The Distribution of Second p-Class Groups on Coclass Graphs

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Abstract.

For a given prime p, Leedham-Green, Newman, and Eick have defined the structure of a directed graph $\mathcal{G}(p)$ on the set of all isomorphism classes of finite p-groups. Two vertices are connected by an edge $G \to H$ if G is isomorphic to the last lower central quotient $H/\gamma_c(H)$ where $c = \mathrm{cl}(H)$ denotes the nilpotency class of H.
If the condition $|H| = p|G|$ is imposed on the edges, $\mathcal{G}(p)$ is partitioned into countably many disjoint subgraphs $\mathcal{G}(p,r)$, $r \geq 0$, called *coclass graphs* of p-groups G of coclass $r = \mathrm{cc}(G) = n - \mathrm{cl}(G)$ where $|G| = p^n$.
A coclass graph $\mathcal{G}(p,r)$ is a forest of finitely many coclass trees $\mathcal{T}(G_i)$ with roots G_i, each with a single infinite main line having a pro-p-group of coclass r as its inverse limit, and additionally contains finitely many sporadic groups outside of coclass trees: $\mathcal{G}(p,r) = (\cup_i \mathcal{T}(G_i)) \cup \mathcal{G}_0(p,r)$.

By Artin's reciprocity law, the *second p-class groups* $G_p^2(K) = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ of algebraic number fields K, where $\mathrm{F}_p^2(K)$ denotes the second Hilbert p-class field of K, are vertices of the *metabelian skeleton* of $\mathcal{G}(p)$.

Our aim is firstly to provide a general algorithm for determining the structure of $G_p^2(K)$ for a given number field K by means of number theoretical invariants of the intermediate fields $K \leq N \leq \mathrm{F}_p^1(K)$ between K and its first Hilbert p-class field $\mathrm{F}_p^1(K)$ and secondly to show that the arithmetic of special types of base fields K gives rise to *selection rules* for $G_p^2(K)$, e.g.

- If $p = 2$ and K is complex quadratic of type $(2,2)$, there are no selection rules and $\mathcal{G}(2,1)$ is entirely populated by the $G_2^2(K)$, apart from the isolated group C_4.
- If $p = 3$ and K is complex quadratic of type $(3,3)$ or real quadratic of type $(3,3)$ without total principalization, then either $G_3^2(K)$ is sporadic or lies on an even branch $\mathcal{B}_{2k}$ of a coclass tree of an even coclass graph $\mathcal{G}(3,2j)$.
- If $p \geq 3$, K is quadratic of type (p,p), and $G_p^2(K)$ is of coclass 1, then K must be real quadratic and $G_p^2(K)$ lies on an odd branch $\mathcal{B}_{2k+1}$ of the unique coclass tree $\mathcal{T}(C_p \times C_p)$ of $\mathcal{G}(p,1)$.

Our aforementioned new algorithm is based on the family of transfers $\mathrm{V}_i : G/G' \to U_i/U_i'$ from a metabelian p-group G to all intermediate groups $G' \leq U_i \leq G$. We prove that the *main lines* of coclass trees, and all other *coclass families* arising from the periodicity of branches, share a common *transfer kernel type* $\varkappa(G) = (\ker(\mathrm{V}_i))$ and that $\varkappa(G)$ is determined by the *transfer target type* $\tau(G) = (\mathrm{str}(U_i/U_i'))$ where $\mathrm{str}(A)$ denotes the multiplet of type invariants of an abelian p-group A. Consequently, the structure of $G_p^2(K)$ and the principalization type of a number field K is determined by the structures of the p-class groups $\mathrm{Cl}_p(N_i)$ of all intermediate fields $K \leq N_i \leq \mathrm{F}_p^1(K)$, according to the Artin reciprocity law.

We have implemented this algorithm in PARI/GP to determine the structure of the second 3-class groups $G_3^2(K) = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the 4 596 quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $-10^6 < D < 10^7$ and 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and to analyze their distribution on the coclass graphs $\mathcal{G}(3,r)$, $1 \leq r \leq 6$.

References.

[1] D. C. Mayer, Transfers of metabelian p-groups, *Monatsh. Math.* (2011), DOI 10.1007/s00605-010-0277-x.
[2] D. C. Mayer, The second p-class group of a number field (preprint 2010).
[3] D. C. Mayer, Principalization algorithm via class group structure (preprint 2011).

Section 0.

Introduction and Notation

Interaction: Class Field Theory $\longleftrightarrow$ Group Theory

$p \geq 2$ prime,
K algebraic number field with p-class rank $\mathrm{r}_p(\mathrm{Cl}_p(K)) \geq 2$.

TABLE 1. Second Hilbert p-class field $\mathrm{F}_p^2(K)$ and second p-class group $G = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid K)$

$\mathrm{F}_p^2(K)$		$1 = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid \mathrm{F}_p^2(K))$		The head's abelianizations contain information on p-class groups
$\mathrm{F}_p^1(N_i)$	Galois correspondence	$U_i' = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid \mathrm{F}_p^1(N_i))$		
$\mathrm{F}_p^1(K)$	$\longleftrightarrow$	$G' = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid \mathrm{F}_p^1(K))$	$\simeq$	$\mathrm{Cl}_p(\mathrm{F}_p^1(K))$
N_i		$U_i = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid N_i)$	$\longrightarrow$	$U_i/U_i' \simeq \mathrm{Gal}(\mathrm{F}_p^1(N_i) \mid N_i) \simeq \mathrm{Cl}_p(N_i)$
K		$G = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid K)$	$\longrightarrow$	$G/G' \simeq \mathrm{Gal}(\mathrm{F}_p^1(K) \mid K) \simeq \mathrm{Cl}_p(K)$

$(U_i)_i$ family of all intermediate normal groups $G' \leq U_i \leq G$, called the *head* of G,
$(N_i)_i$ family of all intermediate fields $K \leq N_i \leq \mathrm{F}_p^1(K)$,
satisfying $U_i = \mathrm{Gal}(\mathrm{F}_p^2(K) \mid N_i)$ and $U_i/G' \simeq \mathrm{Norm}_{N_i|K}(\mathrm{Cl}_p(N_i))$.

Principalization $\longleftrightarrow$ Transfer

TABLE 2. Family of class extensions $\mathrm{j}_{N_i|K}$ and transfers V_{G,U_i}

		$\mathrm{j}_{N_i \mid K}$		
	$\mathrm{Cl}_p(K)$	$\longrightarrow$	$\mathrm{Cl}_p(N_i)$	
Artin isomorphism	$\updownarrow$	$///$	$\updownarrow$	Artin isomorphism
	G/G'	$\longrightarrow$	U_i/U_i'	
		$\mathrm{V}_{G,U_i} = \mathrm{V}_i$		

TABLE 3. Corresponding invariants of K and its second p-class group $G = G_p^2(K)$

p-Principalization Type of K $\varkappa(K) = (\ker(\mathrm{j}_{N_i \mid K}))_i$ Total p-Principalization, for $N_i \neq \mathrm{F}_p^1(K)$ $\nu(K) = \#\{i \mid \ker(\mathrm{j}_{N_i \mid K}) = \mathrm{Cl}_p(K)\}$	*Transfer Kernel Type* (**TKT**) of G $\varkappa(G) = (\ker(\mathrm{V}_i))_i$ Total Transfer, for $U_i \neq G'$ $\nu(G) = \#\{i \mid \ker(\mathrm{V}_i) = G/G'\}$
p-Class Group Structure Type of K $\tau(K) = (\mathrm{str}(\mathrm{Cl}_p(N_i)))_i$ Exceptional p-rank, for $N_i \neq \mathrm{F}_p^1(K)$ $\varepsilon(K) = \#\{i \mid \mathrm{r}_p(\mathrm{Cl}_p(N_i)) \geq p\}$	*Transfer Target Type* (**TTT**) of G $\tau(G) = (\mathrm{str}(U_i/U_i'))_i$ Exceptional p-rank, for $U_i \neq G'$ $\varepsilon(G) = \#\{i \mid \mathrm{r}_p(U_i/U_i') \geq p\}$
p-Class Group Order Type of K $\tau_0(K) = (\mathrm{ord}(\mathrm{Cl}_p(N_i)))_i$	Weak Transfer Target Type of G $\tau_0(G) = (\mathrm{ord}(U_i/U_i'))_i$

Terminology concerning Coclass Graphs $\mathcal{G}(p,r)$

- **Coclass** $\mathrm{cc}(G)$ of a finite p-group G of order $|G| = p^n$ and nilpotency class $\mathrm{cl}(G)$ is defined by $n = \mathrm{cl}(G) + \mathrm{cc}(G)$.
- **Vertex:**
  the isomorphism class of a finite p-group G of coclass $\mathrm{cc}(G) = r$.
- H is **Immediate Descendant** of G,
  if G is isomorphic to the last lower central quotient $H/\gamma_c(H)$,
  with nilpotency class $c = \mathrm{cl}(H)$ and $\gamma_c(H)$ cyclic of order p.
  Then G and H are connected by a **directed edge** $G \to H$.
- **Capable Vertex:** has at least one immediate descendant.
  Terminal Vertex: has no immediate descendants.
- H is **descendant** of G,
  if there is a path of directed edges from G to H.
  In particular, H is descendant of itself, with empty path.
- **Tree** $\mathcal{T}(G)$ with root G: consists of all descendants of G.
- **Coclass Tree:**
  maximal rooted tree containing exactly one infinite path.
- **Main Line:** the unique maximal infinite path of a coclass tree.
- **Branch** $\mathcal{B}(G)$ with root G on a main line: $\mathcal{T}(G) \setminus \mathcal{T}(H)$,
  H denoting the immediate descendant of G on the main line.
- **Depth** $\mathrm{dp}(H)$ of a vertex H on a branch $\mathcal{B}(G)$:
  its distance from the root G on the main line.
  $\mathcal{B}_d(G)$ denotes the **branch of bounded depth** d.
- **Coclass Family** $\mathcal{F}(H)$ of a vertex $H \in \mathcal{B}_d(G_n)$,
  where G_n denotes the vertex of order p^n on the main line
  (n sufficiently large that periodicity has set in already):
  the infinite sequence $(H_i)_{i\ge 0}$ of vertices defined recursively by
  $H_0 = H$ and $H_i = \varphi_{n+(i-1)\ell}(H_{i-1})$ for $i \ge 1$ using the
  periodicity isomorphisms of graphs $\varphi_n : \mathcal{B}_d(G_n) \to \mathcal{B}_d(G_{n+\ell})$
  with period length $\ell = p - 1$.

Summary of Most Recent Fundamental Insights

- Coclass Theory is particularly well suited as a foundation
  not only of p-Group Theory but also of Class Field Theory.
- A Coclass Family is arranged *vertically* on a Coclass Graph and has a Parametrized Presentation for all members
  (whereas an Isoclinism Family is arranged *horizontally*, intersects with infinitely many Coclass Graphs, and does not admit a uniform presentation).
- Members of a Coclass Family share a Common Transfer Kernel Type (TKT) and can be viewed as Excited States $\mathrm{T}\uparrow^n$ of a Ground State T. (There are, however, Isoclinism Families all of whose members have different TKT.)
- New Top-Down Class Number Formulas for certain
  Distinguished Fields N_1, N_2 reveal the Invariants of the Group $G_p^2(K)$.
- The Modern Top-Down Algorithm either determines the Group *uniquely*
  or within a Finite Batch of closely related isoclinic groups
  (whereas the Classical Bottom-Up Principalization Algorithm only
  indicates the Infinite Coclass Family to which the Group $G_p^2(K)$ belongs).
- Total Principalization in the Distinguished Field N_2 determines
  Selection Rules for the Group $G_p^2(K)$ of Quadratic Base Fields K:
  Parity of the Coclass Graph $\mathcal{G}(p,r)$ and of the Branch $\mathcal{B}(j)$.
- Vertices on Main Lines possess at least One Total Principalization $\varkappa(1)=0$.
- Miech's Invariant k together with
  Total Principalization in the Distinguished Field N_1 determines the
  Depth of the Group $G_p^2(K)$ on its Coclass Tree $\mathcal{T}$.
- The recently discovered Connection between the TKT $\varkappa$ and the
  Transfer Target Type (TTT) τ made it possible to
  Extend the Computation of $G_3^2(K)$ to incredible 2020 complex and 2576 real
  Quadratic Fields $K=\mathbb{Q}(\sqrt{D})$ with discriminants $-10^6 < D < 10^7$
  by means of the Top-Down Algorithm.
  (The previous state of the art was that
  Heider and Schmithals computed $\varkappa$, but not $G_3^2(K)$,
  for 13 complex and 5 real K within $-2\cdot 10^4 < D < 10^5$
  using the Bottom-Up Algorithm.)

