Number Fields Sharing a Common Discriminant

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Introduction

In this lecture we present improved theoretical foundations admitting the statement of new formulas for the **number**

$$m_p(d, c)$$

of non-isomorphic dihedral fields $N|\mathbb{Q}$ of degree $2p$, $p \geq 3$ odd prime,

sharing a common conductor c

over their common quadratic subfield K with discriminant d.

$m_p(d, c)$ is called the *p-multiplicity* of c with respect to d.

FIGURE 1. **Multiplet** $(N_i)_{1\leq i\leq m}$ sharing a common conductor c

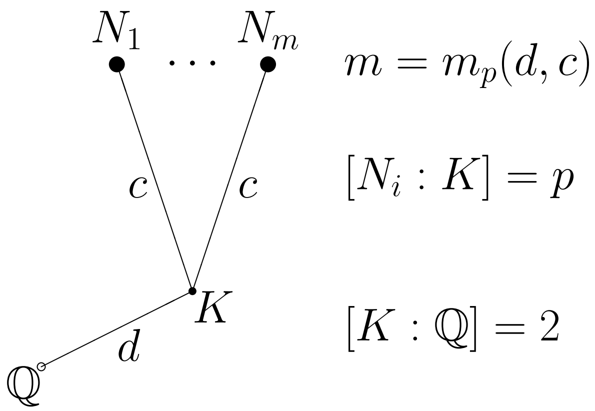


FIGURE 2. p-**ring class field** $\mathrm{F}_{c,p}(K)$ modulo c of K

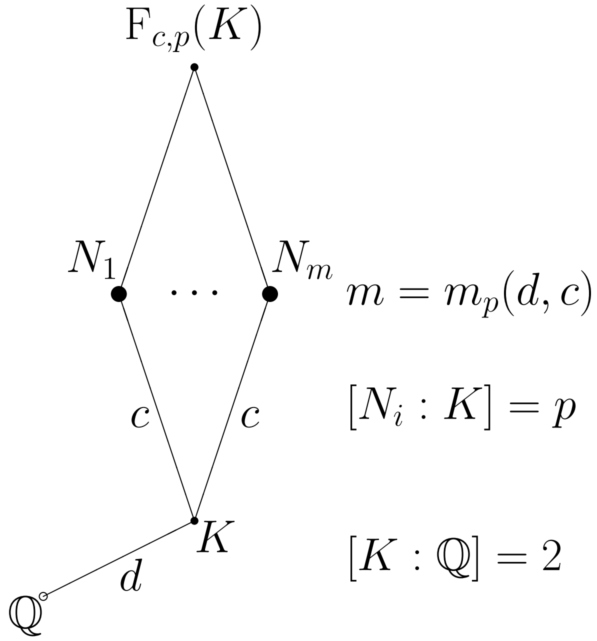
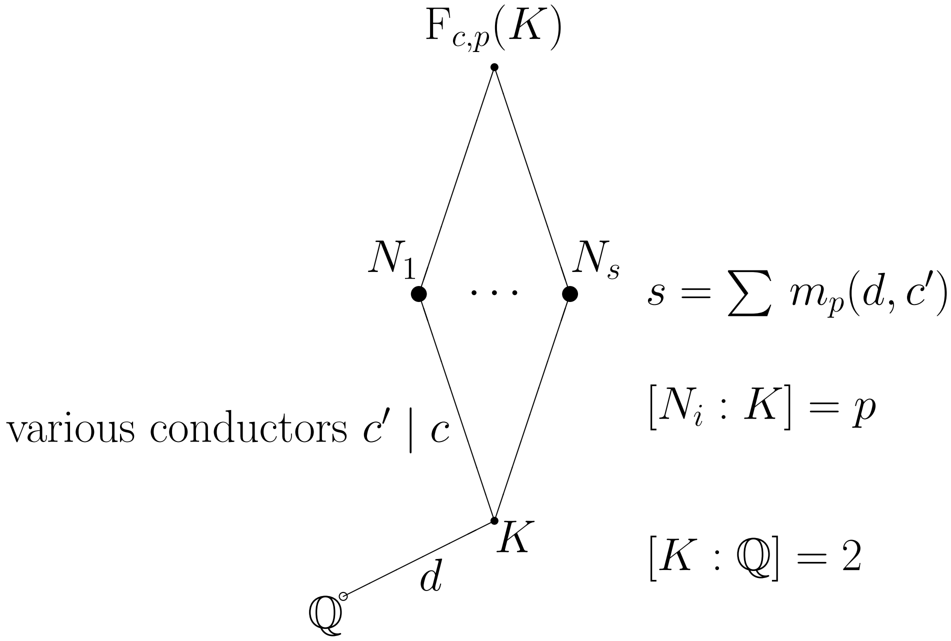


FIGURE 3. p-ring class field $\mathrm{F}_{c,p}(K)$ as **inhomogeneous split extension**

