. For this purpose, we select the generators x,y such that

$$H_1 = \langle y, G'\rangle, \quad H_2 = \langle x, G'\rangle, \quad H_3 = \langle xy, G'\rangle, \quad H_4 = \langle xy^2, G'\rangle, \tag{3.1}$$

and $H_1 = \chi_2(G)$, provided that G is of nilpotency class $\mathrm{cl}(G) \ge 3$. Here we denote by

$$\chi_2(G) := \{g \in G \mid (\forall\ h \in G')\ [g,h] \in \gamma_4(G)\} \tag{3.2}$$

the *two-step centralizer* of G' in G, where we let $(\gamma_i(G))_{i\ge1}$ be the lower central series of $G =: \gamma_1(G)$ with $\gamma_i(G) = [\gamma_{i-1}(G), G]$ for $i \ge 2$, in particular, $\gamma_2(G) = G'$.

The identification of the groups will be achieved with the aid of parametrized polycyclic power-commutator presentations, as given by Blackburn [6], Miech [27], and Nebelung [28]:

$$\begin{aligned} G^n_a(z,w) := \langle\, & x,y,s_2,\ldots,s_{n-1} \mid s_2 = [y,x],\ (\forall_{i=3}^{n})\ s_i = [s_{i-1},x],\ s_n = 1,\ [y,s_2] = s_{n-1}^a, \\ & (\forall_{i=3}^{n-1})\ [y,s_i] = 1,\ x^3 = s_{n-1}^w,\ y^3 s_2^3 s_3 = s_{n-1}^z,\ (\forall_{i=2}^{n-3})\ s_i^3 s_{i+1}^3 s_{i+2} = 1,\ s_{n-2}^3 = s_{n-1}^3 = 1\,\rangle, \end{aligned} \tag{3.3}$$

where $a \in \{0,1\}$ and $w,z \in \{-1,0,1\}$ are bounded parameters, and the *index of nilpotency* $n = \mathrm{cl}(G)+1 = \mathrm{cl}(G)+\mathrm{cc}(G) = \log_3(\mathrm{ord}(G)) =: \mathrm{lo}(G)$ is an unbounded parameter.

Lemma 3.1. *Let G be an arbitrary group with elements $x,y \in G$. Then the second and third power of the product xy are given by*

1. $(xy)^2 = x^2y^2s_2t_3$, *where* $s_2 := [y,x]$, $t_3 := [s_2,y]$,
2. $(xy)^3 = x^3y^3(s_2t_3^2t_4)^2s_3u_4^2u_5s_2t_3$, *where* $s_3 = [s_2,x]$, $t_4 = [t_3,y]$, $u_4 = [s_3,y]$, $u_5 = [u_4,y]$.

If $G \simeq G^n_a(z,w)$, then $(xy)^2 = x^2y^2s_2s_{n-1}^{-a}$ and $(xy)^3 = x^3y^3s_2^3s_3s_{n-1}^{-2a}$, and the second and third power of xy^2 are given by $(xy^2)^2 = x^2y^4s_2^2s_{n-1}^{-2a}$ and $(xy^2)^3 = x^3y^6s_2^6s_3^2s_{n-1}^{-2a}$.

Proof. We prepare the calculation of the powers by proving a few preliminary identities:
$yx = 1\cdot yx = xyy^{-1}x^{-1}\cdot yx = xy\cdot y^{-1}x^{-1}yx = xy\cdot[y,x] = xys_2$, and similarly
$s_2y = ys_2\cdot[s_2,y] = ys_2t_3$ and $t_3y = yt_3\cdot[t_3,y] = yt_3t_4$ and $s_2x = xs_2\cdot[s_2,x] = xs_2s_3$ and
$s_3y = ys_3\cdot[s_3,y] = ys_3u_4$ and $u_4y = yu_4\cdot[u_4,y] = yu_4u_5$. Furthermore,

$yx^2 = yx \cdot x = xys_2 \cdot x = xy \cdot s_2x = xy \cdot xs_2s_3 = x \cdot yx \cdot s_2s_3 = x \cdot xys_2 \cdot s_2s_3 = x^2ys_2^2s_3$,
$s_2y^2 = s_2y \cdot y = ys_2t_3 \cdot y = ys_2 \cdot t_3y = ys_2 \cdot yt_3t_4 = y \cdot s_2y \cdot t_3t_4 = y \cdot ys_2t_3 \cdot t_3t_4 = y^2s_2t_3^2t_4$,
$s_3y^2 = s_3y \cdot y = ys_3u_4 \cdot y = ys_3 \cdot u_4y = ys_3 \cdot yu_4u_5 = y \cdot s_3y \cdot u_4u_5 = y \cdot ys_3u_4 \cdot u_4u_5 = y^2s_3u_4^2u_5$.
Now the second power of xy is
$(xy)^2 = xyxy = x \cdot yx \cdot y = x \cdot xys_2 \cdot y = x^2y \cdot s_2y = x^2y \cdot ys_2t_3 = x^2y^2s_2t_3$
and the third power of xy is
$(xy)^3 = xy \cdot (xy)^2 = xy \cdot x^2y^2s_2t_3 = x \cdot yx^2 \cdot y^2s_2t_3 = x \cdot x^2ys_2^2s_3 \cdot y^2s_2t_3 = x^3ys_2^2 \cdot s_3y^2 \cdot s_2t_3 =$
$= x^3ys_2^2 \cdot y^2s_3u_4^2u_5 \cdot s_2t_3 = x^3ys_2 \cdot s_2y^2 \cdot s_3u_4^2u_5s_2t_3 = x^3ys_2 \cdot y^2s_2t_3^2t_4 \cdot s_3u_4^2u_5s_2t_3 =$
$= x^3y \cdot s_2y^2 \cdot s_2t_3^2t_4s_3u_4^2u_5s_2t_3 = x^3y \cdot y^2s_2t_3^2t_4 \cdot s_2t_3^2t_4s_3u_4^2u_5s_2t_3 = x^3y^3(s_2t_3^2t_4)^2s_3u_4^2u_5s_2t_3$.
If $G \simeq G_a^n(z, w)$, then $t_4 = u_4 = u_5 = 1$, $t_3 = s_{n-1}^{-a}$, $t_3^3 = s_{n-1}^{-3a} = 1$, and G' is abelian. □

Theorem 3.1. *(3-groups G of coclass $\mathrm{cc}(G) = 1$.) Let G be a finite 3-group of coclass $\mathrm{cc}(G) = 1$ and order $\mathrm{ord}(G) = 3^n$ with an integer exponent $n \ge 2$. Then the shallow and deep transfer kernel type of G are given in dependence on the relational parameters a, n, w, z of $G \simeq G_a^n(z, w)$ by Table 1.*