Section 1.

p-Groups G of Coclass $\mathrm{cc}(G) = 1$

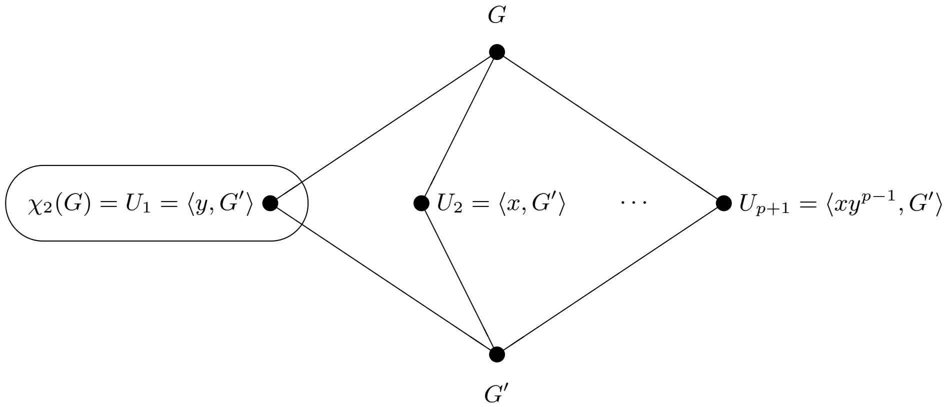
($p \geq 2$ prime)

Asymmetry of the Head of Metabelian p-Groups of Coclass 1

Wiman Blackburn Lemma. $p \geq 2$ prime, $G \in \mathcal{G}(p,1) \Longrightarrow$
1. The abelianization G/G' is of **diamond type** (p,p),
2. The 2-step centralizer $\chi_2(G)$ of $\gamma_2(G) = G'$ with the property

$$[\chi_2(G), \gamma_2(G)] \leq \gamma_4(G)$$

is strictly bigger than $\gamma_2(G)$, provided that $|G| \geq p^4$,
and causes a **polarization** of the **diamond head**:



Parametrized Presentations for Metabelian p-Groups of Coclass 1

Representatives for the vertices of $\mathcal{G}(p,1)$ are the groups

$$G = G_a^n(z,w) = \langle x, y \rangle$$

with 2 generators which satisfy the **Blackburn Miech Relations**

$$x^p = s_{n-1}^w, \quad y^p \prod_{\ell=2}^{p} s_\ell^{\binom{p}{\ell}} = s_{n-1}^z, \quad [y, s_2] = \prod_{r=1}^{k} s_{n-r}^{a(n-r)}, \quad |G| = p^n,$$

where $s_2 = [y,x] \in \gamma_2(G)$ and $s_i = [s_{i-1}, x] \in \gamma_i(G)$ for $i \geq 3$, and

$$x \in G \setminus \chi_2(G), \text{ if } n \geq 4, \quad y \in \chi_2(G) \setminus \gamma_2(G).$$

Miech's invariant $0 \leq k \leq \min(n-4, p-2)$ is defined by

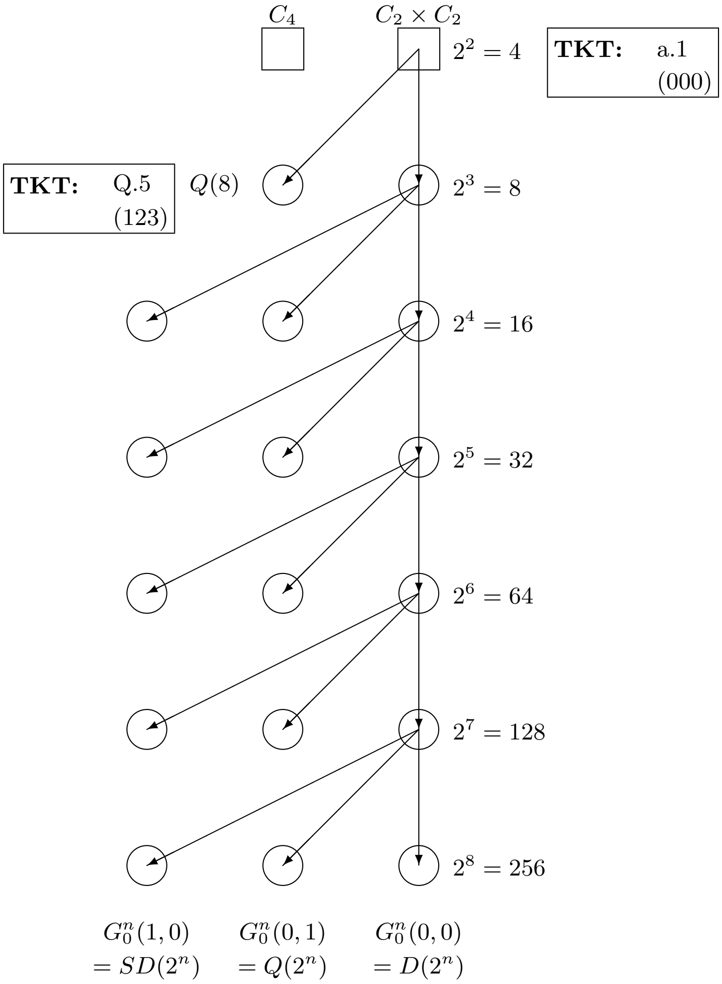
$$[\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G)$$

and provides a measure for the deviation from the maximal degree of commutativity.

The Coclass Graph $\mathcal{G}(2,1)$

Burnside Gorenstein Theorem.

1. $G/G' \simeq (2,2) \Longrightarrow G \in \mathcal{G}(2,1)$.
2. $G \in \mathcal{G}(2,1) \Longrightarrow G$ metabelian.

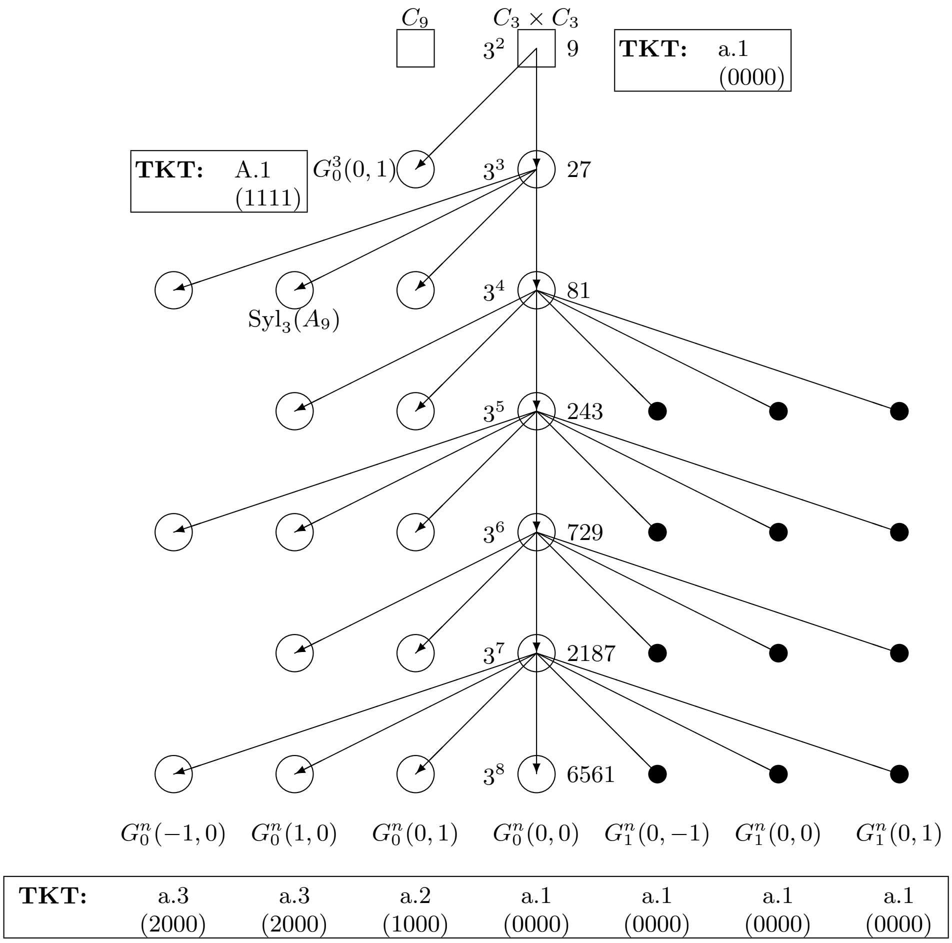


TKT:	S.4	Q.6	d.8
	(232)	(132)	(032)

Main line:	$(C_2 \times C_2, (D(2^n))_{n\geq 3})$.
3 coclass families:	$(D(2^n))_{n\geq 3}$, $(Q(2^n))_{n\geq 4}$, $(SD(2^n))_{n\geq 4}$, with invariant $k = 0$.
Sporadic groups:	C_4, $C_2 \times C_2 = V_4$, $Q(8)$.
Periodicity of branches:	$\mathcal{B}_1(D(2^n)) \simeq \mathcal{B}_1(D(2^{n+1}))$ for $n \geq 3$.
Maximal depth:	$d = 1$.
Period length:	$\ell = 1$.

The Coclass Graph $\mathcal{G}(3,1)$

Wiman Blackburn Theorem.
$G \in \mathcal{G}(3,1) \Longrightarrow G$ metabelian.



Main line:	$(C_3 \times C_3, (G_0^n(0,0))_{n\ge 3})$.
13 coclass families:	one for even n only,
	$(G_0^n(-1,0))_{n\ge 4}$, with $k=0$,
	and the others for n either even or odd,
	$(G_0^n(0,0))_{n\ge 3}$, $(G_0^n(0,1))_{n\ge 4}$, $(G_0^n(1,0))_{n\ge 5}$, with invariant $k=0$,
	$(G_1^n(0,0))_{n\ge 5}$, $(G_1^n(0,1))_{n\ge 5}$, $(G_1^n(0,-1))_{n\ge 5}$, with $k=1$.
Sporadic groups:	C_9, $C_3 \times C_3$, $G_0^3(0,1)$, $G_0^4(1,0) \simeq \mathrm{Syl}_3(A_9)$.
Periodicity of branches:	$\mathcal{B}_1(G_0^n(0,0)) \simeq \mathcal{B}_1(G_0^{n+2}(0,0))$ for $n \ge 4$.
Maximal depth:	$d=1$.
Period length:	$\ell = 2$.

Coclass Families Share a Common Transfer Kernel Type (TKT)

Theorem 1. Transfer Kernel Type $\varkappa(G)$ for $\mathrm{cc}(G) = 1$, $p \geq 3$

TABLE 4. $\varkappa(G)$ in dependence on $G \in \mathcal{G}(p, 1)$ for $p \geq 3$

			p-Group $G_a^n(z,w)$ of Coclass 1							
TKT	$\varkappa$	ν	G	cl(G)	n	a	z	w	dp(G)	tree position
a.1	$(\overbrace{0\ldots0}^{p+1 \text{ times}})$	$p+1$	$C_p \times C_p$	1	2				0	root
A.1	$(\overbrace{1\ldots1}^{p+1 \text{ times}})$	0	$G_0^3(0,1)$	2	3	0	0	1	1	sporadic
a.1	$(\overbrace{0\ldots0}^{p+1 \text{ times}})$	$p+1$	$G_0^n(0,0)$	≥ 2	≥ 3	0	0	0	0	main line
a.2	$(1\overbrace{0\ldots0}^{p \text{ times}})$	p	$G_0^n(0,1)$	≥ 3	≥ 4	0	0	1	1	cc-families
a.3	$(2\overbrace{0\ldots0}^{p \text{ times}})$	p	$G_0^n(z,0)$	≥ 3	≥ 4	0	$\neq 0$	0	1	cc-families
a.1	$(\overbrace{0\ldots0}^{p+1 \text{ times}})$	$p+1$	$G_a^n(z,w)$	≥ 4	≥ 5	$\neq 0$			≥ 1	cc-families

Proof: D. C. Mayer, April 2010, see [1] Transfers of metabelian p-groups, Thm.2.6.

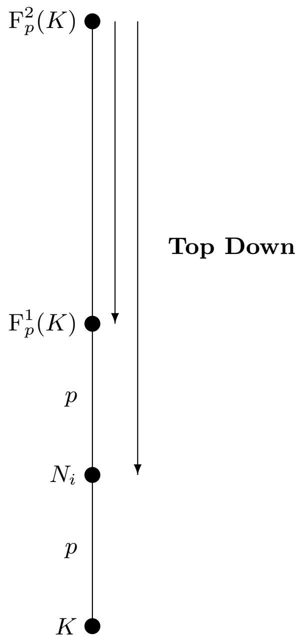
Theorem 2. Transfer Kernel Type $\varkappa(G)$ for $\mathrm{cc}(G) = 1$, $p = 2$

TABLE 5. $\varkappa(G)$ in dependence on $G \in \mathcal{G}(2, 1)$

			2-Group $G_a^n(z,w)$ of Coclass 1							
TKT	$\varkappa$	ν	G	cl(G)	n	a	z	w	dp(G)	tree position
a.1	(000)	3	$C_2 \times C_2$	1	2				0	root
Q.5	(123)	0	$Q(8)$	2	3	0	0	1	1	sporadic
d.8	(032)	1	$D(2^n)$	≥ 2	≥ 3	0	0	0	0	main line
Q.6	(132)	0	$Q(2^n)$	≥ 3	≥ 4	0	0	1	1	coclass family
S.4	(232)	0	$SD(2^n)$	≥ 3	≥ 4	0	1	0	1	coclass family

Proof: D. C. Mayer, October 2009, see [1] Transfers of metabelian p-groups, Thm.2.5.

Branch and Depth on the Coclass Tree $\mathcal{T}(C_p \times C_p)$



Theorem 3. The **Weak TTT** τ_0 for $p \geq 2$, $\mathrm{cc}(G) = 1$ is given by

$$\begin{aligned} \mathrm{h}_p(\mathrm{F}_p^1(K)) &= p^{\mathrm{cl}(G)-1}, \\ \mathrm{h}_p(N_1) &= p^{\mathrm{cl}(G)-k}, \\ \mathrm{h}_p(N_i) &= p^2 \text{ for } 2 \leq i \leq p+1. \end{aligned}$$

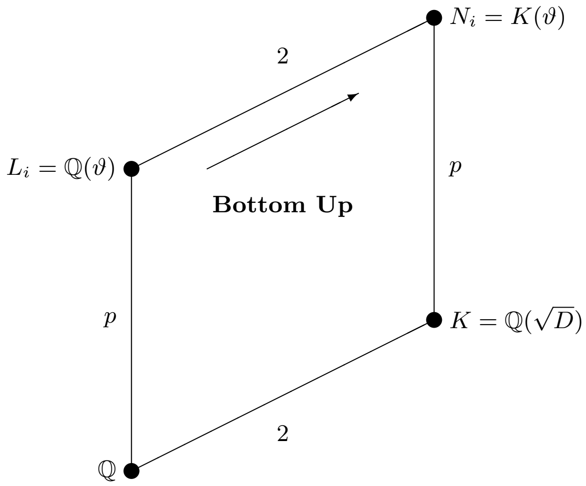
Whereas $\mathrm{h}_p(N_2), \ldots, \mathrm{h}_p(N_{p+1})$ only indicate that $\mathrm{cc}(G) = 1$, $\mathrm{h}_p(\mathrm{F}_p^1(K))$ determines the order p^n, $n = \mathrm{cl}(G) + 1$, and class of G, and the distinguished $\mathrm{h}_p(N_1)$ gives the invariant k of G.