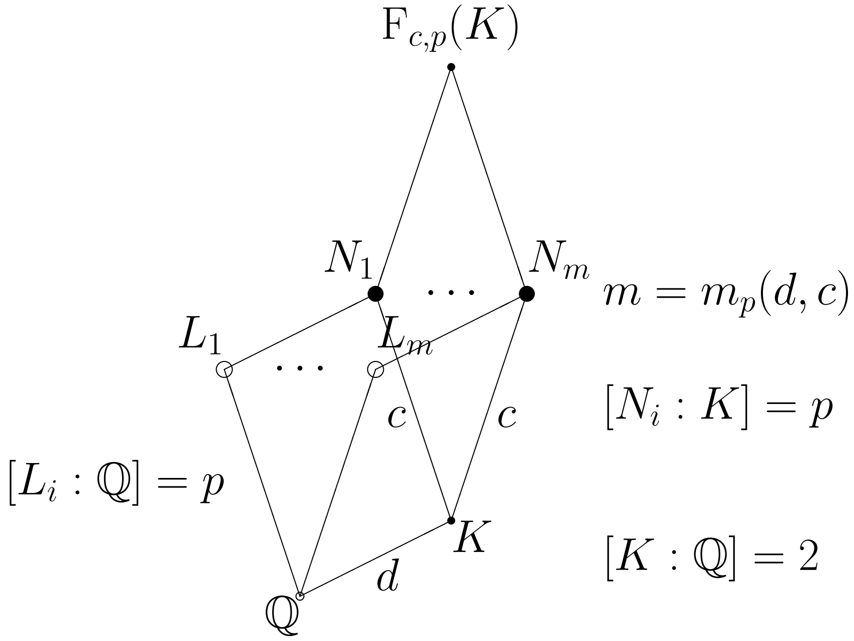


Additive multiplicity formula

$$s = \sum_{c' | c} m_p(d, c') = \frac{p^{\varrho_{c,p}} - 1}{p - 1}, \text{ where}$$

$$\varrho_{c,p} = \varrho_p + t + w - \delta_p(c).$$

FIGURE 4. Families of p conjugate **non-Galois subfields** $(L_i^{(j)})_{0\le j<p}$, $1 \le i \le m$



Fundamental and composite discriminants

- Fundamental discriminant of the **quadratic base field** K (squarefree outside of 2):
$$\mathrm{d}(K) = d$$
- Composite discriminant of the **dihedral field** N with conductor c over K:
$$\mathrm{d}(N) = c^{2(p-1)} \cdot d^p$$
- Composite discriminant of a **non-Galois subfield** L of N of degree p (one of p conjugate fields):
$$\mathrm{d}(L) = c^{p-1} \cdot d^{(p-1)/2} = \left(c^2 \cdot d\right)^{(p-1)/2}$$

1. Historical remarks concerning unramified extensions

Theorem 1.1. Let p be an odd prime and K be a quadratic field of discriminant d and p-class rank $\varrho = \varrho_p$. Then the number of *unramified* non-isomorphic cyclic extensions $N|K$ of degree p, with conductor $c = 1$, is given by

$$m_p(d, 1) = \frac{p^\varrho - 1}{p - 1} \qquad (0.0)$$

Example 1.1. The first examples of non-isomorphic number fields sharing a common discriminant were given around 1930 by Scholz in the form of quartets of unramified cyclic cubic extensions $N_1, \ldots, N_4$ of complex quadratic fields K of 3-class rank two. For $p = 3$, $\varrho_3 = 2$, formula (0.0) yields 3-multiplicity $m_3(d, 1) = \frac{3^2-1}{3-1} = 4$. Minimal absolute discriminant is $d = -3\,299$ and further examples are $d \in \{-3\,896, -4\,027, -9\,748\}$. Later, there were found real quadratic fields having similar properties, starting with $d = 32\,009$.

1.1. **Disproof of Scholz and Taussky's claim about $d = -9748$.** Let p be an odd prime and K be a number field with p-class group $\mathrm{Cl}_p(K)$ of type (p, p). Denote by $\ell_p(K)$ the length of the Hilbert p-class field tower of K and by $G = \mathrm{G}_p^2(K) = \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ the Galois group of the second Hilbert p-class field $\mathrm{F}_p^2(K)$ of K. G is a 2-generated metabelian p-group $G = \langle x, y\rangle$ with main commutator $s_2 = [y, x]$ and associated annihilator ideal $\mathfrak{A} = \{f(X, Y) \in \mathbb{Z}[X, Y] \mid s_2^{f(x-1,y-1)} = 1\}$. Let $(H_i)_{1\le i\le p+1}$ denote the maximal subgroups of G and $T_i : G/G' \to H_i/H_i'$ the corresponding transfers. The family $\varkappa(G) = (\ker(T_i))_{1\le i\le p+1}$ is called transfer kernel type (TKT) of G and $\tau(G) = (H_i/H_i'))_{1\le i\le p+1}$ is called transfer target type (TTT) of G.