TABLE 1. Shallow and deep TKT of 3-groups G with $\mathrm{cc}(G) = 1$

$G \simeq$	n	Type	$\varkappa_s(G)$	$\varkappa_d(G)$
$G_0^n(0,0)$	$= 2$	a.1*	$(0,0,0,0)$	$(3,3,3,3)$
$G_0^n(0,0)$	≥ 3	a.1*	$(0,0,0,0)$	$(9,9,9,9)$
$G_1^n(0,0)$	≥ 5	a.1	$(0,0,0,0)$	$(3,9,3,3)$
$G_1^n(0,-1)$	≥ 5	a.1	$(0,0,0,0)$	$(3,3,9,9)$
$G_1^n(0,1)$	≥ 5	a.1	$(0,0,0,0)$	$(3,3,3,3)$
$G_0^n(0,1)$	≥ 4	a.2	$(1,0,0,0)$	$(9,3,3,3)$
$G_0^n(-1,0)$	≥ 4 even	a.3	$(2,0,0,0)$	$(9,9,3,3)$
$G_0^n(1,0)$	≥ 5	a.3	$(2,0,0,0)$	$(9,9,3,3)$
$G_0^n(1,0)$	$= 4$	a.3*	$(2,0,0,0)$	$(27,9,3,3)$
$G_0^n(0,1)$	$= 3$	A.1	$(1,1,1,1)$	$(9,3,3,3)$

Proof. The shallow TKT $\varkappa_s(G)$ of all 3-groups G of coclass $\mathrm{cc}(G) = 1$ has been determined in [13], where the designations a.n of the types were introduced with $n \in \{1, 2, 3\}$. Here, we indicate a capable mainline vertex of the tree $\mathcal{T}^1(R)$ with root $R = C_3 \times C_3$ [26, Fig. 1–2, pp. 142–143] by the type a.1* with a trailing asterisk. As usual, type a.3* indicates the unique 3-group $G \simeq \mathrm{Syl}_3 A_9$ with $\tau(G) = [(3,3,3),(3,3)^3]$. Now we want to determine the deep TKT $\varkappa_d(G)$, using the presentation of $G \simeq G_a^n(z, w)$ in Formula (3.3). For this purpose, we need expressions for the images of the deep Artin transfers $T_i := T_{H_i,G'} : H_i/H_i' \to G'$, for each $1 \le i \le 4$. (Observe that $p = 3$ implies $G'' = 1$ by [5].) Generally, we have to distinguish *outer* transfers, $T_i(g \cdot H_i') = g^3$ if $g \in H_i \setminus G'$ [13, Eqn. (4), p. 470], and *inner* transfers, $T_i(g \cdot H_i') = g^{1+h+h^2} = g^3 \cdot [g,h]^3 \cdot [[g,h],h]$ if $g \in G'$ and h is selected in $H_i \setminus G'$ [13, Eqn. (6), p. 486].

First, we consider the distinguished two-step centralizer $H_1 = \chi_2(G)$ with $i = 1$. Then $H_1 = \langle y, G' \rangle$ and $H_1' = 1$ if $a = 0$ (H_1 abelian), but $H_1' = \gamma_{n-1}(G) = \langle s_{n-1} \rangle$ if $a = 1$ (H_1 non-abelian) [13, Eqn. (3), p. 470]. The outer transfer is determined by $T_1(y \cdot H_1') = y^3 = s_2^{-3}s_3^{-1}s_{n-1}^z$. For the inner transfer, we have $T_1(s_j \cdot H_1') = s_j^{1+y+y^2} = s_j^3 \cdot [s_j, y]^3 \cdot [[s_j, y], y] = s_j^3 \cdot 1^3 \cdot [1, y] = s_j^3$ for all $j \ge 3$, but $T_1(s_2 \cdot H_1') = s_2^3 \cdot s_{n-1}^{-3a} \cdot [s_{n-1}^{-a}, y] = s_2^3$ for $j = 2$, since $s_{n-1}^{-a} \in \langle s_{n-1} \rangle = \gamma_{n-1}(G) = \zeta_1(G)$ lies in the centre of G. The first kernel equation $s_2^{-3}s_3^{-1}s_{n-1}^z = 1$ is solvable by either $n = 3$, where $z = 0$, $s_3 = 1$, $s_2^3 = 1$, or $n = 4$, $z = 1$, where $s_2^3 = 1$, $s_{n-1}^z = s_3$. The second kernel equation

$s_i^3 = 1$ is solvable by either $i = n-1$ or $i = n-2$. Thus, the deep transfer kernel is given by

$$\ker(T_1) = \begin{cases} H_1 = \langle y, s_2\rangle \simeq C_3 \times C_3 \text{ if } n = 3 \ (G \text{ extra special}), \\ H_1 = \langle y, s_2, s_3\rangle \simeq C_3 \times C_3 \times C_3 \text{ if } n = 4,\ z = 1 \ (G \simeq \mathrm{Syl}_3 A_9), \\ \gamma_{n-2}(G) = \langle s_{n-2}, s_{n-1}\rangle \simeq C_3 \times C_3 \text{ if } n = 4,\ z \neq 1 \text{ or } n \geq 5,\ a = 0, \\ \gamma_{n-2}(G)/\gamma_{n-1}(G) \simeq \langle s_{n-2}\rangle \simeq C_3 \text{ if } n \geq 5,\ a = 1 \ (H_1 \text{ non-abelian}). \end{cases} \tag{3.4}$$

Second, we put $i = 2$. Then $H_2 = \langle x, G'\rangle$ and $H_2' = \gamma_3(G) = \langle s_3, \ldots, s_{n-1}\rangle$. The outer transfer is determined by $T_2(x \cdot H_2') = x^3 = s_{n-1}^w$. The inner transfer is given by $T_2(s_j \cdot H_2') = s_j^{1+x+x^2} = s_j^3 \cdot [s_j, x]^3 \cdot [[s_j, x], x] = s_j^3 s_{j+1}^3 s_{j+2} = 1$, for all $j \geq 2$, independently of a, n, w, z. Consequently, the deep transfer kernel is given by

$$\ker(T_2) = \begin{cases} H_2/H_2' = \langle x, s_2, \ldots, s_{n-1}\rangle/\langle s_3, \ldots, s_{n-1}\rangle \simeq \langle x, s_2\rangle \simeq C_3 \times C_3 \text{ if } w = 0, \\ G'/H_2' = \langle s_2, \ldots, s_{n-1}\rangle/\langle s_3, \ldots, s_{n-1}\rangle \simeq \langle s_2\rangle \simeq C_3 \text{ if } w = \pm 1. \end{cases} \tag{3.5}$$