The **Branch Root Order** of G is given by $\mathrm{cl}(G) + 1 - \mathrm{dp}(G)$,

where the **Depth** of G is $\mathrm{dp}(G) = \begin{cases} k, & \text{if } \varkappa(1) = 0, \\ k+1, & \text{if } \varkappa(1) \neq 0. \end{cases}$

Proof: D. C. Mayer, April 2010, see [2] The second p-class group of a number field, Thm.3.2.

Selection Rules for $K = \mathbb{Q}(\sqrt{D})$, $p \geq 3$, $\mathrm{cc}(G) = 1$



Theorem 4. If $G \in \mathcal{G}(p, 1)$, then K must be real quadratic, $D > 0$, and
the p-class numbers of the non-Galois subfields L_i of N_i are given by

$$\begin{aligned} \mathrm{h}_p(L_1) &= p^{\frac{\mathrm{cl}(G)-\mathrm{dp}(G)}{2}}, \\ \mathrm{h}_p(L_i) &= p \text{ for } 2 \leq i \leq p+1. \end{aligned}$$

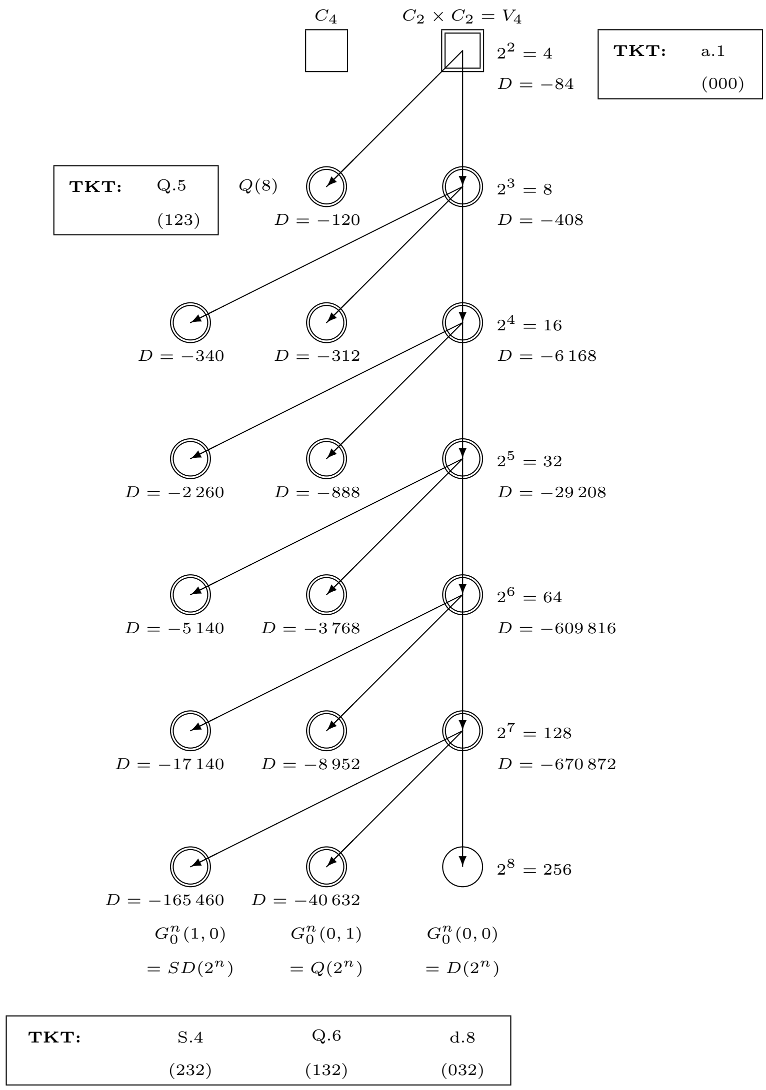
Whereas $\mathrm{h}_p(L_2), \ldots, \mathrm{h}_p(L_{p+1})$ do not give any information,
the distinguished $\mathrm{h}_p(L_1)$ enforces $\mathrm{cl}(G) - \mathrm{dp}(G) \equiv 0 \pmod 2$.

The **Branch Root Order** of G is **odd**,

$$\mathrm{cl}(G) + 1 - \mathrm{dp}(G) \equiv 1 \pmod 2.$$

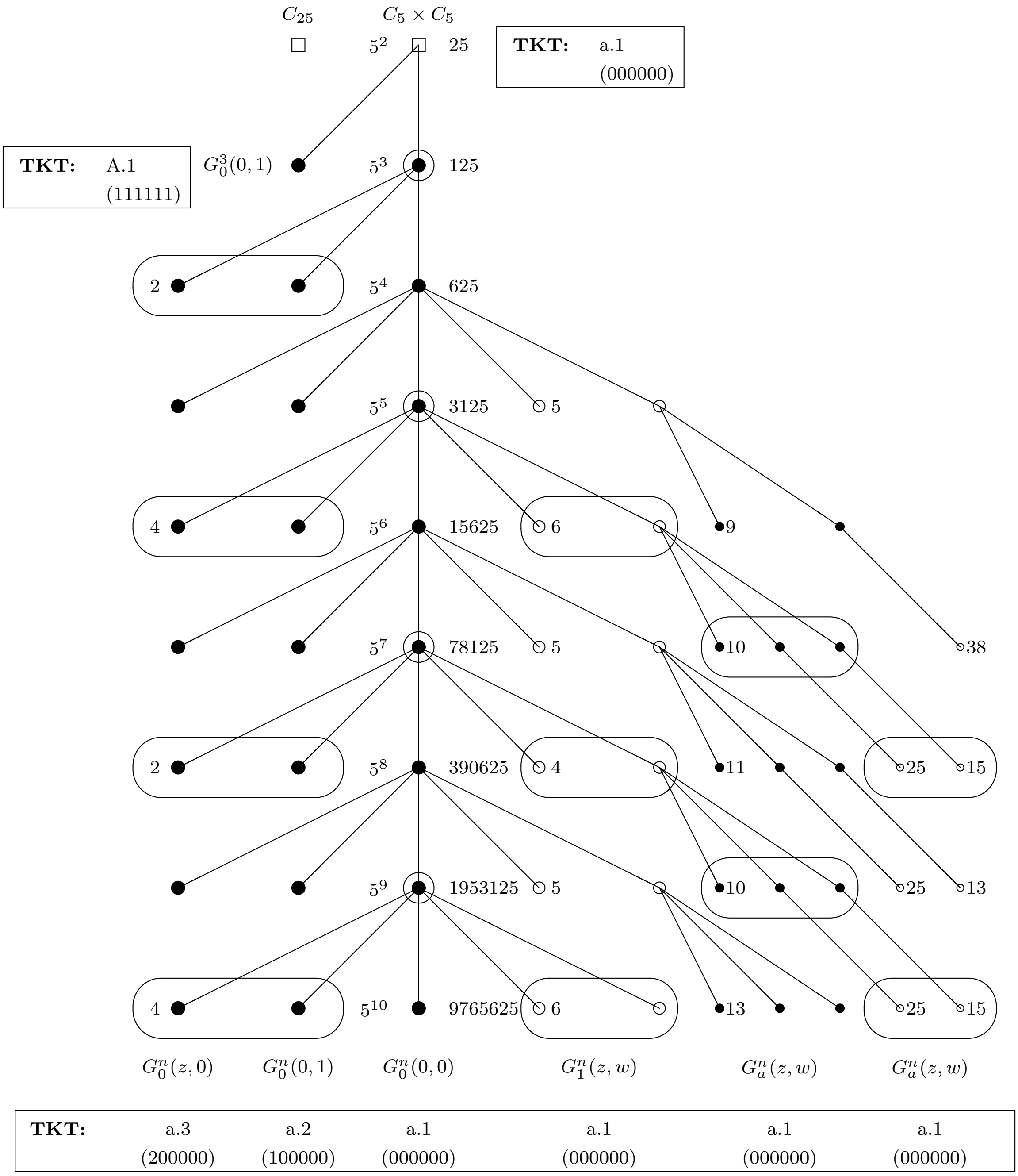
Proof: D. C. Mayer, April 2010, see [2] The second p-class group of a number field, Thm.4.1.

Population of the Coclass Graph $\mathcal{G}(2,1)$



- The numerical results suggest the conjecture that the tree $\mathcal{T}(V_4)$ is covered entirely by second 2-class groups $G_2^2(K)$ of complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$, $D < 0$.

FIGURE 1. TKTs and selection rule for $G_5^2(K)$, $K = \mathbb{Q}(\sqrt{D})$, $D > 0$, on the metabelian skeleton of $\mathcal{G}(5,1)$, where bigger values of $0 \le k \le 3$ occur.



Section 2.

3-**Groups** G **of Coclass** $\mathrm{cc}(G) = r$

$(r \geq 2)$

Parametrized Presentations for Metabelian 3-Groups of Coclass at least 2 with Abelianization of type $(3,3)$

Nebelung's Lemma. $G \in \mathcal{G}(3,r)$, $r \geq 2 \Longrightarrow$
The smallest integer $s \geq 2$ such that the 2-step centralizer $\chi_s(G)$ of $\gamma_s(G)$ with the property

$$[\chi_s(G), \gamma_s(G)] \leq \gamma_{s+2}(G)$$

is strictly bigger than $\gamma_2(G) = G'$ satisfies the inequalities

$$3 \leq r+1 \leq s \leq r+2.$$

Representatives for the vertices of $\mathcal{G}(3,r)$, $r \geq 2$, are the groups

$$G = G_\rho^{m,n}(\alpha,\beta,\gamma,\delta) = \langle x, y \rangle$$

with 2 generators x, y which satisfy the **Nebelung Relations**

$$s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1}, \quad s_3\sigma_3\sigma_4 = \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma} \tau_e^{\delta}, \quad t_3^{-1}\tau_3\tau_4 = \sigma_{m-2}^{\rho\delta} \sigma_{m-1}^{\alpha} \tau_e^{\beta}, \quad \tau_{e+1} = \sigma_{m-1}^{-\rho},$$

$$\mathrm{cl}(G) = m-1, \quad |G| = 3^n, \quad e = n-m+2 = \mathrm{cc}(G)+1, \text{ where}$$

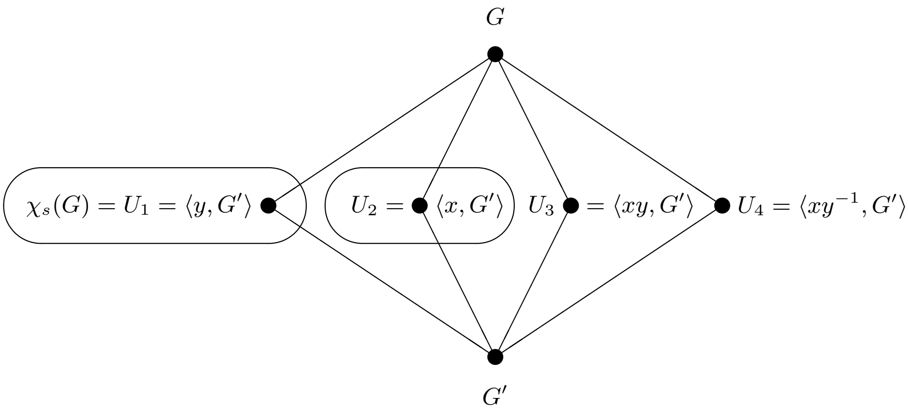
$$s_2 = t_2 = [y,x] \in \gamma_2(G), \quad s_i = [s_{i-1},x], t_i = [t_{i-1},y] \in \gamma_i(G) \text{ for } i \geq 3,$$

$$\sigma_3 = y^3, \tau_3 = x^3 \in \gamma_3(G), \quad \sigma_i = [\sigma_{i-1},x], \tau_i = [\tau_{i-1},y] \in \gamma_i(G) \text{ for } i \geq 4,$$

$$\text{and } \gamma_3(G)/\gamma_4(G) = \langle y^3, x^3 \rangle, \quad x \in G \setminus \chi_s(G), \text{ if } e < m-1, \quad y \in \chi_s(G) \setminus G'.$$