Theorem 1.2. Let $p = 3$ and let K be a complex quadratic field. Then the following statements are equivalent.

(1) The annihilator of G is $\mathfrak{A} = \mathfrak{X}_\alpha$ for some $\alpha \ge 3$, where $\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3\rangle$.

(2) The TKT of G belongs to section E, that is

$$\varkappa(G) \in \{(1, 3, 1, 3), (2, 3, 1, 3), (1, 2, 3, 1), (2, 2, 3, 1)\}.$$

Theorem 1.3. Let $p = 3$, then the following statements are equivalent for any number field K.

(1) The TKT and TTT of G are given by $\varkappa(G) = (2, 2, 3, 1)$ and $\tau(G) = ((9, 27), (3, 9)^3)$, where the power denotes iteration.

(2) G is isomorphic to one of the groups $\langle 3^7, 302\rangle$ and $\langle 3^7, 306\rangle$ in the SmallGroups library.

Example 1.2. The complex quadratic field K with discriminant $d = -9\,748$ is characterized by the properties in Theorem 1.3.

Incorrect claims.

(1) (Scholz and Taussky, 1934)
$K = \mathbb{Q}(\sqrt{-9748})$ has 3-class field tower of length $\ell_3(K) = 2$.

(2) (Heider and Schmithals, 1982)
A complex quadratic field K with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and one of the four capitulation types in section E,

$$\varkappa(K) \in \{(1,3,1,3),(2,3,1,3),(1,2,3,1),(2,2,3,1)\},$$

has 3-class field tower of length $\ell_3(K) = 2$.

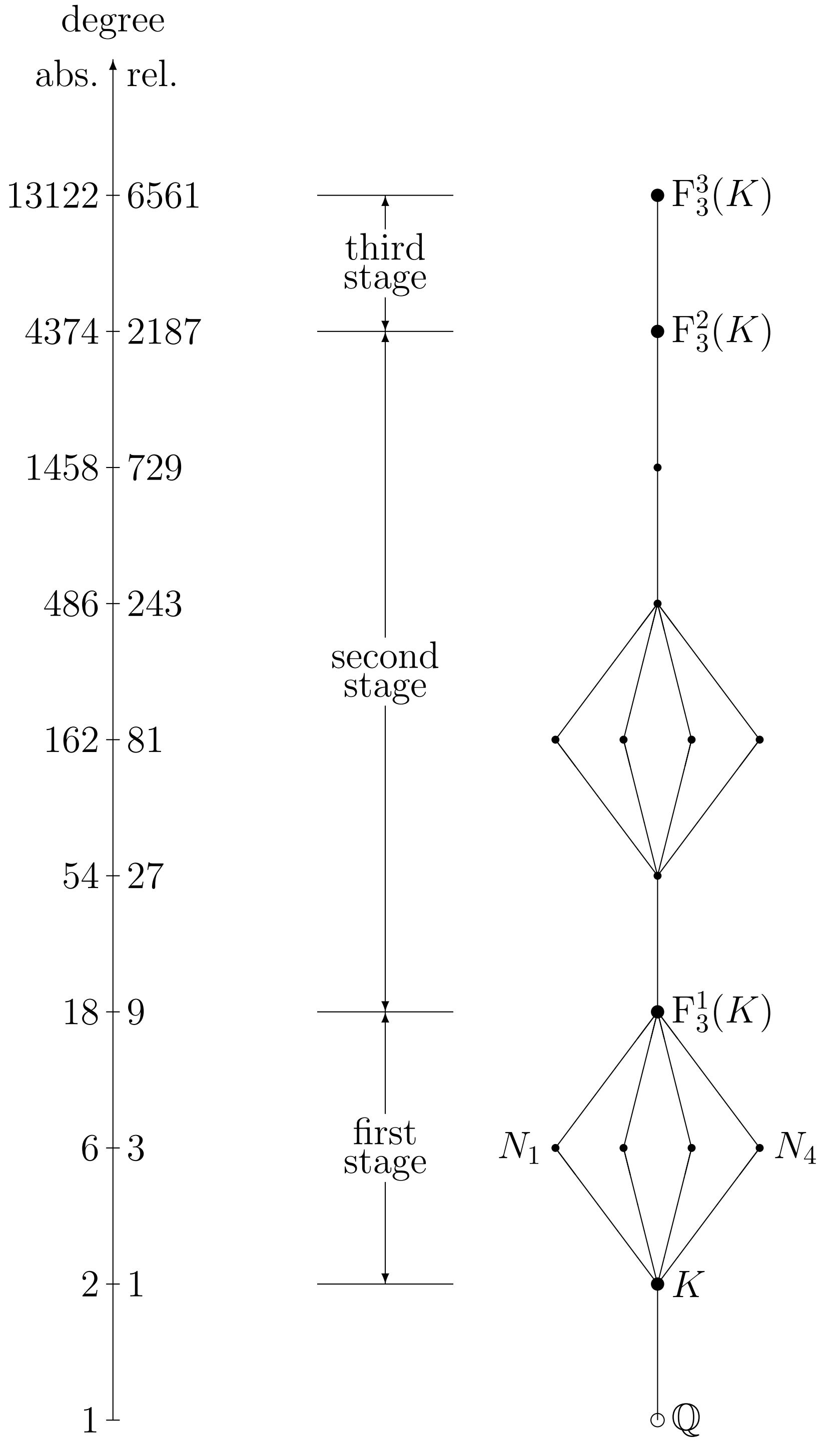
Caveat. (Brink, 1984, resp. Brink and Gold, 1987)
Let K be a complex quadratic field having 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$ and second 3-class group $G = \langle x, y\rangle$ with main commutator $s_2 = [y,x]$. If the annihilator ideal $\mathfrak{A}$ or 'symbolic order' of s_2 is given by $\mathfrak{A} = \langle X^\alpha, XY, Y^2, X^2+3X+3\rangle = \mathfrak{X}_\alpha$ for some $\alpha \geq 3$, then K has 3-class field tower of length $\ell_3(K) \geq 2$.