Next, we put $i = 3$. Then $H_3 = \langle xy, G'\rangle$ and $H_3' = \gamma_3(G) = \langle s_3, \ldots, s_{n-1}\rangle$. The outer transfer is determined by $T_3(xy \cdot H_3') = (xy)^3 = x^3 y^3 s_2^3 s_3 s_{n-1}^{-2a} = s_{n-1}^{w+z-2a}$. For the inner transfer, we have $T_3(s_j \cdot H_3') = s_j^{1+xy+(xy)^2} = s_j^3 \cdot [s_j, xy]^3 \cdot [[s_j, xy], xy] = s_j^3 s_{j+1}^3 s_{j+2} = 1$, for all $j \geq 3$, independently of a, n, w, z. The first kernel equation $s_{n-1}^{w+z-2a} = 1 \iff w + z - 2a \equiv 0 \pmod 3$ is solvable by either $a = w = z = 0$ or $a = 1$, $w = -1$.

Therefore, the deep transfer kernel is given by

$$\ker(T_3) = \begin{cases} H_3/H_3' \simeq \langle xy, s_2\rangle \simeq C_3 \times C_3 \text{ if either } a = w = z = 0 \text{ or } a = 1,\ w = -1, \\ G'/H_3' \simeq \langle s_2\rangle \simeq C_3 \text{ otherwise }. \end{cases} \tag{3.6}$$

Finally, we put $i = 4$. Then $H_4 = \langle xy^2, G'\rangle$ and $H_4' = \gamma_3(G) = \langle s_3, \ldots, s_{n-1}\rangle$. The outer transfer is determined by $T_4(xy^2 \cdot H_4') = (xy^2)^3 = x^3 y^6 s_2^6 s_3^2 s_{n-1}^{-2a} = s_{n-1}^{w+2z-2a}$. The inner transfer is given by $T_4(s_j \cdot H_4') = s_j^{1+xy^2+(xy^2)^2} = s_j^3 \cdot [s_j, xy^2]^3 \cdot [[s_j, xy^2], xy^2] = s_j^3 s_{j+1}^3 s_{j+2} = 1$, for all $j \geq 3$, independently of a, n, w, z. The first kernel equation $s_{n-1}^{w+2z-2a} = 1 \iff w + 2z - 2a \equiv 0 \pmod 3$ is solvable by either $a = w = z = 0$ or $a = 1$, $w = -1$.

Thus, the deep transfer kernel is given by

$$\ker(T_4) = \begin{cases} H_4/H_4' \simeq \langle xy^2, s_2\rangle \simeq C_3 \times C_3 \text{ if either } a = w = z = 0 \text{ or } a = 1,\ w = -1, \\ G'/H_4' \simeq \langle s_2\rangle \simeq C_3 \text{ otherwise }. \end{cases} \tag{3.7}$$

These finer results are summarized in terms of coarser cardinalities in Table 1. □

4. Arithmetical application to 3-class tower groups

4.1. **Real quadratic fields.** As a final highlight of our progressive innovations, we come to a number theoretic application of Theorem 3.1, more precisely, the unambiguous identification of the pro-3 Galois group $G_3^\infty F = \mathrm{Gal}(F_3^{(\infty)}/F)$ of the maximal unramified pro-3 extension $F_3^{(\infty)}$, that is the Hilbert 3-class field tower, of certain real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d > 1$, 3-class group $\mathrm{Cl}_3(F)$ of elementary type $(3,3)$, and shallow transfer kernel type a.1, $\varkappa_s(F) = (0,0,0,0)$, in its *ground state* with $\tau(F) \sim [(9,9),(3,3)^3]$ or in a higher *excited state* with $\tau(F) \sim [(3^e,3^e),(3,3)^3]$, $e \geq 3$.

The first field of this kind with $d = 62\,501$ was discovered by Heider and Schmithals in 1982 [9]. They computed the sTKT $\varkappa_s(F) = (0,0,0,0)$ with four total 3-principalizations in the unramified cyclic cubic extensions E_i/F, $1 \leq i \leq 4$, on a CDC Cyber mainframe. The fact that $d = 62\,501$ is a triadic irregular discriminant (in the sense of Gauss) with non-cyclic 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$ has been pointed out earlier in 1936 by Pall [29] already. The second field of this kind with $d = 152\,949$ was discovered by ourselves in 1991 by computing $\varkappa_s(F)$ on an AMDAHL mainframe [11]. In 2006, there followed $d = 252\,977$ and $d = 358\,285$, and many other cases in 2009 [12, 16].

Generally, there are three contestants for the group $G = G_3^\infty F$, for any assigned state $\tau(F) \sim [(3^e,3^e),(3,3)^3]$, $e \geq 2$, and the following **Main Theorem** admits their identification by means of the deep transfer kernel type. (See their statistical distribution at the end of section 4.1.)

Theorem 4.1. ***(3-class tower groups G of coclass $\mathrm{cc}(G)=1$ and type a.1.)*** *Let $F=\mathbb{Q}(\sqrt{d})$ be a quadratic field with fundamental discriminant d, 3-class group $\mathrm{Cl}_3(F)\simeq C_3\times C_3$, and shallow transfer kernel type a.1, $\varkappa_s(F)=(0,0,0,0)$.*

Then F is real with $d>1$, the 3-class tower group $G=G_3^\infty F$ of F has coclass $\mathrm{cc}(G)=1$, and the relational parameters $n\ge 5$ and $w\in\{-1,0,1\}$ of $G\simeq G_1^n(0,w)$ are given in dependence on the deep transfer kernel type $\varkappa_d(F)$ as follows:

$$
\begin{aligned}
&G\simeq G_1^{2(e+1)}(0,0) && \text{with } n=2(e+1),\ w=0 && \Longleftrightarrow && \varkappa_d(F)\sim(3,9,3,3),\\
&G\simeq G_1^{2(e+1)}(0,-1) && \text{with } n=2(e+1),\ w=-1 && \Longleftrightarrow && \varkappa_d(F)\sim(3,3,9,9),\\
&G\simeq G_1^{2(e+1)}(0,1) && \text{with } n=2(e+1),\ w=1 && \Longleftrightarrow && \varkappa_d(F)\sim(3,3,3,3),
\end{aligned}
\tag{4.1}
$$

where we suppose that the state of type a.1 is determined by the transfer target type $\tau(F)\sim[(3^e,3^e),(3,3)^3]$ with $e\ge 2$.