Asymmetry of the Head of Metabelian 3-Groups of Coclass at least 2

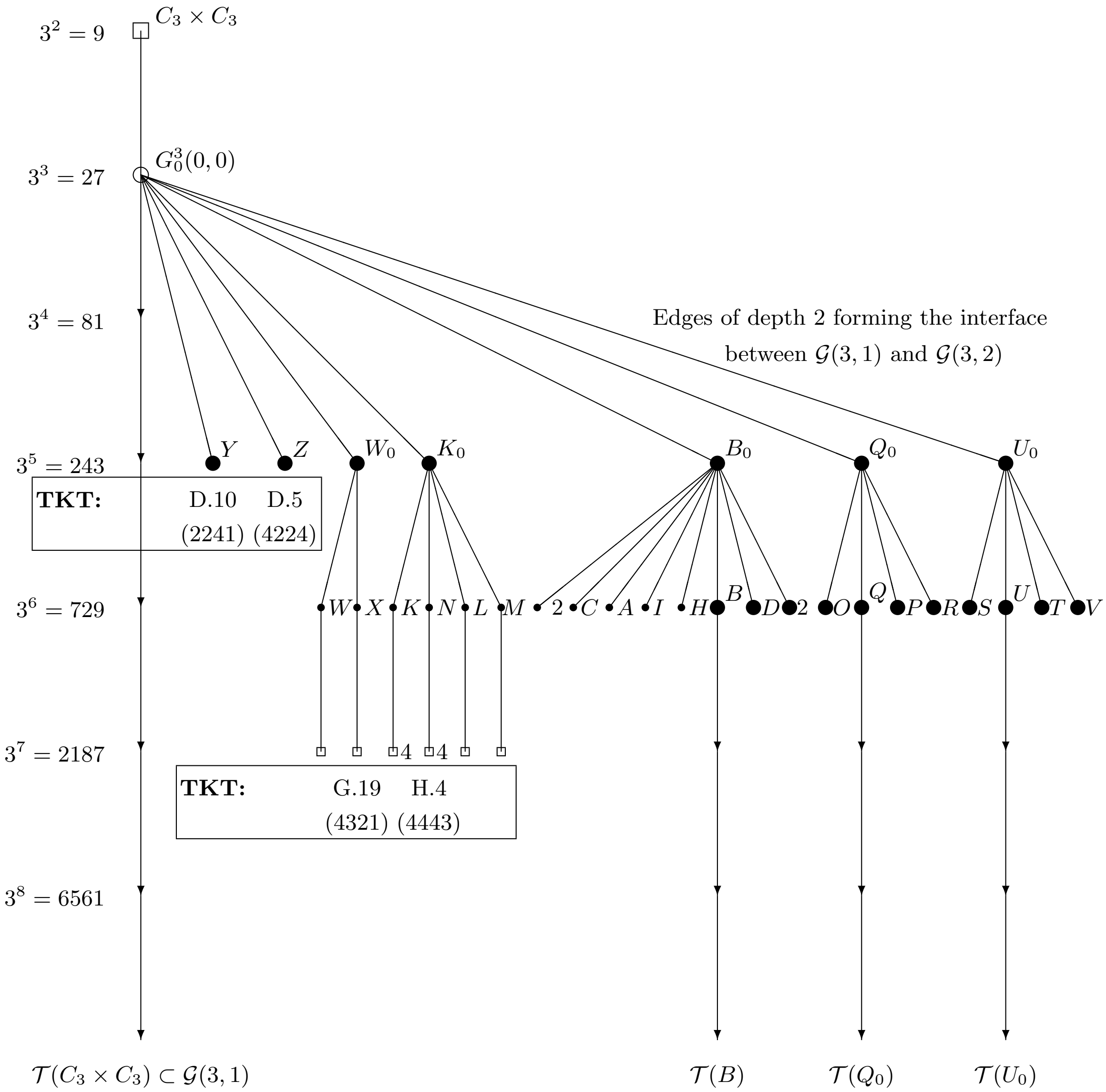
The choice of y, x causes a **bipolarization** of the **diamond head**:



Miech's invariant $0 \leq k \leq 1$ is defined by

$$[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G).$$

Top of Coclass Graph $\mathcal{G}(3,2)$ restricted to Groups with Abelianization of Type $(3,3)$

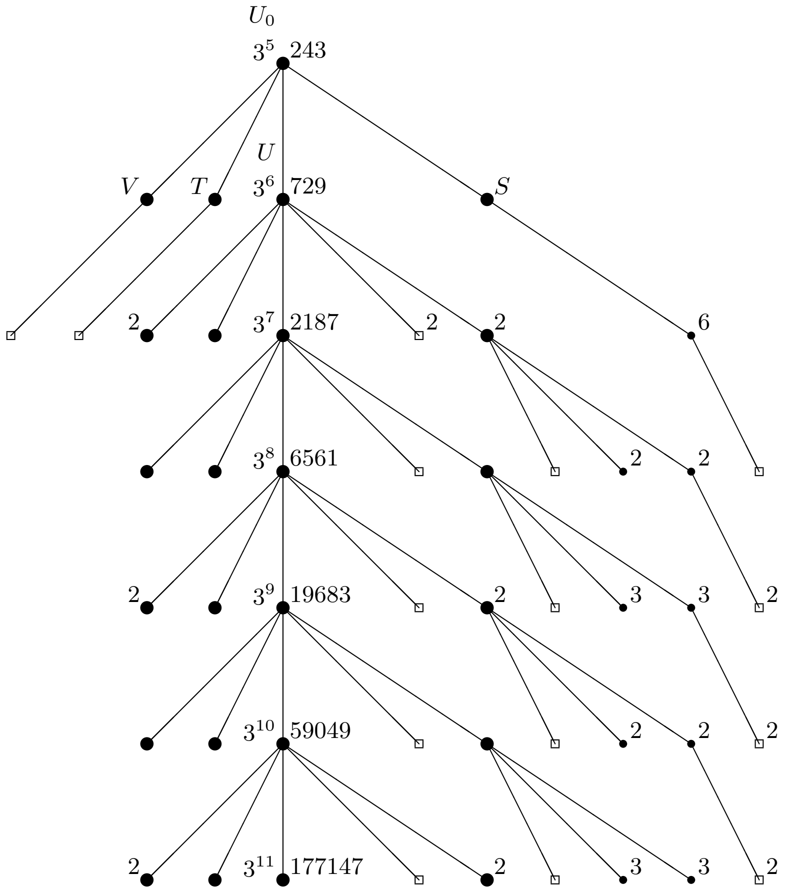


3 roots of coclass trees with metabelian main lines:
$B = G_0^{5,6}(0,0,0,0)$, $Q_0 = G_0^{4,5}(0,-1,0,1)$, $U_0 = G_0^{4,5}(0,0,0,1)$.
Isolated and sporadic groups:
$Y = G_0^{4,5}(0,0,-1,1)$, $Z = G_0^{4,5}(1,1,-1,1)$; $W_0 = G_0^{4,5}(-1,0,0,1)$, $K_0 = G_0^{4,5}(1,1,1,1)$.

The Coclass Tree $\mathcal{T}(U_0) \subset \mathcal{G}(3,2)$

Structure Theorem.

$G \in \mathcal{T}(U_0) \Longrightarrow \varepsilon(G) = 0.$



TKT:	E.9	E.8	c.21	G.16	G.16	G.16
	(2231)	(1231)	(0231)	(4231)	(4231)	(4231)

Main line: $(G_0^{n-1,n}(0,0,0,1))_{n\geq 5}$, where $U_0 = G_0^{4,5}(0,0,0,1)$, $U = G_0^{5,6}(0,0,0,1)$.

10 coclass families: metabelian with invariant $k = 0$: two for odd n only, $(G_0^{n-1,n}(0,0,-1,1))_{n\geq 7}$, $(G_0^{n-1,n}(-1,0,0,1))_{n\geq 7}$, and the others for n either even or odd, $(G_0^{n-1,n}(0,0,0,1))_{n\geq 5}$, $(G_0^{n-1,n}(1,0,-1,1))_{n\geq 6}$, $(G_0^{n-1,n}(0,0,1,1))_{n\geq 6}$, $(G_0^{n-1,n}(1,0,0,1))_{n\geq 6}$, where $T = G_0^{5,6}(1,0,-1,1)$, $V = G_0^{5,6}(0,0,1,1)$, $S = G_0^{5,6}(1,0,0,1)$.

Periodicity of branches: $\mathcal{B}_3(G_0^{n-1,n}(0,0,0,1)) \simeq \mathcal{B}_3(G_0^{n+1,n+2}(0,0,0,1))$ for $n \geq 7$.

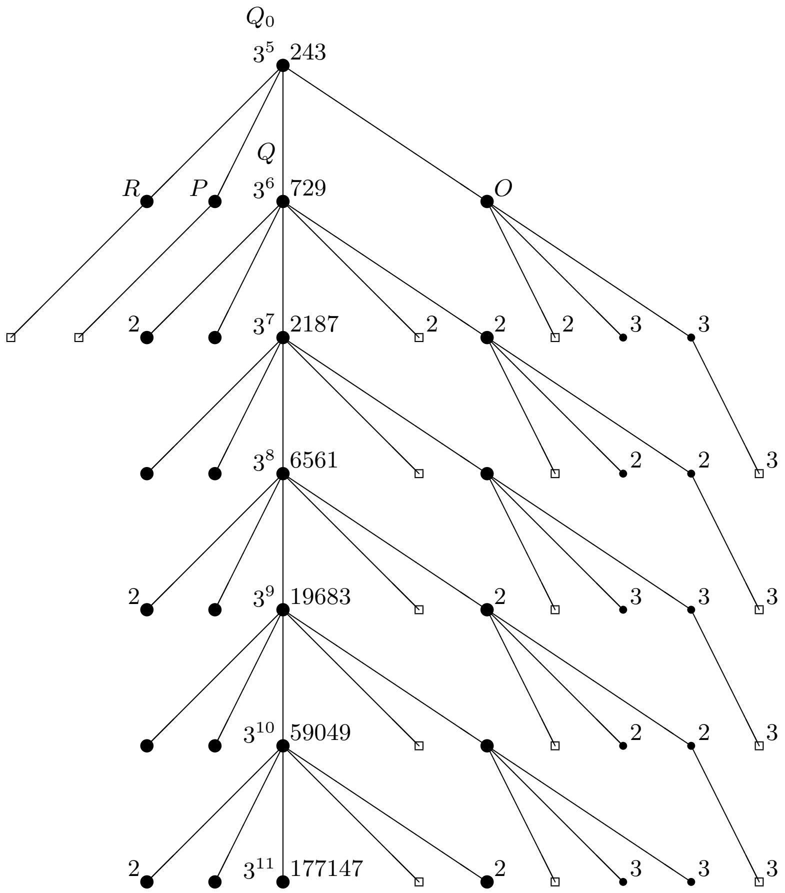
Maximal depth: $d = 3$.

Period length: $\ell = 2$.

The Coclass Tree $\mathcal{T}(Q_0) \subset \mathcal{G}(3,2)$

Structure Theorem.

$G \in \mathcal{T}(Q_0) \Longrightarrow \varepsilon(G) = 1.$



TKT: E.14	E.6	c.18	H.4	H.4	H.4
(2313)	(1313)	(0313)	(3313)	(3313)	(3313)

Main line: $(G_0^{n-1,n}(0,-1,0,1))_{n\geq 5}$, where $Q_0 = G_0^{4,5}(0,-1,0,1)$, $Q = G_0^{5,6}(0,-1,0,1)$.

10 coclass families: metabelian with invariant $k = 0$: two for odd n only, $(G_0^{n-1,n}(0,-1,-1,1))_{n\geq 7}$, $(G_0^{n-1,n}(-1,-1,1,1))_{n\geq 7}$, and the others for n either even or odd, $(G_0^{n-1,n}(0,-1,0,1))_{n\geq 5}$, $(G_0^{n-1,n}(1,-1,1,1))_{n\geq 6}$, $(G_0^{n-1,n}(0,-1,1,1))_{n\geq 6}$, $(G_0^{n-1,n}(1,-1,-1,1))_{n\geq 6}$, where $P = G_0^{5,6}(1,-1,1,1)$, $R = G_0^{5,6}(0,-1,1,1)$, $O = G_0^{5,6}(1,-1,-1,1)$.

Periodicity of branches: $\mathcal{B}_3(G_0^{n-1,n}(0,-1,0,1)) \simeq \mathcal{B}_3(G_0^{n+1,n+2}(0,-1,0,1))$ for $n \geq 7$.

Maximal depth: $d = 3$.

Period length: $\ell = 2$.

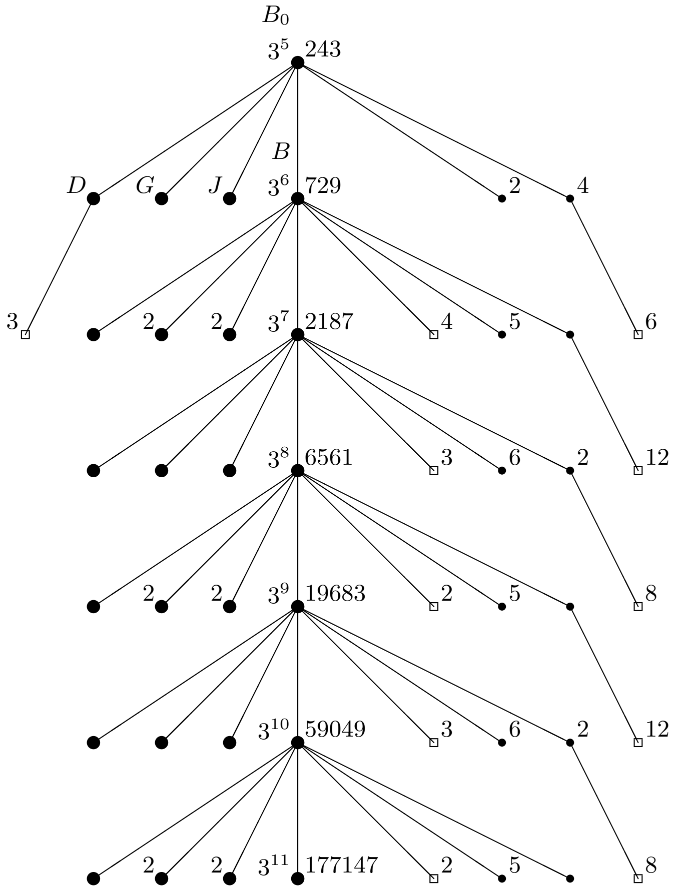
The Coclass Tree $\mathcal{T}(B) \subset \mathcal{T}(B_0) \subset \mathcal{G}(3,2)$

Structure Theorem.

$G \in \mathcal{T}(B_0) \Longrightarrow \varepsilon(G) = 2.$

Selection Rule.