Theorem 1.4 (Boston, Bush, and Mayer, Aug. 24, 2012)**.**
Let K be a **complex quadratic** field with 3-class group $\mathrm{Cl}_3(K)$ **of type** $(3,3)$.
Suppose the Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field $\mathrm{F}_3^2(K)$ of K has
transfer kernel type (TKT) $\varkappa(G) = (2,2,3,1)$ and
transfer target type (TTT) $\tau(G) = \big((9,27),(3,9)^3\big)$, where the power denotes iteration.
Then K has 3-class field tower of length $\ell_3(K) = 3$.

Priority Claim. Theorem 1.4 implies that we, Boston, Bush, and Mayer, are definitely the first, who

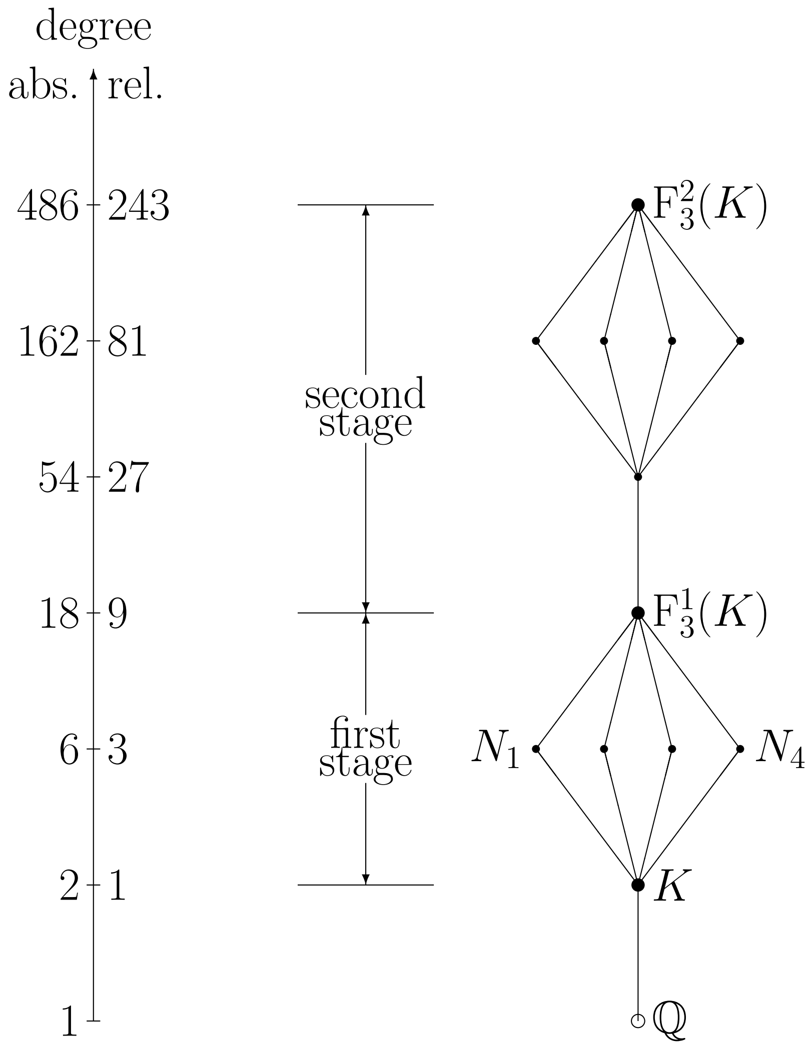
(1) have given a faultless disproof of Scholz and Taussky's erroneous statement about the length of the 3-tower of $\mathbb{Q}(\sqrt{-9\,748})$ on the last page of their famous 1934 paper,
(2) know an infinite family of complex quadratic fields K, in fact, a family having an asymptotic density of nearly 10%, whose 3-class field towers are of exact length $\ell_3(K) = 3$.