Proof. Let $F=\mathbb{Q}(\sqrt{d})$ be a quadratic field with 3-class group $\mathrm{Cl}_3(F)\simeq C_3\times C_3$, denote by $E_1,\dots,E_4$ its four unramified cyclic cubic extensions and by $T_{E_i/F}:\mathrm{Cl}_3(F)\to\mathrm{Cl}_3(E_i)$ $(1\le i\le 4)$ the transfer homomorphisms of 3-classes.

If the 3-principalization is total, that is $\ker(T_{E_i/F})=\mathrm{Cl}_3(F)$, for each $1\le i\le 4$, then F must be a *real* quadratic field with positive fundamental discriminant $d>1$, since the order of the principalization kernels $\ker(T_{E_i/F})$ of an imaginary quadratic field F is bounded from above by $(U_F:N_{E_i/F}U_{E_i})\cdot[E_i:F]=1\cdot 3=3$, according to the Theorem on the Herbrand quotient of the unit groups U_{E_i}.

By the Artin reciprocity law of class field theory [1, 2], the principalization type $\varkappa(F)=(0000)$ of the field F corresponds to the shallow transfer kernel type $\varkappa_s(G)=(0000)$ of the 3-class tower group $G=\mathrm{Gal}(F_3^{(\infty)}/F)$ of F, and the abelian type invariants $\mathrm{Cl}_3(F)\simeq 1^2$ of the 3-class group of F correspond to the abelian quotient invariants $G/G'\simeq 1^2$ of G.

According to [13], a finite 3-group G with $G/G'\simeq 1^2$ and $\varkappa_s(G)=(0000)$ must be of coclass $\mathrm{cc}(G)=1$. Table 1 shows that either $G\simeq G_0^n(0,0)$ of type a.1* with $n\ge 2$ or $G\simeq G_1^n(0,w)$ of type a.1 with $n\ge 5$ and $-1\le w\le 1$.

For a real quadratic field F, the relation rank $d_2(G)=\dim_{\mathbb{F}_3}H_2(G,\mathbb{F}_3)$ of the 3-class tower group $G=G_3^{(\infty)}F$ is bounded by $d_2(G)\le 3$ [24, Thm. 1.3, pp. 75–76]. Consequently, G cannot be a non-abelian mainline vertex $G_0^n(0,0)$ with $n\ge 3$ of the coclass-1 tree $\mathcal{T}^1(R)$ with root $R=C_3\times C_3$, since all these vertices have the relation rank 4. According to [12, Thm. 4.1 (1), p. 486], G cannot be the abelian root $R=G_0^2(0,0)$ either, and we must have $G\simeq G_1^n(0,w)$ with $n\ge 5$ and $w\in\{-1,0,1\}$.

Now the claim is a consequence of Theorem 3.1 and Table 1. □

Table 2 shows that the *ground state* $\tau(F)=[(9,9),(3,3)^3]$ of the sTKT $\varkappa_s(F)=(0,0,0,0)$ has the nice property that the smallest three discriminants already realize three different 3-class tower groups $G=G_3^\infty F\simeq\langle 729,i\rangle$ with $i\in\{99,100,101\}$, identified by their dTKT $\varkappa_d(F)=\varkappa_d(G)$.

TABLE 2. Deep TKT of 3-class tower groups G with $\tau(G)=[(9,9),(3,3)^3]$

G	$\varkappa_d(G)$	MD
$\langle 729,99\rangle\simeq G_1^6(0,0)$	$(3,9,3,3)$	62 501
$\langle 729,100\rangle\simeq G_1^6(0,-1)$	$(3,3,9,9)$	152 949
$\langle 729,101\rangle\simeq G_1^6(0,1)$	$(3,3,3,3)$	252 977

In Table 3, we see that the *first excited state* $\tau(F)=[(27,27),(3,3)^3]$ of the sTKT $\varkappa_s(F)=(0,0,0,0)$ does not behave so well: although the smallest two discriminants [12, 16, 22, 23] already realize two different 3-class tower groups $G=G_3^\infty F\simeq\langle 6561,i\rangle$ with $i\in\{2225,2227\}$, we have to wait for the seventh occurrence until $\langle 6561,2226\rangle$ is realized, as the dTKT $\varkappa_d(F)=\varkappa_d(G)$ shows. The counter 7 is a typical example of a *statistic delay.*

Table 3. Deep TKT of 3-class tower groups G with $\tau(G) = [(27,27),(3,3)^3]$

G	$\varkappa_d(G)$	MD	further discriminants
$\langle 6561, 2225\rangle \simeq G_1^8(0,0)$	$(3,9,3,3)$	10 399 596	16 613 448
$\langle 6561, 2226\rangle \simeq G_1^8(0,-1)$	$(3,3,9,9)$	27 780 297	
$\langle 6561, 2227\rangle \simeq G_1^8(0,1)$	$(3,3,3,3)$	2 905 160	14 369 932, 15 019 617, 21 050 241

The *second excited state* $\tau(F) = [(81,81),(3,3)^3]$ of the sTKT $\varkappa_s(F) = (0,0,0,0)$, however, is well-behaved again: the smallest three discriminants already realize three different 3-class tower groups $G = G_3^\infty F \simeq G_1^{10}(0,w)$ with $w \in \{0,-1,1\}$, identified by their dTKT $\varkappa_d(F) = \varkappa_d(G)$. (For logarithmic orders ≥ 9, no SmallGroup identifiers exist.) See Table 4.

Table 4. Deep TKT of 3-class tower groups G with $\tau(G) = [(81,81),(3,3)^3]$

G	$\varkappa_d(G)$	MD
$G_1^{10}(0,0)$	$(3,9,3,3)$	63 407 037
$G_1^{10}(0,-1)$	$(3,3,9,9)$	62 565 429
$G_1^{10}(0,1)$	$(3,3,3,3)$	40 980 808

In all tables, the shortcut MD means the *minimal discriminant* [26, Dfn. 6.2, p. 148].

The diagram in Figure 1 visualizes the initial eight branches of the coclass tree $\mathcal{T}^1(R)$ with abelian root $R = \langle 9,2\rangle \simeq C_3 \times C_3$. Basic definitions, facts, and notation concerning general descendant trees of finite p-groups are summarized briefly in [15, § 2, pp. 410–411], [14]. They are discussed thoroughly in the broadest detail in the initial sections of [17]. Descendant trees are crucial for recent progress in the theory of p-class field towers [21, 24, 25], in particular for describing the mutual location of the second p-class group $\mathrm{G}_p^2 F$ and the p-class tower group $\mathrm{G}_p^\infty F$ of a number field F. Generally, the vertices of the coclass tree in the figure represent isomorphism classes of finite 3-groups. Two vertices are connected by a directed edge $G \to H$ if H is isomorphic to the last lower central quotient $G/\gamma_c(G)$, where $c = \mathrm{cl}(G) = n-1$ denotes the nilpotency class of G, and $|G| = 3|H|$, that is, $\gamma_c(G) \simeq C_3$ is cyclic of order 3. See also [15, § 2.2, p. 410–411] and [17, § 4, p. 163–164].