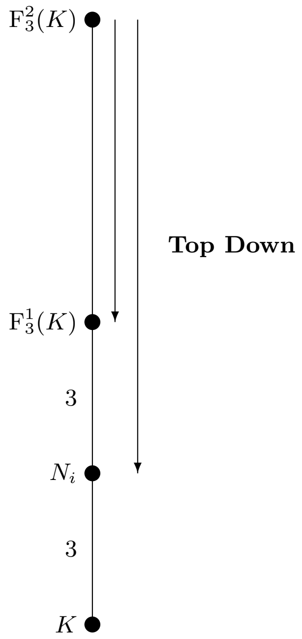
$K = \mathbb{Q}(\sqrt{D}) \Longrightarrow G_3^2(K) \notin \mathcal{T}(B_0).$



TKT:	d.23	d.25	d.19	b.10		b.10	b.10
	(1043)	(2043)	(4043)	(0043)		(0043)	(0043)

Main line:	$(G_0^{n-1,n}(0,0,0,0))_{n\ge 5}$, where $B_0 = G_0^{4,5}(0,0,0,0)$, $B = G_0^{5,6}(0,0,0,0)$.
10 coclass families:	metabelian with invariant $k=0$: two for odd n only, $(G_0^{n-1,n}(0,0,-1,0))_{n\ge 7}$, $(G_0^{n-1,n}(1,0,-1,0))_{n\ge 7}$, and the others for n either even or odd, $(G_0^{n-1,n}(0,0,0,0))_{n\ge 5}$, $(G_0^{n-1,n}(1,0,0,0))_{n\ge 6}$, $(G_0^{n-1,n}(0,0,1,0))_{n\ge 6}$, $(G_0^{n-1,n}(1,0,1,0))_{n\ge 6}$, where $D = G_0^{5,6}(1,0,0,0)$, $G = G_0^{5,6}(0,0,1,0)$, $J = G_0^{5,6}(1,0,1,0)$.
Periodicity of branches:	$\mathcal{B}_2(G_0^{n-1,n}(0,0,0,0)) \simeq \mathcal{B}_2(G_0^{n+1,n+2}(0,0,0,0))$ for $n \ge 7$.
Maximal depth:	$d = 2$.
Period length:	$\ell = 2$.

Branch and Depth on a Coclass Tree



Theorem 5. The **Weak TTT** τ_0 for $p = 3$, $\mathrm{cc}(G) \geq 2$ is given by

$$\begin{aligned} \mathrm{h}_3(\mathrm{F}_3^1(K)) &= 3^{\mathrm{cl}(G)+\mathrm{cc}(G)-2}, \\ \mathrm{h}_3(N_1) &= 3^{\mathrm{cl}(G)-k}, \\ \mathrm{h}_3(N_2) &= 3^{\mathrm{cc}(G)+1}, \\ \mathrm{h}_3(N_i) &= 3^3 \text{ for } 3 \leq i \leq 4. \end{aligned}$$

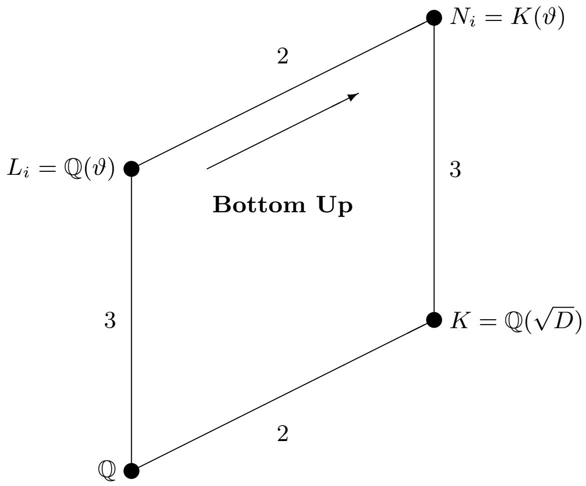
Whereas $\mathrm{h}_3(N_3)$ and $\mathrm{h}_3(N_4)$ only indicate that $\mathrm{cc}(G) \geq 2$,
the distinguished $\mathrm{h}_3(N_2)$ gives the precise coclass of G,
$\mathrm{h}_3(\mathrm{F}_3^1(K))$ determines the order 3^n, $n = \mathrm{cl}(G) + \mathrm{cc}(G)$, and class of G,
and finally the distinguished $\mathrm{h}_3(N_1)$ yields the invariant k of G.

The **Branch Root Order** of G is given by $\mathrm{cl}(G) + \mathrm{cc}(G) - \mathrm{dp}(G)$,

where the **Depth** of non-sporadic G is $\mathrm{dp}(G) = \begin{cases} k, & \text{if } \varkappa(1) = 0, \\ k+1, & \text{if } \varkappa(1) \neq 0. \end{cases}$

Proof: D. C. Mayer, May 2003, see [2] The second p-class group of a number field, Thm.3.4.

Selection Rules for $K = \mathbb{Q}(\sqrt{D})$, $p = 3$, $\mathrm{cc}(G) \geq 2$



Theorem 6. The 3-class numbers of the non-Galois subfields L_i of N_i are given by

$$\begin{aligned}
\mathrm{h}_3(L_1) &= \begin{cases} 3^{\frac{\mathrm{cl}(G)-(k+1)}{2}}, & \text{for sporadic } G, \\ 3^{\frac{\mathrm{cl}(G)-\mathrm{dp}(G)}{2}}, & \text{otherwise,} \end{cases} \\
\mathrm{h}_3(L_2) &= \begin{cases} 3^{\frac{\mathrm{cc}(G)+1}{2}}, & \text{if } \varkappa(2) = 0, \\ 3^{\frac{\mathrm{cc}(G)}{2}}, & \text{if } \varkappa(2) \neq 0, \end{cases} \\
\mathrm{h}_3(L_i) &= 3 \text{ for } 3 \leq i \leq 4.
\end{aligned}$$

Whereas $\mathrm{h}_3(L_3)$ and $\mathrm{h}_3(L_4)$ do not give any information,
the distinguished $\mathrm{h}_3(L_2)$ indicates the coclass of G and enforces

$$\mathrm{cc}(G) \equiv \begin{cases} 1 \pmod 2, & \text{if } \varkappa(2) = 0, \\ 0 \pmod 2, & \text{if } \varkappa(2) \neq 0, \end{cases}$$

and the distinguished $\mathrm{h}_3(L_1)$ demands $\mathrm{cl}(G) - \mathrm{dp}(G) \equiv 0 \pmod 2$.

The **Branch Root Order** of non-sporadic G is given by

$$\mathrm{cl}(G) + \mathrm{cc}(G) - \mathrm{dp}(G) \equiv \begin{cases} 1 \pmod 2, & \text{if } \varkappa(2) = 0, \\ 0 \pmod 2, & \text{if } \varkappa(2) \neq 0. \end{cases}$$

Proof: D. C. Mayer, October 2005, see [2] The second p-class group of a number field, Thm.4.2.

The TTT τ determines the TKT $\varkappa$ of densely populated sporadic groups

Theorem 7. Structures of Transfer Targets for $\mathrm{cc}(G) = 2$, $\mathrm{cl}(G) = 3$

TABLE 6. $\varkappa$ in dependence on τ for $p = 3$, $n = 5$, $k = 0$ (Isoclinism family Φ_6)

			Transfer Target Type τ					
TKT	$\varkappa$	ν	$\mathrm{Cl}_3(\mathrm{F}_3^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε
b.10	(0043)	2	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2
c.21	(0231)	1	$(3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
c.18	(0313)	1	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	1
D.10	(2241)	0	$(3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	1
G.19	(2143)	0	$(3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
H.4	(4443)	0	$(3,3,3)$	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	3
D.5	(4224)	0	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	$(9,3)$	2

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.2.4.

Theorem 8. Structures of Transfer Targets for $\mathrm{cc}(G) = 2$, $\mathrm{cl}(G) = 4$

TABLE 7. $\varkappa$ in dependence on τ for $p = 3$, $n = 6$, $k = 1$ (Isoclinism families $\Phi_{40}, \dots, \Phi_{43}$)

			Transfer Target Type τ					
TKT	$\varkappa$	ν	$\mathrm{Cl}_3(\mathrm{F}_3^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	$\mathrm{Cl}_3(N_4)$	ε
b.10	(0043)	2	$(9,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2
H.4	(4443)	0	$(9,3,3)$	$(3,3,3)$	$(3,3,3)$	$(9,3)$	$(3,3,3)$	3
G.19	(2143)	0	$(3,3,3,3)$	$(9,3)$	$(9,3)$	$(9,3)$	$(9,3)$	0
b.10	(0043)	2	$(3,3,3,3)$	$(9,3)$	$(9,3)$	$(3,3,3)$	$(3,3,3)$	2

Proof: D.C.Mayer, December 2009, [3] Principalisation algorithm via class group structure, Thm.2.5.

For $p = 2$, however, $\varkappa$ is not determined by τ.

Theorem 9. Structures of Transfer Targets for $p = 2$, $\mathrm{cc}(G) = 1$

TABLE 8. Different TKT's $\varkappa$ sharing the same TTT τ, for each $n \geq 4$

			Transfer Target Type τ				
TKT	$\varkappa$	ν	$\mathrm{Cl}_2(\mathrm{F}_2^1(K))$	$\mathrm{Cl}_3(N_1)$	$\mathrm{Cl}_3(N_2)$	$\mathrm{Cl}_3(N_3)$	ε
a.1	(000)	3	1	(2)	(2)	(2)	0
Q.5	(123)	0	(2)	(4)	(4)	(4)	0
d.8	(032)	1	(2^{n-2})	(2^{n-1})	$(2,2)$	$(2,2)$	2
Q.6	(132)	0	(2^{n-2})	(2^{n-1})	$(2,2)$	$(2,2)$	2
S.4	(232)	0	(2^{n-2})	(2^{n-1})	$(2,2)$	$(2,2)$	2

Proof: D. C. Mayer, April 2010, see [2] The second p-class group of a number field, Sec.9.

Section 3.

Second 3-Class Groups $G_3^2(K)$

of Quadratic Fields $K = \mathbb{Q}(\sqrt{D})$

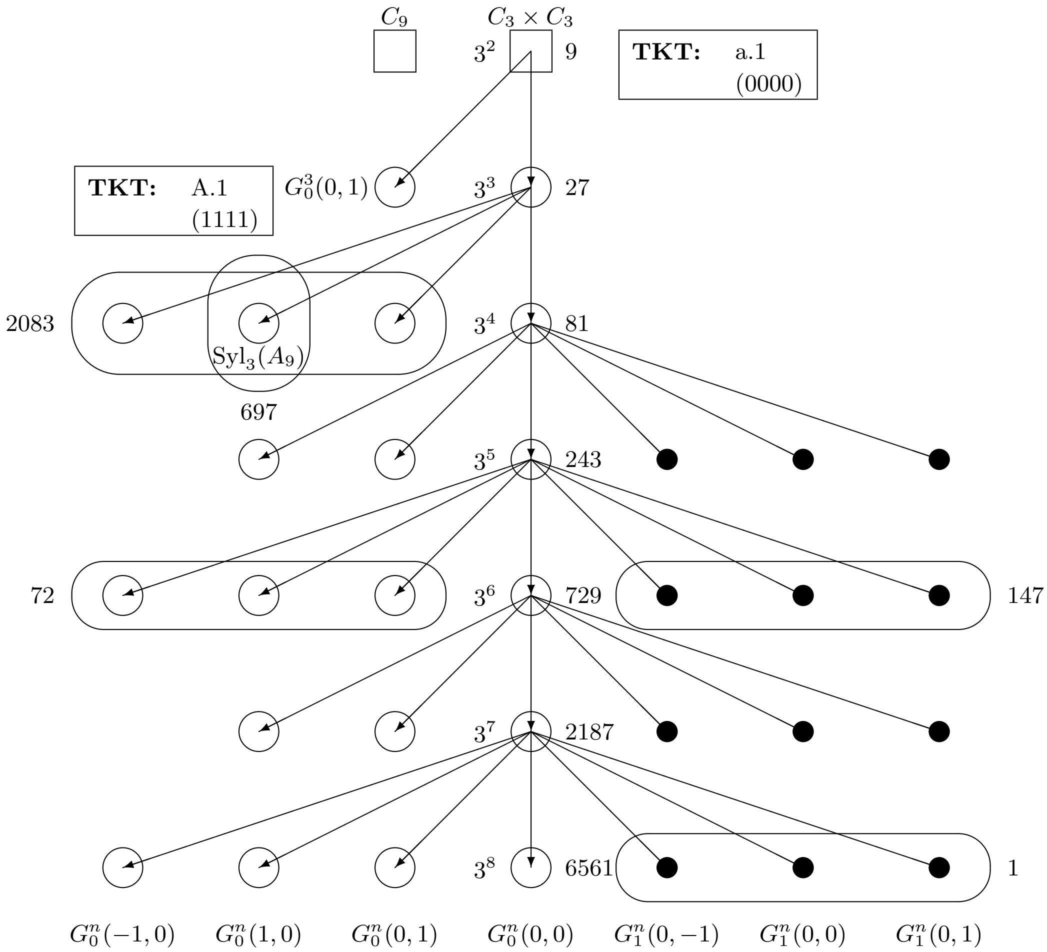
Methods for Determining the Group $G_3^2(K)$

The following table shows the fineness of resolution, i.e. the accuracy, in determining the position of $G_3^2(K)$ on the coclass graphs $\mathcal{G}(3,r)$, obtained by Scholz and Taussky's Classical Bottom-Up Algorithm, by our Recent Top-Down Algorithm, and by a combination of both algorithms, for each transfer kernel type (TKT) $\varkappa$. By a **family** we understand an infinite coclass family and by an m-**batch** we understand a multiplet of $m \geq 2$ immediate descendants of a common parent.