FIGURE 5. **3-class field tower** of $K = \mathbb{Q}(\sqrt{-9\,748})$



For comparison, Figure 1.1 shows the well-known two-stage 3-tower of the complex quadratic field $\mathbb{Q}(\sqrt{-4\,027})$. This fact was proved correctly in three independent ways by Scholz and Taussky, by Heider and Schmithals, and by Brink and Gold.

FIGURE 6. **3-class field tower** of $K = \mathbb{Q}(\sqrt{-4\,027})$



2. Quadratic Invariants

Definition 2.1. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with discriminant d, maximal order $\mathcal{O}$, ideal group $\mathcal{I}$, class group Cl, unit group U.

- For an arbitrary prime $p \geq 2$ let
$$I_p = \{\alpha \in K \mid \alpha\mathcal{O} = \mathfrak{i}^p \text{ for some } \mathfrak{i} \in \mathcal{I}\}$$
be the group of *principal pth powers of ideals* of K. I_p contains the product $U \cdot K^p$ as a subgroup, and $I_p/U \cdot K^p \simeq \mathrm{Cl}/\mathrm{Cl}^p$ is the p-elementary class group.
- For an integer $c \geq 1$, the subgroup
$$S_c = \{\alpha \in K^\times \mid \alpha \equiv 1 \pmod{c}\}$$
of the multiplicative group $K^\times = K \setminus \{0\}$ is called *ray* (Strahl) modulo c of K.
- For any system X of numbers or ideals of K, $X[c]$ denotes the elements of X coprime to c.
- The *ring* modulo c of K with generators
$$R_c = \mathbb{Q}[c] \cdot S_c$$
extends the ray S_c.

Definition 2.2. Let p be a prime, and
K be a quadratic field with p-class rank ϱ_p.
$f \in \mathbb{Z}$, $f \geq 1$, will denote any module of declaration.

- The vector space
$$V_p = I_p/K^p \simeq I_p[f]/K[f]^p$$
over $\mathbb{F}_p$ is called the vector space of
non-trivial principal pth powers of ideals of K.
We have $V_p \simeq (\mathrm{Cl}/\mathrm{Cl}^p) \times (U/U^p)$.
- Its dimension over $\mathbb{F}_p$ is called the
modified p-class rank $\sigma_p = \dim_{\mathbb{F}_p}(V_p)$ of K,
bounded by $\varrho_p \leq \sigma_p \leq \varrho_p + 1$, if $p \geq 3$.
- The subspace
$$V_p(c) \simeq I_p[f] \cap R_c \cdot K[f]^p/K[f]^p,$$
where $c \in \mathbb{Z}$, $c \geq 1$, $c \mid f$,
is called the *p-ring space* modulo c of K.
- Its codimension in V_p is called the *p-defect* of c,
$$\delta_p(c) = \mathrm{codim}(V_p(c)) = \dim_{\mathbb{F}_p}(V_p/V_p(c)).$$

Definition 2.3. A nearly squarefree integer $c \geq 1$ is called a *p-admissible conductor* over K if

$$c = p^e \cdot q_1 \cdots q_t$$

with $t \geq 0$, pairwise distinct primes $q_1, \ldots, q_t \in \mathbb{P} \setminus \{p\}$ such that $q_i \equiv \left(\frac{d}{q_i}\right) \pmod{p}$, and

$$e \in \begin{cases} \{0,2\} & \text{if } \left(\frac{d}{p}\right) = \pm 1, \\ \{0,1\} & \text{if } p \geq 5,\ p \mid d \\ & \text{or } p = 3,\ d \equiv +3 \pmod{9}, \\ \{0,1,2\} & \text{if } p = 3,\ d \equiv -3 \pmod{9}. \end{cases}$$

Formally, we write $c = q_1 \cdots q_\tau$,
where $\tau = t$ if $e = 0$
and $\tau = t+1$, $q_{t+1} = p^e$ if $e \geq 1$.

2.1. **Special types of conductors.**

Definition 2.4. Let $c = p^e \cdot q_1 \cdots q_t$ be p-admissible over K.

- With respect to the p-contribution,
  c is *irregular* if $p = 3$, $e = 2$, and $d \equiv -3 \pmod 9$,
  marked by an irregularity flag $\omega = 1$.
  Otherwise, c is *regular* and we put $\omega = 0$.
- With respect to the p-defect,
  c is *free* if $\delta_p(c) = 0$ (in the sense of restriction-free)
  and *restrictive* if $\delta_p(c) \geq 1$.
  An irregular c is *tamely irregular* if $\delta_3(3) = 0$,
  and *wildly irregular* if $\delta_3(3) = 1$.
- With respect to the p-multiplicity,
  c is *capable* if $m_p(d, c) \geq 1$ (gives rise to extensions),
  and *incapable* if $m_p(d, c) = 0$.