The vertices of the tree diagram in Figure 1 are classified by using various symbols:

(1) big contour squares $\square$ represent abelian groups,
(2) big full discs $\bullet$ represent metabelian groups with at least one abelian maximal subgroup,
(3) small full discs $\bullet$ represent metabelian groups without abelian maximal subgroups.

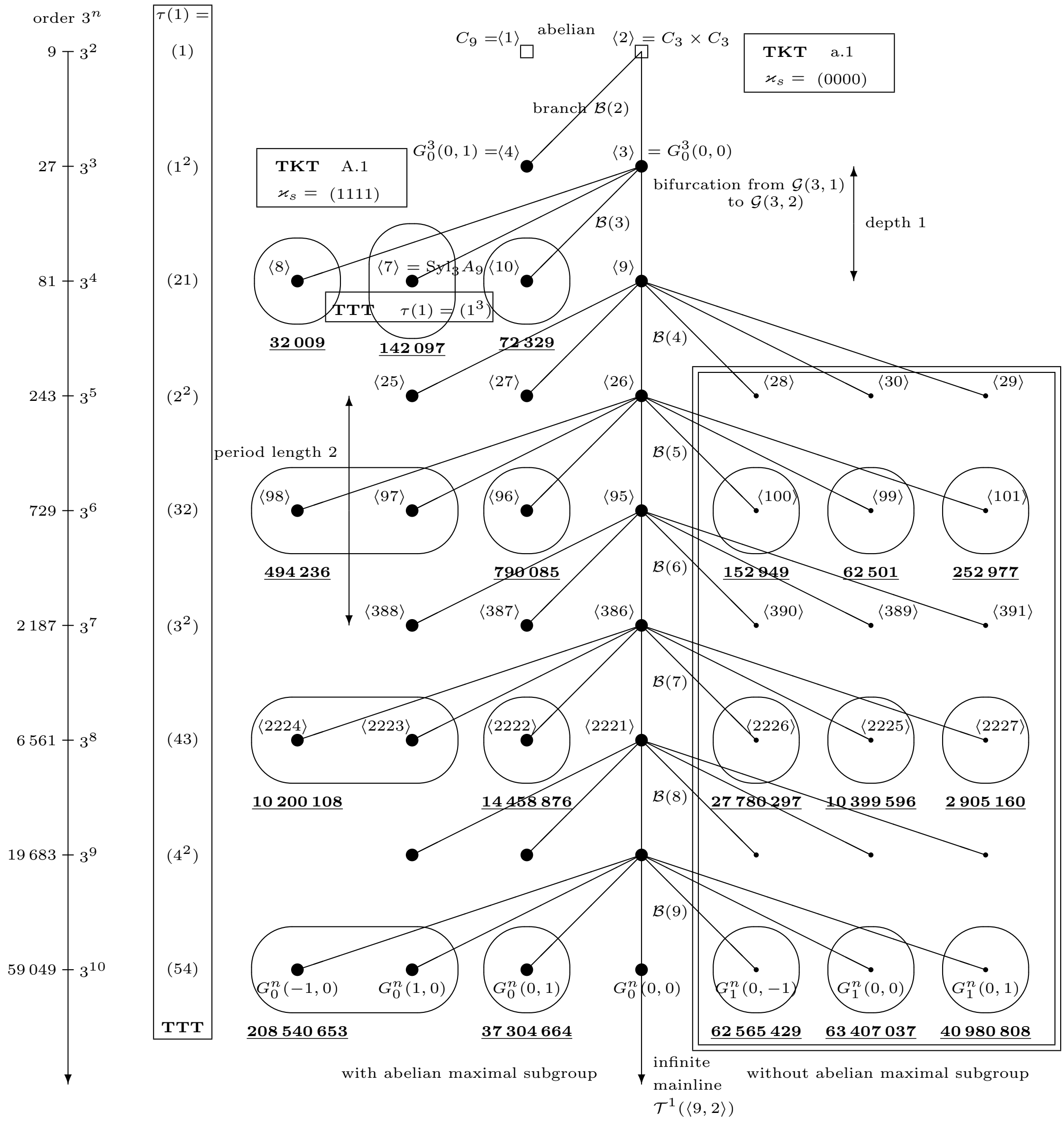
The groups of particular importance are labelled by a number in angles, which is the identifier in the SmallGroups Library [3, 4] of MAGMA [10]. We omit the orders, which are given on the left hand scale. The sTKT $\varkappa_s$ [13, Thm. 2.5, Tbl. 6–7], in the bottom rectangle concerns all vertices located vertically above. The first component $\tau(1)$ of the TTT [18, Dfn. 3.3, p. 288] in the left rectangle concerns vertices G on the same horizontal level containing an abelian maximal subgroup. It is given in logarithmic notation. The periodicity with length 2 of branches, $\mathcal{B}(j) \simeq \mathcal{B}(j+2)$ for $j \geq 4$, sets in with branch $\mathcal{B}(4)$, having a root of order 3^4.

3-class tower groups $G = G_3^\infty F$ with coclass $\mathrm{cc}(G) = 1$ of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ are located as arithmetically realized vertices on the tree diagram in Figure 1. The minimal fundamental discriminants d, i.e. the MDs, are indicated by underlined boldface integers adjacent to the oval surrounding the realized vertex [4, 10, 20].

The double contour rectangle surrounds the vertices which became distinguishable by the progressive innovations in the present paper and were inseparable up to now.