Table 9. Comparison of the Bottom-Up and Top-Down Algorithm

	Algorithm		
TKT	Bottom-Up	Combined	Top-Down
a.1	3 families	3-batch	3-batch
a.2	family	vertex	3-batch with a.3
a.3*	2 families with a.3	vertex	vertex
a.3	2 families with a.3*	vertex	2-batch with a.2
a.3↑	2 families	2-batch	3-batch with a.2
b.10	infinitely many families	6- or 9-batch	6- or 9-batch
c.18	main line	vertex	vertex
c.21	main line	vertex	vertex
d*.19	infinitely many main lines	2 vertices on different trees	5 vertices on different trees
d*.23	infinitely many main lines	vertex	5 vertices on different trees
d*.25	infinitely many main lines	2 vertices on different trees	5 vertices on different trees
d.19	infinitely many families	2-batch	5-batch with d.23,25
d.23	infinitely many families	vertex	5-batch with d.19,25
d.25	infinitely many families	2-batch	5-batch with d.19,23
A.1	impossible	impossible	impossible
D.5	vertex	vertex	vertex
D.10	vertex	vertex	vertex
G.19	infinitely many families	2-batch	2-batch
H.4	infinitely many families	4-batch	4-batch
E.6	family	vertex	3-batch with E.14
E.14	2 families	2-batch	3-batch with E.6
E.8	family	vertex	3-batch with E.9
E.9	2 families	2-batch	3-batch with E.8
G.16	infinitely many families	two 4-batches	two 4-batches
H.4↑	infinitely many families	two 4-batches	two 4-batches
F.7	infinitely many families	3-batch	13-batch with F.11,12,13
F.11	infinitely many families	2-batch	13-batch with F.7,12,13
F.12	infinitely many families	4-batch	13-batch with F.7,11,13
F.13	infinitely many families	4-batch	13-batch with F.7,11,12
G.16r	infinitely many families	4-batch	18 vertices with G.19r,H.4r
G.19r	infinitely many families	two 3-batches	18 vertices with G.16r,H.4r
H.4r	infinitely many families	two 4-batches	18 vertices with G.16r,G.19r
G.16i	infinitely many families	3-batch	12 vertices with G.19i,H.4i
G.19i	infinitely many families	4-batch	12 vertices with G.16i,H.4i
H.4i	infinitely many families	5-batch	12 vertices with G.16i,G.19i
F.7↑	infinitely many families	4 vertices on different trees	24 vertices on different trees
F.11↑	infinitely many families	4 vertices on different trees	24 vertices on different trees
F.12↑	infinitely many families	8 vertices on different trees	24 vertices on different trees
F.13↑	infinitely many families	8 vertices on different trees	24 vertices on different trees

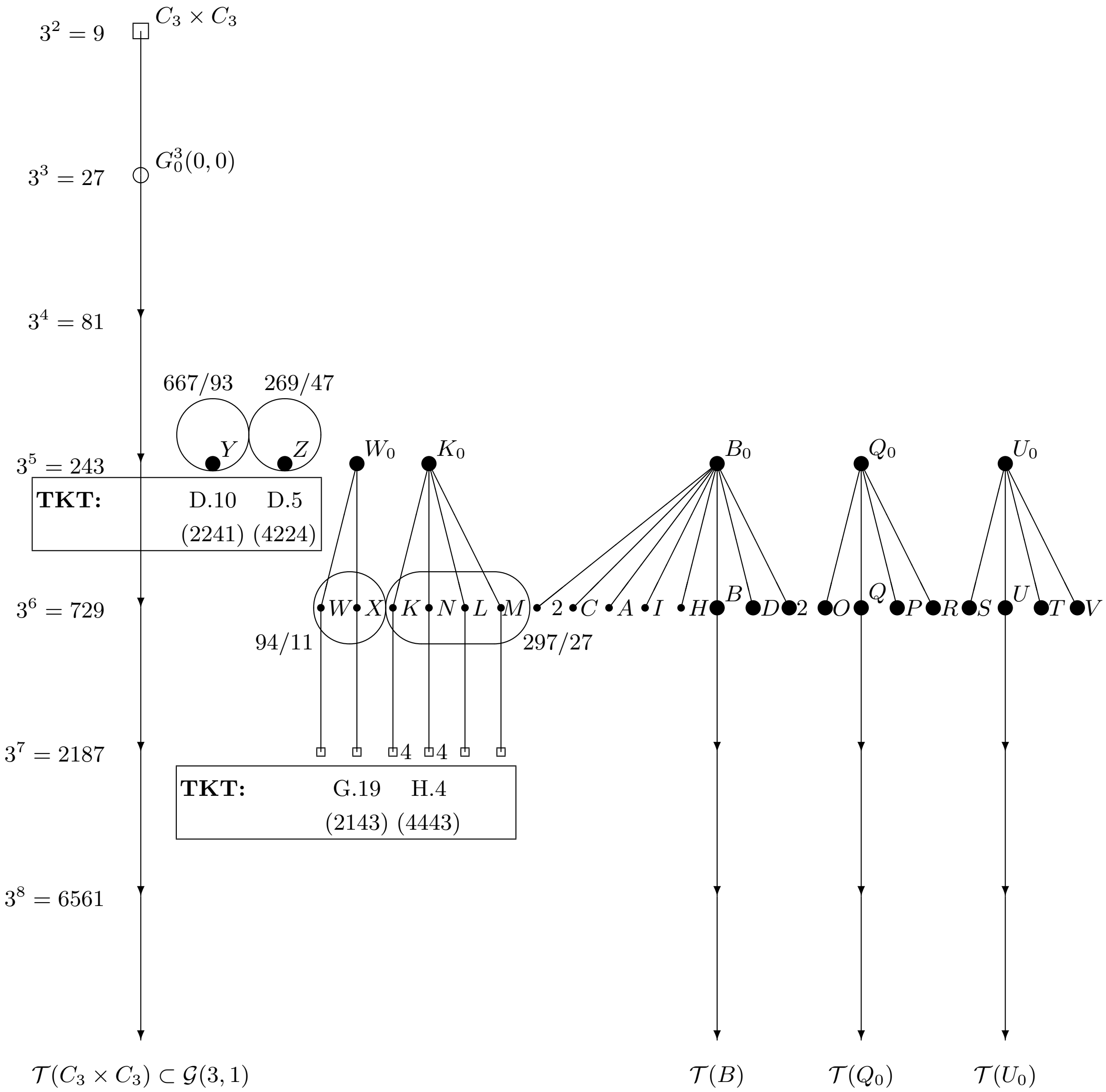
Distribution on the Coclass Graph $\mathcal{G}(3,1)$



TKT:	a.3	a.3	a.2	a.1	a.1	a.1	a.1
	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

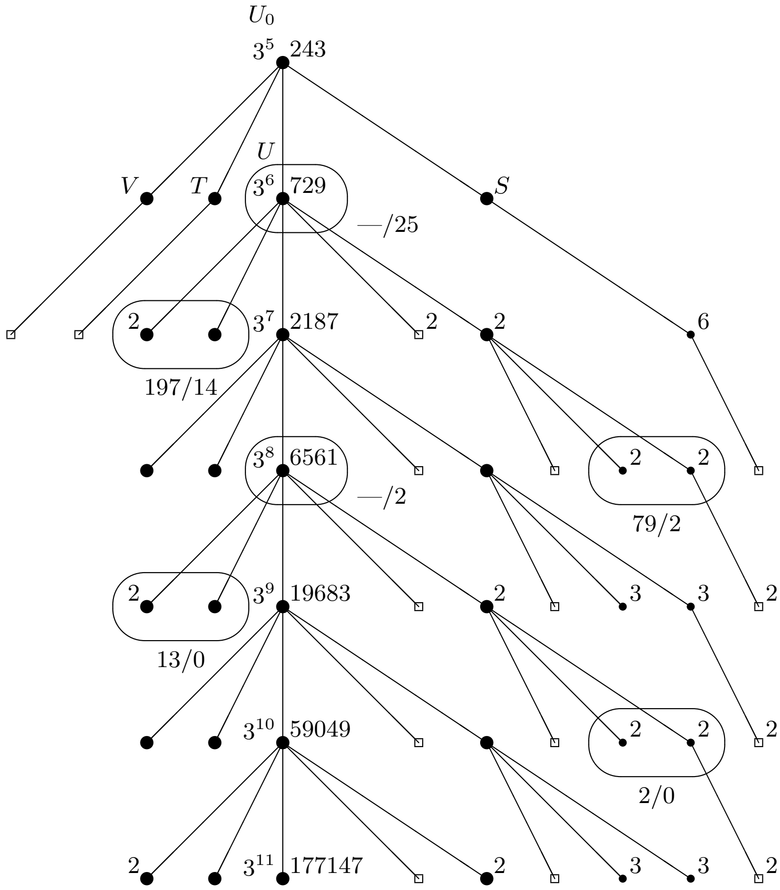
- $G_3^2(K) \in \mathcal{G}(3,1)$ for 2303 (**89.4**%) of the 2576 discriminants $0 < D < 10^7$.
- Since the Transfer Kernel Types $\varkappa$ of all coclass families are **total** with $\nu \geq 3$, there occur $G_3^2(K)$ of **real quadratic fields** $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ only.
- Due to the **Selection Rule** (Theorem 4), the $G_3^2(K)$ are distributed on **odd branches** only. This is a restriction from 13 to 7 coclass families:
  the main line $(G_0^n(0,0))_{n\geq 3}$ for odd n, and the others for even n,
  $(G_0^n(0,1))_{n\geq 4}$, $(G_0^n(1,0))_{n\geq 4}$, $(G_0^n(-1,0))_{n\geq 4}$, with invariant $k = 0$,
  $(G_1^n(0,0))_{n\geq 6}$, $(G_1^n(0,1))_{n\geq 6}$, $(G_1^n(0,-1))_{n\geq 6}$, with $k = 1$.
  Only one of these groups is **intrinsically sporadic**: $G_0^4(1,0) \simeq \mathrm{Syl}_3(A_9)$.
- **Open Problem:** It is unknown why there is no actual hit of the main line.

Distribution among Sporadic Groups of $\mathcal{G}(3,2)$



- $G_3^2(K) \in \mathcal{G}_0(3,2)$ for 1327 (**65.7**%) of the 2020 discriminants $-10^6 < D < 0$.
- $G_3^2(K) \in \mathcal{G}_0(3,2)$ for 178 (**6.9**%) of the 2576 discriminants $0 < D < 10^7$.
- Isolated and sporadic groups:
  $Y = G_0^{4,5}(0,0,-1,1)$, $Z = G_0^{4,5}(1,1,-1,1)$; $W_0 = G_0^{4,5}(-1,0,0,1)$, $K_0 = G_0^{4,5}(1,1,1,1)$.
- It is **unknown** why there is no actual hit of the vertices W_0 and K_0.

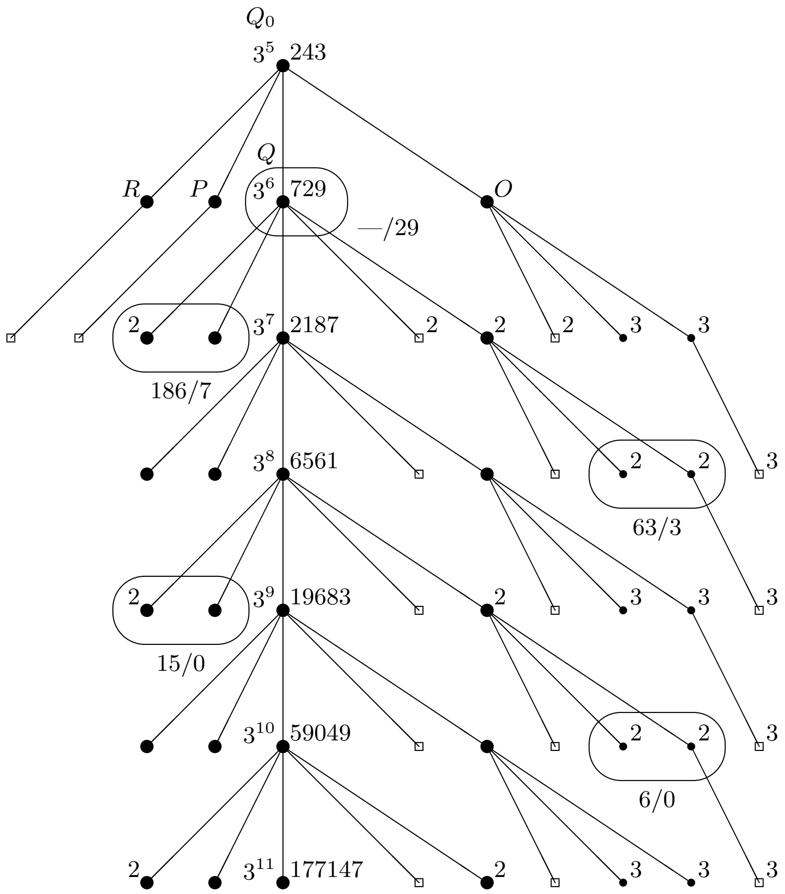
Distribution on the Coclass 2 Tree $\mathcal{T}(U_0)$



TKT:	E.9	E.8	c.21	G.16	G.16	G.16
	(2231)	(1231)	(0231)	(4231)	(4231)	(4231)

- $G_3^2(K) \in \mathcal{T}(U_0)$ for 291 (**14.4**%) of the 2020 discriminants $-10^6 < D < 0$.
- $G_3^2(K) \in \mathcal{T}(U_0)$ for 43 (**1.7**%) of the 2576 discriminants $0 < D < 10^7$.
- Since the Transfer Kernel Type $\varkappa = (0231)$ of the main line (c.21) is **total** with $\varkappa(1) = 0$, there only occur $G_3^2(K)$ of **real quadratic fields** $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ on the main line.
- Due to the **Selection Rule** (Theorem 6), the $G_3^2(K)$ are distributed on **even branches** only. This is a restriction from 10 to 6 metabelian coclass families with invariant $k = 0$: the main line $(G_0^{n-1,n}(0,0,0,1))_{n\geq 6}$ for even n, and the others for odd n, $(G_0^{n-1,n}(0,0,-1,1))_{n\geq 7}$, $(G_0^{n-1,n}(0,0,1,1))_{n\geq 7}$, $(G_0^{n-1,n}(1,0,-1,1))_{n\geq 7}$, $(G_0^{n-1,n}(-1,0,0,1))_{n\geq 7}$, $(G_0^{n-1,n}(1,0,0,1))_{n\geq 7}$.
- It is **unknown** why there is no actual hit of the vertices $(G_0^{n-1,n}(\pm 1,0,0,1))_{n\geq 7}$.