Theorem 2.1 (**Regulator Quotient Criterion**)**.** Let $p \geq 3$ be an odd prime and K be a **real quadratic field** with discriminant $d > 0$ and **class number not divisible by** p. Then K has modified p-class rank $\sigma_p = 1$ and the following characterization of p-admissible conductors q over K holds.

(1) A prime q, or the prime power $q = p^2$, is a **free regular** conductor over K, $\delta_p(q) = 0$, if and only if the quotient of the regulator $\mathrm{R}(q)$ of the suborder $\mathcal{O}_q$ by the regulator $\mathrm{R}(1)$ of the maximal order $\mathcal{O}$ of K satisfies the condition

$$v_p(\mathrm{R}(q)/\mathrm{R}(1)) < v_p(E(q)),$$

where the expression $E(q)$ in terms of q is given by

$$E(q) = \begin{cases} q-1 & \text{for } q \equiv +1 \pmod{p},\ \left(\frac{d}{q}\right) = +1, \\ q+1 & \text{for } q \equiv -1 \pmod{p},\ \left(\frac{d}{q}\right) = -1, \\ q & \text{for } q = p,\ p \mid d, \\ \sqrt{q} & \text{for } q = p^2,\ \left(\frac{d}{p}\right) = \pm 1. \end{cases}$$

(2) If $p = 3$ and K has discriminant $d \equiv -3 \pmod 9$ and thus enables irregular 3-admissible conductors, then the prime power $q = 3^2 = 9$ is a **free irregular** conductor over K, $\delta_3(9) = 0$, if and only if the quotient of the regulator $\mathrm{R}(9)$ of the suborder $\mathcal{O}_9$ by the regulator $\mathrm{R}(1)$ of the maximal order $\mathcal{O}$ of K satisfies the condition

$$v_3(\mathrm{R}(9)/\mathrm{R}(1)) = 0.$$

3. Main Theorems

3.1. **Regular and tamely irregular conductors.**

Theorem 3.1. Let $p \geq 3$ be an odd prime and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with modified p-class rank $\sigma_p \geq 2$ and discriminant d. Assume that $c = p^e \cdot q_1 \cdots q_t$ is a restrictive p-admissible conductor over K with p-**defect** $\delta_p(c)$ **equal two**. Denote by H_i, $1 \leq i \leq p+1$, the **hyperplanes** of the vector space V_p which contain the p-ring space $V_p(c)$ and by $n_i = \#\{1 \leq k \leq \tau \mid V_p(q_k) = H_i\}$ their **occupation numbers**, for $1 \leq i \leq p+1$.
If c is either **regular** or **tamely irregular**, that is $p = 3$, $d \equiv -3 \pmod 9$, $e = 2$, and $\delta_3(3) = 0$, then the p-multiplicity of c with respect to d is given by

$$m_p(d,c) = p^{\varrho} \cdot p^{\omega} (p-1)^u \cdot \frac{1}{p^2} \left[(p-1)^{v-1} + \sum_{i=1}^{p+1} (-1)^{v-n_i} (p-1)^{n_i} \right], \tag{2.1}$$

where $\varrho = \varrho_p$ denotes the ordinary p-class rank of K, $\omega = 0$ if c is regular, and $\omega = 1$ otherwise, $u = \#\{1 \leq k \leq \tau \mid V_p(q_k) = V_p\}$, and $v = \tau - u$.

Example 3.1. Concerning the actual occurrence of a multiplet of 9 complex cubic fields L of discriminant $d_L = c^2 d$ with regular conductor of 3-defect $\delta_3(c) = 2$, the smallest case we found (not necessarily minimal) is $d_L = -32\,618\,700$ with associated $d = -4027$, $\varrho_3 = 2$. Since $\alpha_2 \in \mathcal{O}_5$, $\alpha_3 \in \mathcal{O}_9$, $\alpha_4 \in \mathcal{O}_2$, the occupation numbers of the four hyperplanes $H_i < V_3$ with respect to the capable restrictive 3-admissible conductor $c = 3^2 \cdot 2 \cdot 5 = 90$ are given by $(n_1, n_2, n_3, n_4) = (0, 1, 1, 1)$, whence formula (2.1) yields $m_3(-4027, 90) = 9$.