In Table 5, we give the isomorphism type of the 3-class tower group $G = G_3^\infty F$ of all real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$ and shallow transfer kernel type a.1, $\varkappa_s = (0,0,0,0)$, in its *ground state* $\tau(F) = [(9,9),(3,3)^3]$, for the complete range $1 < d < 10^7$ of 150 fundamental discriminants d. It was determined by means of Theorem 4.1, applied to the

FIGURE 1. Distribution of minimal discriminants for $G_3^\infty F$ on the coclass-1 tree $\mathcal{T}^1(\langle 9,2\rangle)$



TKT	a.3	a.3	a.2	a.1*	a.1	a.1	a.1
$\varkappa_s =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

results of computing the (restricted) deep transfer kernel type $\varkappa_d(F) = (\#\ker(T_{F_3^{(1)}/E_i}))_{2\le i\le 4}$, consisting of the orders of the 3-principalization kernels of those unramified cyclic cubic extensions E_i, $2 \le i \le 4$, in the Hilbert 3-class field $F_3^{(1)}$ of F whose 3-class group $\mathrm{Cl}_3(E_i)$ is of type $(3,3)$. These trailing three components of the TTT $\tau(F) = [(9,9),(3,3)^3]$ were called its *stable part* in [20, Dfn. 5.5, p. 84]. The computations were done with the aid of the computational algebra system MAGMA [10]. The 3-principalization kernel of the remaining extension E_1 with 3-class group $\mathrm{Cl}_3(E_1)$ of type $(9,9)$ does not contain essential information and can be omitted. This leading component of the TTT $\tau(F) = [(9,9),(3,3)^3]$ was called its *polarized part* in [20, Dfn. 5.5, p. 84]. For more details on the concepts *stabilization* and *polarization*, see [20, § 6, pp. 90–95].

Table 5. Statistics of 3-class tower groups G with $\tau(G) = [(9,9),(3,3)^3]$

No.	d	G	No.	d	G	No.	d	G
1	62 501	$\langle 729,99\rangle$	51	3 995 004	$\langle 729,101\rangle$	101	7 313 928	$\langle 729,99\rangle$
2	152 949	$\langle 729,100\rangle$	52	4 045 265	$\langle 729,101\rangle$	102	7 391 212	$\langle 729,99\rangle$
3	252 977	$\langle 729,101\rangle$	53	4 183 205	$\langle 729,100\rangle$	103	7 406 249	$\langle 729,101\rangle$
4	358 285	$\langle 729,101\rangle$	54	4 196 840	$\langle 729,100\rangle$	104	7 415 841	$\langle 729,101\rangle$
5	531 437	$\langle 729,99\rangle$	55	4 199 901	$\langle 729,101\rangle$	105	7 447 697	$\langle 729,100\rangle$
6	586 760	$\langle 729,101\rangle$	56	4 220 977	$\langle 729,100\rangle$	106	7 502 501	$\langle 729,100\rangle$
7	595 009	$\langle 729,100\rangle$	57	4 233 608	$\langle 729,99\rangle$	107	7 601 081	$\langle 729,101\rangle$
8	726 933	$\langle 729,99\rangle$	58	4 252 837	$\langle 729,100\rangle$	108	7 623 320	$\langle 729,101\rangle$
9	801 368	$\langle 729,100\rangle$	59	4 409 313	$\langle 729,100\rangle$	109	7 630 645	$\langle 729,100\rangle$
10	940 593	$\langle 729,100\rangle$	60	4 429 612	$\langle 729,101\rangle$	110	7 634 065	$\langle 729,100\rangle$
11	966 489	$\langle 729,99\rangle$	61	4 533 032	$\langle 729,99\rangle$	111	7 643 993	$\langle 729,100\rangle$
12	1 177 036	$\langle 729,99\rangle$	62	4 586 797	$\langle 729,100\rangle$	112	7 683 308	$\langle 729,101\rangle$
13	1 192 780	$\langle 729,101\rangle$	63	4 662 917	$\langle 729,100\rangle$	113	7 704 653	$\langle 729,100\rangle$
14	1 313 292	$\langle 729,99\rangle$	64	4 680 701	$\langle 729,99\rangle$	114	7 713 961	$\langle 729,99\rangle$
15	1 315 640	$\langle 729,99\rangle$	65	4 766 309	$\langle 729,99\rangle$	115	7 804 828	$\langle 729,100\rangle$
16	1 358 556	$\langle 729,100\rangle$	66	4 782 664	$\langle 729,99\rangle$	116	7 936 316	$\langle 729,100\rangle$
17	1 398 829	$\langle 729,101\rangle$	67	4 783 697	$\langle 729,100\rangle$	117	8 037 645	$\langle 729,100\rangle$
18	1 463 729	$\langle 729,101\rangle$	68	4 965 009	$\langle 729,100\rangle$	118	8 101 277	$\langle 729,101\rangle$
19	1 580 709	$\langle 729,100\rangle$	69	5 039 692	$\langle 729,99\rangle$	119	8 235 965	$\langle 729,101\rangle$
20	1 595 669	$\langle 729,100\rangle$	70	5 048 988	$\langle 729,99\rangle$	120	8 248 953	$\langle 729,99\rangle$
21	1 722 344	$\langle 729,99\rangle$	71	5 111 669	$\langle 729,100\rangle$	121	8 263 020	$\langle 729,99\rangle$
22	1 751 909	$\langle 729,101\rangle$	72	5 119 637	$\langle 729,99\rangle$	122	8 320 764	$\langle 729,99\rangle$
23	1 831 097	$\langle 729,99\rangle$	73	5 154 385	$\langle 729,100\rangle$	123	8 375 228	$\langle 729,99\rangle$
24	1 942 385	$\langle 729,101\rangle$	74	5 226 941	$\langle 729,100\rangle$	124	8 501 541	$\langle 729,101\rangle$
25	2 021 608	$\langle 729,99\rangle$	75	5 337 341	$\langle 729,99\rangle$	125	8 523 385	$\langle 729,101\rangle$
26	2 042 149	$\langle 729,101\rangle$	76	5 350 569	$\langle 729,100\rangle$	126	8 578 617	$\langle 729,99\rangle$
27	2 076 485	$\langle 729,99\rangle$	77	5 353 240	$\langle 729,99\rangle$	127	8 623 704	$\langle 729,101\rangle$
28	2 185 465	$\langle 729,101\rangle$	78	5 362 136	$\langle 729,101\rangle$	128	8 637 717	$\langle 729,99\rangle$
29	2 197 669	$\langle 729,101\rangle$	79	5 400 712	$\langle 729,101\rangle$	129	8 674 397	$\langle 729,99\rangle$
30	2 314 789	$\langle 729,99\rangle$	80	5 478 321	$\langle 729,99\rangle$	130	8 723 237	$\langle 729,99\rangle$
31	2 409 853	$\langle 729,99\rangle$	81	5 827 564	$\langle 729,99\rangle$	131	8 737 913	$\langle 729,101\rangle$
32	2 433 221	$\langle 729,101\rangle$	82	5 891 701	$\langle 729,100\rangle$	132	8 748 764	$\langle 729,99\rangle$
33	2 539 129	$\langle 729,101\rangle$	83	5 909 217	$\langle 729,99\rangle$	133	8 816 389	$\langle 729,99\rangle$
34	2 555 249	$\langle 729,100\rangle$	84	5 982 269	$\langle 729,101\rangle$	134	8 957 485	$\langle 729,101\rangle$
35	2 710 072	$\langle 729,100\rangle$	85	6 105 693	$\langle 729,100\rangle$	135	8 993 409	$\langle 729,100\rangle$
36	2 851 877	$\langle 729,99\rangle$	86	6 155 861	$\langle 729,99\rangle$	136	9 006 397	$\langle 729,101\rangle$
37	2 954 929	$\langle 729,99\rangle$	87	6 337 340	$\langle 729,99\rangle$	137	9 051 665	$\langle 729,99\rangle$
38	3 005 369	$\langle 729,101\rangle$	88	6 429 997	$\langle 729,100\rangle$	138	9 058 892	$\langle 729,101\rangle$
39	3 197 864	$\langle 729,100\rangle$	89	6 618 085	$\langle 729,99\rangle$	139	9 130 973	$\langle 729,99\rangle$
40	3 197 944	$\langle 729,101\rangle$	90	6 658 973	$\langle 729,100\rangle$	140	9 185 153	$\langle 729,101\rangle$
41	3 258 120	$\langle 729,101\rangle$	91	6 792 365	$\langle 729,99\rangle$	141	9 195 769	$\langle 729,101\rangle$
42	3 323 065	$\langle 729,99\rangle$	92	6 806 152	$\langle 729,99\rangle$	142	9 328 597	$\langle 729,99\rangle$
43	3 342 785	$\langle 729,99\rangle$	93	6 882 737	$\langle 729,99\rangle$	143	9 379 849	$\langle 729,100\rangle$
44	3 644 357	$\langle 729,99\rangle$	94	6 927 452	$\langle 729,101\rangle$	144	9 380 744	$\langle 729,99\rangle$
45	3 658 421	$\langle 729,100\rangle$	95	6 953 513	$\langle 729,99\rangle$	145	9 419 704	$\langle 729,99\rangle$
46	3 692 717	$\langle 729,99\rangle$	96	6 974 609	$\langle 729,99\rangle$	146	9 511 580	$\langle 729,100\rangle$
47	3 721 565	$\langle 729,99\rangle$	97	7 010 133	$\langle 729,101\rangle$	147	9 615 813	$\langle 729,100\rangle$
48	3 799 597	$\langle 729,100\rangle$	98	7 019 717	$\langle 729,99\rangle$	148	9 645 393	$\langle 729,99\rangle$
49	3 821 244	$\langle 729,99\rangle$	99	7 075 740	$\langle 729,101\rangle$	149	9 801 773	$\langle 729,99\rangle$
50	3 869 909	$\langle 729,99\rangle$	100	7 263 365	$\langle 729,99\rangle$	150	9 834 557	$\langle 729,99\rangle$