Distribution on the Coclass 2 Tree $\mathcal{T}(Q_0)$



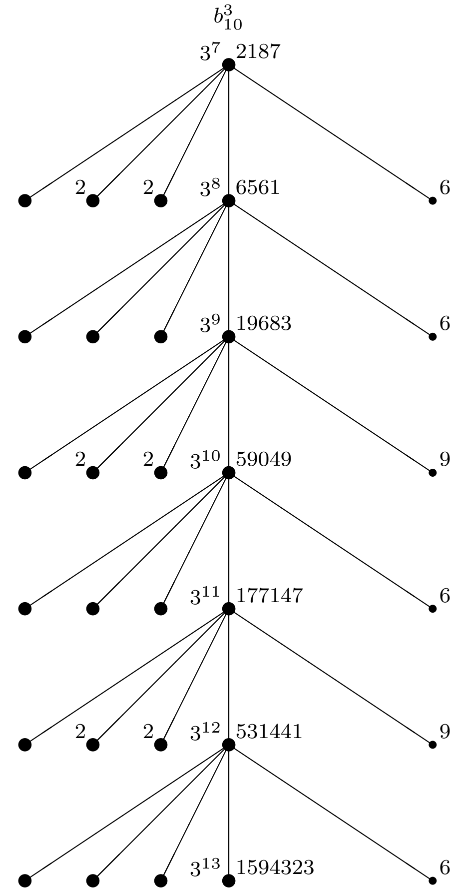
TKT: E.14	E.6	c.18	H.4	H.4	H.4
(2313)	(1313)	(0313)	(3313)	(3313)	(3313)

- $G_3^2(K) \in \mathcal{T}(Q_0)$ for 270 (**13.4**%) of the 2020 discriminants $-10^6 < D < 0$.
- $G_3^2(K) \in \mathcal{T}(Q_0)$ for 39 (**1.5**%) of the 2576 discriminants $0 < D < 10^7$.
- Since the Transfer Kernel Type $\varkappa = (0313)$ of the main line (c.18) is **total** with $\varkappa(1) = 0$, there only occur $G_3^2(K)$ of **real quadratic fields** $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ on the main line.
- Due to the **Selection Rule** (Theorem 6), the $G_3^2(K)$ are distributed on **even branches** only. This is a restriction from 10 to 6 metabelian coclass families with invariant $k = 0$: the main line $(G_0^{n-1,n}(0,-1,0,1))_{n\geq 6}$ for even n, and the others for odd n, $(G_0^{n-1,n}(0,-1,-1,1))_{n\geq 7}$, $(G_0^{n-1,n}(0,-1,1,1))_{n\geq 7}$, $(G_0^{n-1,n}(1,-1,1,1))_{n\geq 7}$, $(G_0^{n-1,n}(-1,-1,1,1))_{n\geq 7}$, $(G_0^{n-1,n}(1,-1,-1,1))_{n\geq 7}$.
- It is **unknown** why there is no actual hit of the vertices $(G_0^{n-1,n}(\pm 1,-1,\mp 1,1))_{n\geq 7}$.

Metabelian Skeleton of Coclass Tree $\mathcal{T}(b^3_{10}) \subset \mathcal{G}(3,3)$

Structure Theorem.
$G \in \mathcal{G}(3,3) \Longrightarrow \varepsilon(G) = 2.$

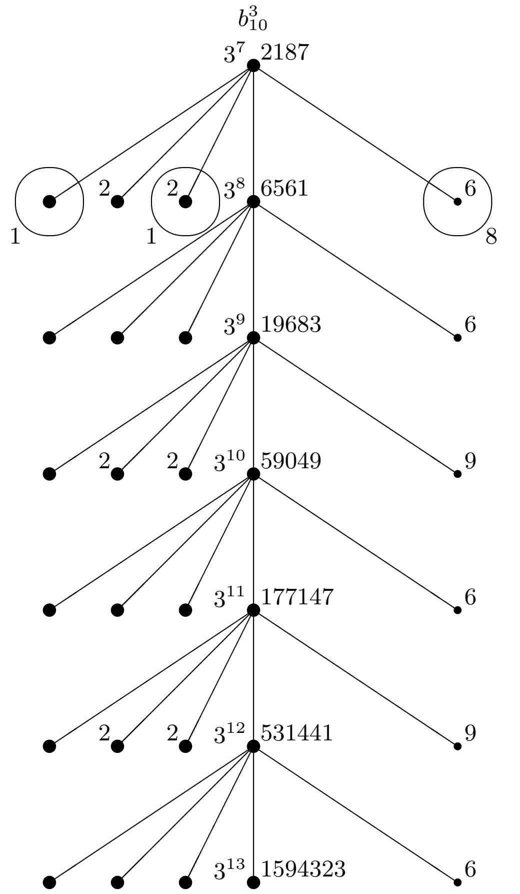
Selection Rule.
$K = \mathbb{Q}(\sqrt{D}),\ G^2_3(K) \in \mathcal{G}(3,3) \Longrightarrow D > 0,\ G^2_3(K) \in \mathcal{T}(b^3_{10}).$



TKT:	d.23	d.25	d.19	b.10	b.10
	(1043)	(2043)	(4043)	(0043)	(0043)

Main line:	$(G_0^{n-2,n}(0,0,0,0))_{n\ge 7}$, with root $b^3_{10} = G_0^{5,7}(0,0,0,0)$.
10 coclass families:	metabelian with invariant $k=0$: two for even n only, $(G_0^{n-2,n}(0,0,-1,0))_{n\ge 8}$, $(G_0^{n-2,n}(1,0,-1,0))_{n\ge 8}$, and the others for n either even or odd, $(G_0^{n-1,n}(0,0,0,0))_{n\ge 7}$, $(G_0^{n-2,n}(1,0,0,0))_{n\ge 8}$, $(G_0^{n-2,n}(0,0,1,0))_{n\ge 8}$, $(G_0^{n-2,n}(1,0,1,0))_{n\ge 8}$.
Periodicity of branches:	$\mathcal{B}_1(G_0^{n-2,n}(0,0,0,0)) \simeq \mathcal{B}_1(G_0^{n,n+2}(0,0,0,0))$ for $n \ge 8$.
Maximal depth:	$d = 1$ (restricted to the metabelian skeleton).
Period length:	$\ell = 2$.

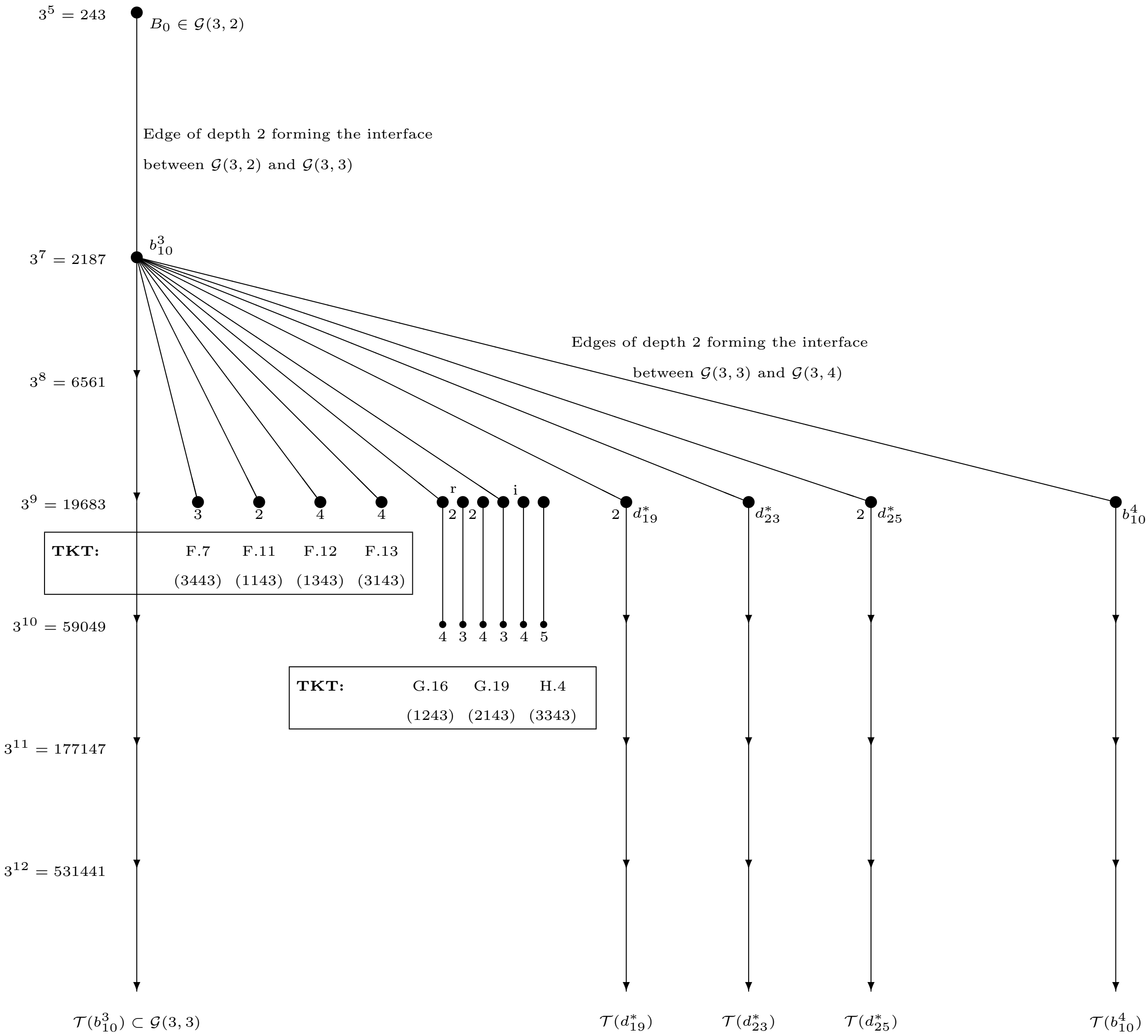
Distribution on the Coclass 3 Tree $\mathcal{T}(b_{10}^3)$



TKT:	d.23	d.25	d.19	b.10	b.10
	(1043)	(2043)	(4043)	(0043)	(0043)

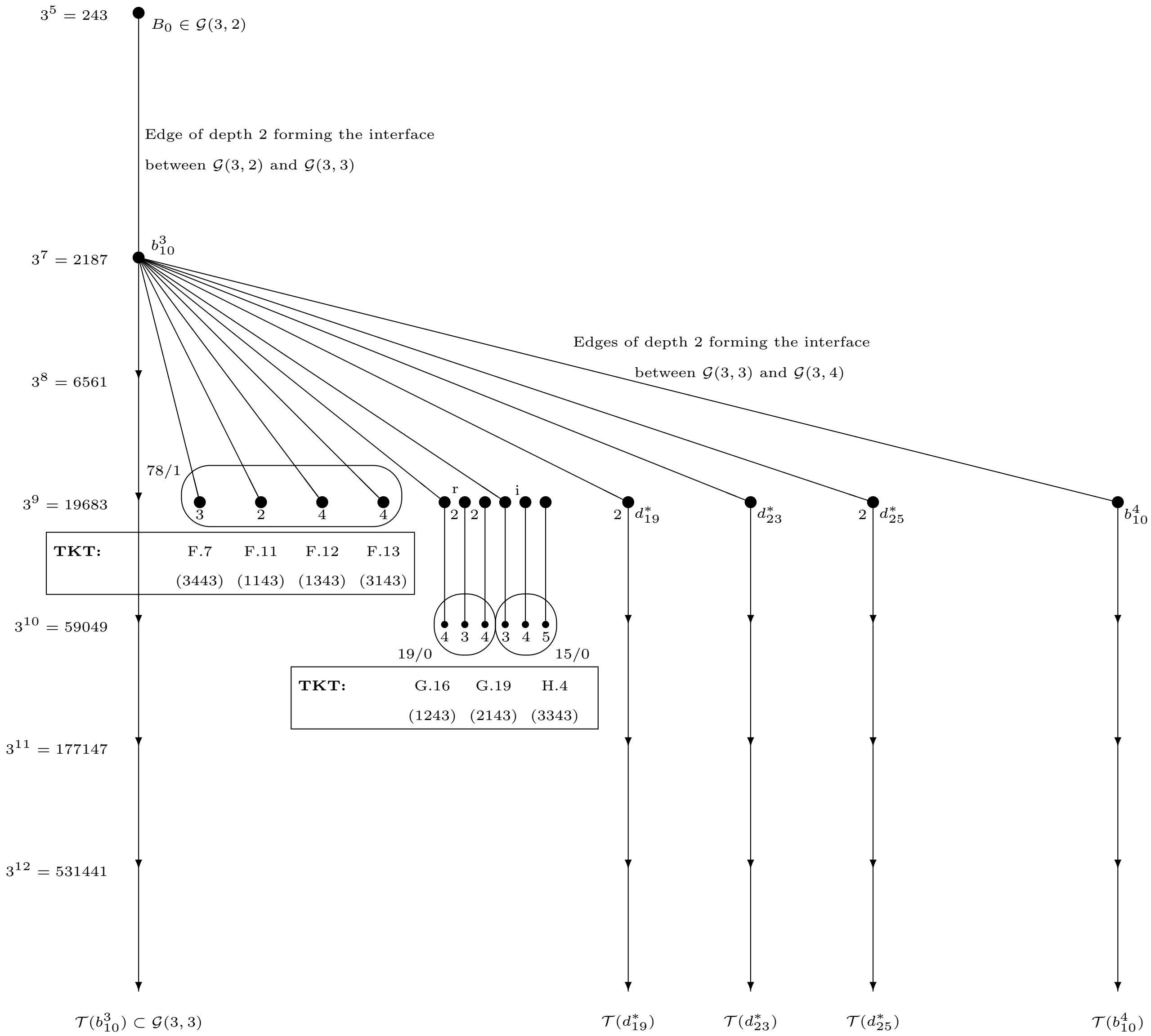
- $G_3^2(K) \in \mathcal{G}(3,3)$ for 10 (**0.4**%) of the 2576 discriminants $0 < D < 10^7$.
- Since the Transfer Kernel Types $\varkappa$ of all coclass families are **total** with $\varkappa(2) = 0$, there occur $G_3^2(K)$ of **real quadratic fields** $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ only.
- Due to the **Selection Rule** (Theorem 6), the $G_3^2(K)$ are distributed on **odd branches** only.
- It is **unknown** why there is no actual hit of the main line $(G_0^{n-2,n}(0,0,0,0))_{n\geq 7}$.