TABLE 1. Small example of a capable restrictive conductor c with $\delta_3(c) = 2$

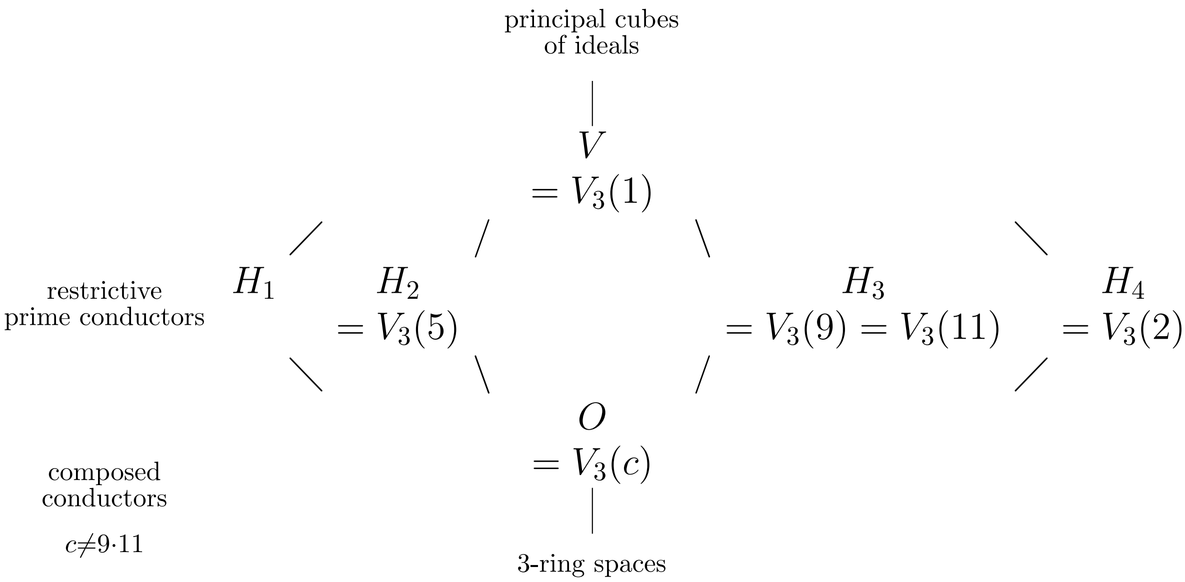
d	$F_i = (A, B, C)$	r_i	(x_i, y_i)	c	$d_L = c^2 d$	$m_3(d, c)$
-4027	$(13, 9, 79)$	13	$(69, 1)$	90	$-32\,618\,700$	9
	$(17, 11, 61)$	61	$(43, 15)$			
	$(19, 1, 53)$	229	$(2927, 99)$			
	$(29, -27, 41)$	43	$(416, 6)$			

Local matrix ϵ of the conductor $c = 3^2 \cdot 2 \cdot 5 \cdot 11 = 990$ over $\mathbb{Q}(\sqrt{-4\,027})$:

$$\epsilon_{k,\ell} = +1 \iff H_\ell \le V_3(q_k),\ 1 \le k \le \tau,\ 1 \le \ell \le 4.$$

$d = -4\,027$	q_k	F_1	$F_3 = F_1 F_2$	$F_4 = F_1 F_2^2$	F_2	ℓ
$(F_1, \dots, F_4)$		$(13, 9, 79)$	$(19, 1, 53)$	$(29, -27, 41)$	$(17, 11, 61)$	
$(r_1, \dots, r_4)$		13	229	43	61	
$(\alpha_1, \dots, \alpha_4)$		$(69, 1)$	$(2927, 99)$	$(416, 6)$	$(43, 15)$	
		local matrix				
$d \equiv 5 \pmod 8$	2	0	0	$+1$	0	4
$(d/5) = -1$	5	0	0	0	$+1$	2
$(d/11) = -1$	11	0	$+1$	0	0	3
$d \equiv -1 \pmod 3$	3^2	0	$+1$	0	0	3

Positions of 3-ring spaces of conductors $c \mid 990$ over $\mathbb{Q}(\sqrt{-4\,027})$:



Comparing 3-multiplicities of conductors $c \mid 990$ over $d = -4\,027$:

c	τ	$\delta_3(c)$	$m_3(d,c)$	$(n_1,\ldots,n_4)$	$d_L = c^2 d$
1	0	0	4	$(0,0,0,0)$	$-4\,027$
$2, 5, 3^2, 11$	1	1	0	$(1,0,0,0)$	
$2\cdot 5, 2\cdot 3^2, 2\cdot 11, 5\cdot 3^2, 5\cdot 11$	2	2	0	$(1,1,0,0)$	
$3^2\cdot 11$	2	1	9	$(2,0,0,0)$	$-39\,468\,627$
$2\cdot 3^2\cdot 11, 5\cdot 3^2\cdot 11$	3	2	0	$(2,1,0,0)$	
$2\cdot 5\cdot 3^2$	3	2	9	$(1,1,1,0)$	$-32\,618\,700$
$2\cdot 5\cdot 11$	3	2	9	$(1,1,1,0)$	$-48\,726\,700$
$2\cdot 5\cdot 3^2\cdot 11$	4	2	9	$(2,1,1,0)$	$-3\,946\,862\,700$