A systematic statistical evaluation of Table 5 shows that, with respect to the complete range $1 < d < 10^7$, the group $G \simeq \langle 729,99\rangle$ occurs most often with a clearly elevated relative frequency of 44%, whereas $G \simeq \langle 729,100\rangle$ and $G \simeq \langle 729,101\rangle$ share the common lower percentage of 28%, although the automorphism group $\mathrm{Aut}(G)$ of all three groups has the same order. However, the proportion $44 : 28 : 28$ for the upper bound 10^7 is obviously not settled yet, because there are remarkable fluctuations, as Table 6 shows. According to Boston, Bush and Hajir [7, 8], we have to expect an asymptotic limit $33 : 33 : 33$ of the proportions for $d \to \infty$.

TABLE 6. Proportions of 3-class tower groups $G \simeq \langle 729, i\rangle$ with $i \in \{99, 100, 101\}$

G	for $d < 10^6\times$	1	2	3	4	5	6	7	8	9	10
$\langle 729, 99\rangle$		36%	38%	41%	43%	40%	42%	45%	41%	43%	44%
$\langle 729, 100\rangle$		36%	29%	24%	24%	31%	31%	30%	32%	29%	28%
$\langle 729, 101\rangle$		27%	33%	35%	33%	29%	27%	25%	27%	28%	28%

4.2. **Totally real dihedral fields.** In fact, we have computed much more information with MAGMA than mentioned at the end of the previous section 4.1. To understand the actual scope of our numerical results it is necessary to recall that each unramified cyclic cubic relative extension E_i/F, $1 \le i \le 4$, gives rise to a dihedral absolute extension $E_i/\mathbb{Q}$ of degree 6, that is an S_3-extension [12, Prp. 4.1, p. 482]. For the trailing three fields E_i, $2 \le i \le 4$, in the stable part of the TTT $\tau(F) = [(9, 9), (3, 3)^3]$, i.e. with $\mathrm{Cl}_3(E_i)$ of type $(3, 3)$, we have constructed the unramified cyclic cubic extensions $\tilde{E}_{i,j}/E_i$, $1 \le j \le 4$, and determined the Artin pattern $\mathrm{AP}(E_i)$ of E_i, in particular, the 3-principalization type of E_i in the fields $\tilde{E}_{i,j}$. The dihedral fields E_i of degree 6 share a common polarization $\tilde{E}_{i,1} = F_3^{(1)}$, the Hilbert 3-class field of F, which is contained in the relative 3-genus field $(E_i/F)^*$, whereas the other extensions $\tilde{E}_{i,j}$ with $2 \le j \le 4$ are non-abelian over F, for each $2 \le i \le 4$. Our computational results suggest the following conjecture concerning the infinite family of totally real dihedral fields E_i for varying real quadratic fields F.

Conjecture 4.1. *(3-class tower groups $\mathcal{G}$ of totally real dihedral fields.) Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with fundamental discriminant $d > 1$, 3-class group $\mathrm{Cl}_3(F) \simeq C_3 \times C_3$, and shallow transfer kernel type* a.1, $\varkappa_s(F) = (0, 0, 0, 0)$, *in the ground state with transfer target type $\tau(F) \sim [(9, 9), (3, 3)^3]$. Let E_2, E_3, E_4 be the three unramified cyclic cubic relative extensions of F with 3-class group $\mathrm{Cl}_3(E_i)$ of type $(3, 3)$.*

Then $E_i/\mathbb{Q}$ is a totally real dihedral extension of degree 6, for each $2 \le i \le 4$, and the connection between the component $\varkappa_d(F)_i = \#\ker(T_{F_3^{(1)}/E_i})$ of the deep transfer kernel type $\varkappa_d(F)$ of F and the 3-class tower group $\mathcal{G}_i = G_3^\infty E_i = \mathrm{Gal}((E_i)_3^{(\infty)}/E_i)$ of E_i is given in the following way:

$$(4.2)\qquad \begin{aligned} \varkappa_d(F)_i = 3 &\iff \mathcal{G}_i \simeq \langle 243, 27\rangle \quad \text{with } \varkappa_s(\mathcal{G}_i) = (1, 0, 0, 0), \\ \varkappa_d(F)_i = 9 &\iff \mathcal{G}_i \simeq \langle 243, 26\rangle \quad \text{with } \varkappa_s(\mathcal{G}_i) = (0, 0, 0, 0). \end{aligned}$$