Top of Coclass Graph $\mathcal{G}(3,4)$ restricted to Groups with Abelianization of Type $(3,3)$



6 roots of coclass trees with metabelian main lines:
$b_{10}^4 = G_0^{6,9}(0,0,0,0)$, $d_{19}^* = G_0^{6,9}(0,1,0,1)$, $d_{19}^*(-) = G_0^{6,9}(0,-1,0,1)$,
$d_{23}^* = G_0^{6,9}(0,0,0,1)$, $d_{25}^* = G_0^{6,9}(0,1,0,0)$, $d_{25}^*(-) = G_0^{6,9}(0,-1,0,0)$.
51 isolated and sporadic groups.

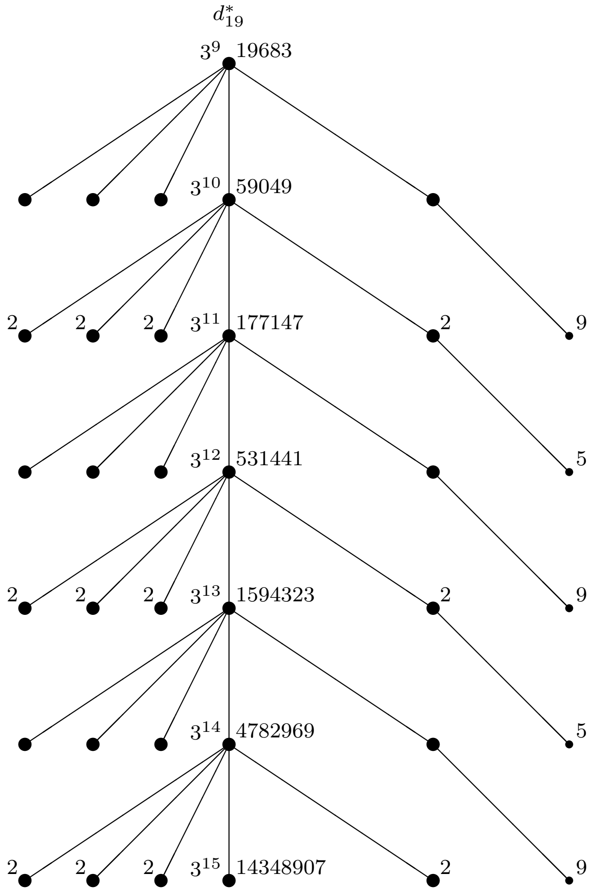
Distribution among Sporadic Groups of $\mathcal{G}(3,4)$



- $G_3^2(K) \in \mathcal{G}_0(3,4)$ for 112 (**5.5**%) of the 2020 discriminants $-10^6 < D < 0$.
- $G_3^2(K) \in \mathcal{G}_0(3,4)$ for 1 of the 2576 discriminants $0 < D < 10^7$.
- It is **unknown** why there is no actual hit of the roots of the sporadic trees.

Metabelian Skeleton of Coclass Tree $\mathcal{T}(d_{19}^*) \subset \mathcal{G}(3,4)$

Structure Theorem.
$G \in \mathcal{G}(3,4) \Longrightarrow \varepsilon(G) = 2.$

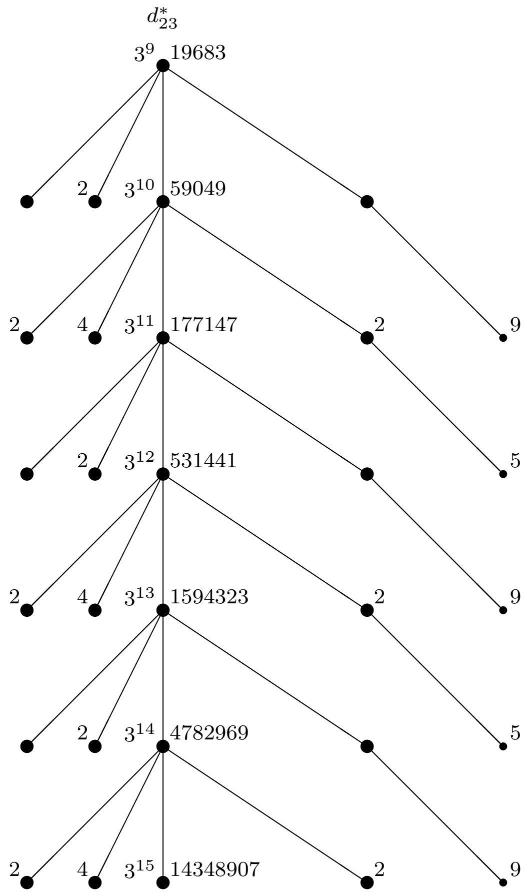


TKT:	F.7	F.12	F.13	d*.19	H.4	H.4
	(3443)	(1343)	(3143)	(0443)	(3343)	(3343)

Main line:	$(G_0^{n-3,n}(0,1,0,1))_{n\ge 9}$, with root $d_{19}^* = G_0^{6,9}(0,1,0,1)$.
14 coclass families:	metabelian with invariant $k=0$: four for odd n only, $(G_0^{n-3,n}(-1,1,1,1))_{n\ge 11}$, $(G_0^{n-3,n}(-1,1,0,1))_{n\ge 11}$, $(G_0^{n-3,n}(0,1,-1,1))_{n\ge 11}$, $(G_0^{n-3,n}(-1,1,-1,1))_{n\ge 11}$, and the others for n either even or odd, $(G_0^{n-3,n}(0,1,0,1))_{n\ge 9}$, $(G_0^{n-3,n}(1,1,-1,1))_{n\ge 10}$, $(G_0^{n-3,n}(1,1,0,1))_{n\ge 10}$, $(G_0^{n-3,n}(0,1,1,1))_{n\ge 10}$, $(G_0^{n-3,n}(1,1,1,1))_{n\ge 10}$.
Periodicity of branches:	$\mathcal{B}_2(G_0^{n-3,n}(0,1,0,1)) \simeq \mathcal{B}_2(G_0^{n-1,n+2}(0,1,0,1))$ for $n \ge 9$.
Maximal depth:	$d = 2$ (restricted to the metabelian skeleton).
Period length:	$\ell = 2$.

Metabelian Skeleton of Coclass Tree $\mathcal{T}(d_{23}^{*}) \subset \mathcal{G}(3,4)$

Structure Theorem.
$G \in \mathcal{G}(3,4) \Longrightarrow \varepsilon(G) = 2.$

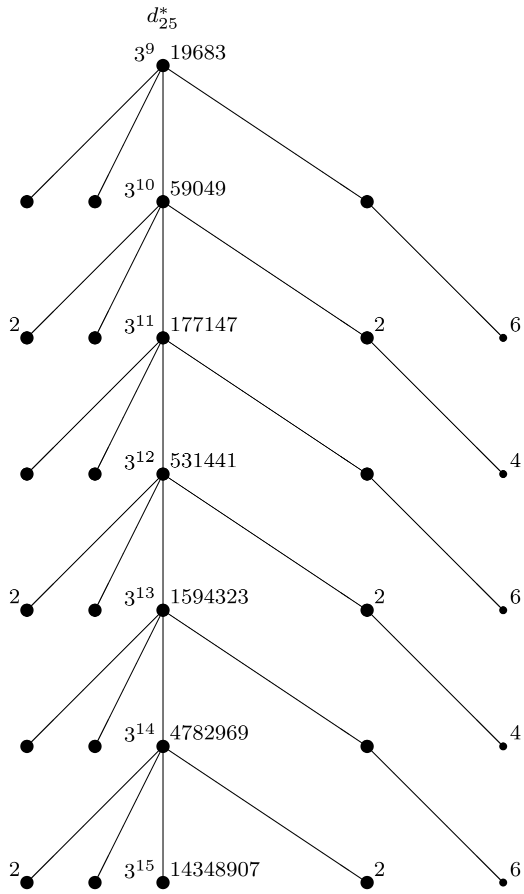


TKT:	F.11	F.12	d*.23	G.16	G.16
	(1143)	(1343)	(0243)	(1243)	(1243)

Main line:	$(G_0^{n-3,n}(0,0,0,1))_{n\geq 9}$, with root $d_{23}^{*} = G_0^{6,9}(0,0,0,1)$.
14 coclass families:	metabelian with invariant $k = 0$: four for odd n only, $(G_0^{n-3,n}(0,0,-1,1))_{n\geq 11}$, $(G_0^{n-3,n}(-1,0,1,1))_{n\geq 11}$, $(G_0^{n-3,n}(-1,0,-1,1))_{n\geq 11}$, $(G_0^{n-3,n}(-1,0,0,1))_{n\geq 11}$, and the others for n either even or odd, $(G_0^{n-3,n}(0,0,0,1))_{n\geq 9}$, $(G_0^{n-3,n}(0,0,1,1))_{n\geq 10}$, $(G_0^{n-3,n}(1,0,1,1))_{n\geq 10}$, $(G_0^{n-3,n}(1,0,-1,1))_{n\geq 10}$, $(G_0^{n-3,n}(1,0,0,1))_{n\geq 10}$.
Periodicity of branches:	$\mathcal{B}_2(G_0^{n-3,n}(0,0,0,1)) \simeq \mathcal{B}_2(G_0^{n-1,n+2}(0,0,0,1))$ for $n \geq 9$.
Maximal depth:	$d = 2$ (restricted to the metabelian skeleton).
Period length:	$\ell = 2$.

Metabelian Skeleton of Coclass Tree $\mathcal{T}(d^*_{25}) \subset \mathcal{G}(3,4)$

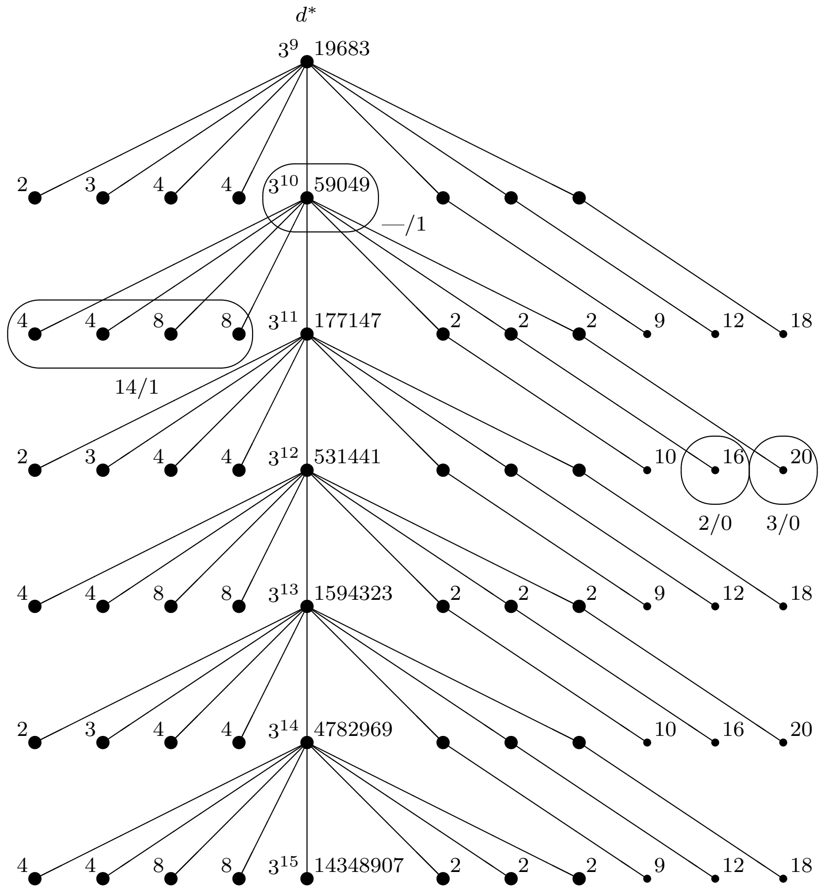
Structure Theorem.
$G \in \mathcal{G}(3,4) \Longrightarrow \varepsilon(G) = 2.$



TKT:	F.13	F.11	d*.25	G.19	G.19
	(3143)	(1143)	(0143)	(2143)	(2143)

Main line:	$(G_0^{n-3,n}(0,1,0,0))_{n\geq 9}$, with root $d^*_{25} = G_0^{6,9}(0,1,0,0)$.
10 coclass families:	metabelian with invariant $k=0$: two for odd n only, $(G_0^{n-3,n}(1,1,-1,0))_{n\geq 11}$, $(G_0^{n-3,n}(0,1,-1,0))_{n\geq 11}$, and the others for n either even or odd, $(G_0^{n-3,n}(0,1,0,0))_{n\geq 9}$, $(G_0^{n-3,n}(1,1,0,0))_{n\geq 10}$, $(G_0^{n-3,n}(1,1,1,0))_{n\geq 10}$, $(G_0^{n-3,n}(0,1,1,0))_{n\geq 10}$.
Periodicity of branches:	$\mathcal{B}_2(G_0^{n-3,n}(0,1,0,0)) \simeq \mathcal{B}_2(G_0^{n-1,n+2}(0,1,0,0))$ for $n \geq 9$.
Maximal depth:	$d = 2$ (restricted to the metabelian skeleton).
Period length:	$\ell = 2$.

Distribution on the Accumulated Coclass 4 Tree



TKT:	F.7	F.11	F.12	F.13	d*	G.16	G.19	H.4
	(3143)	(1143)	(3143)	(1143)		(1243)	(2143)	(3343)

- $G_3^2(K) \in \mathcal{G}(3,4) \setminus \mathcal{G}_0(3,4)$ for 19 (**0.9**%) of the 2020 discriminants $-10^6 < D < 0$.
- $G_3^2(K) \in \mathcal{G}(3,4) \setminus \mathcal{G}_0(3,4)$ for 2 of the 2576 discriminants $0 < D < 10^7$.
- The accumulated main line d* contains 2 main lines of type d_{19}^*, a single main line of type d_{23}^*, and 2 main lines of type d_{25}^*.
- Due to the **Selection Rule** (Theorem 6), the $G_3^2(K)$ are distributed on **even branches** only.
- It is **unknown** why there is no actual hit of the parents of vertices at depth 2 with invariant $k = 1$.