Remark 4.1. The conjecture is supported by all $3 \cdot 150 = 450$ totally real dihedral fields E_i which were involved in the computation of Table 5. A provable argument for the truth of the conjecture is the fact that $\varkappa_d(F)_i = \#\ker(T_{F_3^{(1)}/E_i}) = \#\varkappa_s(E_i)_1 = \#\varkappa_s(\mathcal{G}_i)_1$, for $2 \le i \le 4$, but it does not explain why the sTKT $\varkappa_s(\mathcal{G}_i)$ is a.2 with a fixed point if $\varkappa_d(F)_i = 3$. It is interesting that a dihedral field E_i of degree 6 is satisfied with a non-σ group, such as $\langle 243, 27\rangle$, as its 3-class tower group. On the other hand, it is not surprising that a mainline group, such as $\langle 243, 26\rangle$ with sTKT a.1* and relation rank $d_2 = 4$, is possible as $\mathcal{G}_i = G_3^\infty E_i$, since the upper Shafarevich bound for the relation rank of the 3-class tower group of a totally real dihedral field E_i of degree 6 with $\mathrm{Cl}_3(E_i) \simeq C_3 \times C_3$ is given by $\varrho + r_1 + r_2 - 1 = 2 + 6 + 0 - 1 = 7 > 4$ [24, Thm. 1.3, p. 75].

Assuming an asymptotic limit $33 : 33 : 33$ of the proportion of the real quadratic 3-class tower groups $G \in \{\langle 729, 99\rangle, \langle 729, 100\rangle, \langle 729, 101\rangle\}$ for the ground state of sTKT a.1, we can also conjecture an asymptotic limit $33 : 66$ of the corresponding totally real dihedral 3-class tower groups $\mathcal{G}_i \in \{\langle 243, 26\rangle, \{\langle 243, 27\rangle\}$, since the restricted dTKTs $(9, 3, 3)$, $(3, 9, 9)$, $(3, 3, 3)$ together contain three times the 9 and six times the 3 in Equation (4.2).

5. Acknowledgements

The author gratefully acknowledges that his research was supported by the Austrian Science Fund (FWF): P 26008-N25.

References

[1] E. Artin, *Beweis des allgemeinen Reziprozitätsgesetzes*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 353–363, DOI 10.1007/BF02952531.

[2] E. Artin, *Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 46–51, DOI 10.1007/BF02941159.

[3] H. U. Besche, B. Eick and E. A. O'Brien, *A millennium project: constructing small groups*, Int. J. Algebra Comput. **12** (2002), 623-644, DOI 10.1142/s0218196702001115.

[4] H. U. Besche, B. Eick and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.

[5] N. Blackburn, *On prime-power groups in which the derived group has two generators*, Proc. Camb. Phil. Soc. **53** (1957), 19–27.

[6] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.

[7] N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of imaginary quadratic fields*, Math. Ann. **368** (2017), no. 1, 633–669, DOI 10.1007/s00208-016-1449-3.

[8] N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of real quadratic fields*, to appear.

[9] F.-P. Heider und B. Schmithals, *Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen*, J. Reine Angew. Math. **336** (1982), 1–25.

[10] The MAGMA Group, MAGMA *Computational Algebra System*, Version 2.23-3, Sydney, 2017, (`http://magma.maths.usyd.edu.au`).

[11] D. C. Mayer, *List of discriminants $d_L < 200\,000$ of totally real cubic fields L, arranged according to their multiplicities m and conductors f*, Computer Centre, Department of Computer Science, University of Manitoba, Winnipeg, Canada, 1991, Austrian Science Fund, Project Nr. J0497-PHY.

[12] D. C. Mayer, *The second p-class group of a number field*, Int. J. Number Theory **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.

[13] D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.

[14] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, 27ièmes Journées Arithmétiques, Faculty of Math. and Informatics, Univ. of Vilnius, Lithuania, presentation delivered on July 01, 2011.

[15] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456, DOI 10.5802/jtnb.842.

[16] D. C. Mayer, *Principalization algorithm via class group structure*, J. Théor. Nombres Bordeaux **26** (2014), no. 2, 415–464, DOI 10.5802/jtnb.874.

[17] D. C. Mayer, *Periodic bifurcations in descendant trees of finite p-groups*, Adv. Pure Math. **5** (2015), no. 4, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015.

[18] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015) no. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015.

[19] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, 29ièmes Journées Arithmétiques, Univ. of Debrecen, Hungary, presentation delivered on July 09, 2015.

[20] D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66–104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.

[21] D. C. Mayer, *Three-stage towers of 5-class fields*, (arXiv: 1604.06930v1 [math.NT] 23 Apr 2016.)

[22] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Math. Phys. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135.

[23] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, 2nd International Conference on Groups and Algebras (ICGA) 2016, Suzhou, China, invited lecture delivered on July 26, 2016.

[24] D. C. Mayer, *Recent progress in determining p-class field towers*, Gulf J. Math. **4** (2016), no. 4, 74–102, ISSN 2309-4966.

[25] D. C. Mayer, *Recent progress in determining p-class field towers*, 1st International Colloquium of Algebra, Number Theory, Cryptography and Information Security (ANCI) 2016, Faculté Polydisciplinaire de Taza, Université Sidi Mohamed Ben Abdellah, Fès, Morocco, invited keynote delivered on November 12, 2016, `http://www.algebra.at/ANCI2016DCM.pdf`.

[26] D. C. Mayer, *Criteria for three-stage towers of p-class fields*, Adv. Pure Math. **7** (2017), no. 2, 135 – 179, DOI 10.4236/apm.2017.72008, Special Issue on Number Theory, February 2017.

[27] R. J. Miech, *Metabelian p-groups of maximal class*, Trans. Amer. Math. Soc. **152** (1970), 331–373.

[28] B. Nebelung, *Klassifikation metabelscher 3-Gruppen mit Faktorkommutatorgruppe vom Typ $(3,3)$ und Anwendung auf das Kapitulationsproblem*, Inauguraldissertation, Universität zu Köln (W. Jehne), 1989.

[29] G. Pall, *Note on irregular determinants*, J. London Math. Soc. **11** (1936), 34–35.

NAGLERGASSE 53, 8010 GRAZ, AUSTRIA
E-mail address: algebraic.number.theory@algebra.at
URL: http://www.algebra.at