3.2. **Wildly irregular conductors.**

Theorem 3.2. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with modified 3-class rank $\sigma_3 \geq 2$ and discriminant $d \equiv -3 \pmod 9$. Assume that $c = 3^2 \cdot q_1 \cdots q_t$ is a restrictive 3-admissible conductor over K with 3-**defect** $\delta_3(c)$ **equal two**. Let c be **wildly irregular**, that is $\delta_3(3) = 1$ and the 3-ring space $V_3(3)$ coincides with some **fixed hyperplane** H containing $V_3(c)$ in the vector space V_3. Denote by $n = \#\{1 \leq k \leq t+1 \mid V_3(q_k) = H\}$ the **occupation number of** H. Then the 3-multiplicity of c with respect to d is given by the **degenerate formula**

$$m_3(d,c) = 3^{\varrho} \cdot 2^{u+n} \cdot \frac{1}{3}\left[2^{v-n-1} - (-1)^{v-n-1}\right], \tag{2.2}$$

where $\varrho = \varrho_3$ denotes the 3-class rank of K, $u = \#\{1 \leq k \leq t \mid V_3(q_k) = V_3\}$, and $v = \tau - u$.

Example 3.2. Concerning the actual occurrence of a multiplet of 9 complex cubic fields L of discriminant $d_L = c^2 d$ with restrictively irregular conductor of 3-defect $\delta_3(c) = \delta_3(3^2) = 2$, the smallest case we found (quite certainly non-minimal) is $d_L = -1\,565\,807\,679$ with associated $d = -8751$, $\varrho_3 = 2$. Since $\alpha_2 \in \mathcal{O}_3$, $\alpha_4 \in \mathcal{O}_{47}$, the occupation number n of the distinguished hyperplane $H_2 = V_3(3)$ and the position counter v of all proper subspaces of V_3, with respect to the capable restrictive 3-admissible conductor $c = 3^2 \cdot 47 = 423$, are given by $n = 0$ and $v = 2$, whence formula (2.2) yields $m_3(-8751, 423) = 9$.

TABLE 2. Small example of a capable restrictive conductor c with $\delta_3(c) = 2$

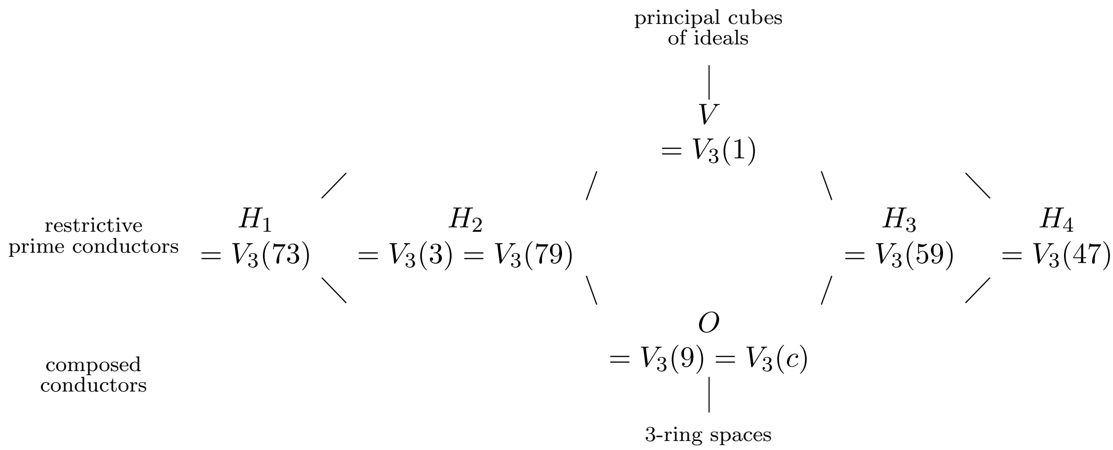
d	$F_i = (A, B, C)$	r_i	(x_i, y_i)	c	$d_L = c^2 d$	$m_3(d, c)$
-8751	$(30, 3, 73)$	3739	$(172700, 4526)$	423	$-1\,565\,807\,679$	9
	$(40, 7, 55)$	5647	$(102584, 9006)$			
	$(46, 9, 48)$	3187	$(254996, 2714)$			
	$(43, 35, 58)$	3259	$(92060, 3854)$			

Local matrix ϵ of the conductor $c = 3^2 \cdot 47 \cdot 59 \cdot 73 \cdot 79$ over $\mathbb{Q}(\sqrt{-8\,751})$:

$$\epsilon_{k,\ell} = +1 \Longleftrightarrow H_\ell \le V_3(q_k),\ 1 \le k \le \tau,\ 1 \le \ell \le 4.$$

$d = -8\,751$	q_k	F_1	$F_3 = F_1F_2$	$F_4 = F_1F_2^2$	F_2	ℓ
$(F_1,\ldots,F_4)$		$(30,3,73)$	$(46,9,48)$	$(43,35,58)$	$(40,7,55)$	
$(r_1,\ldots,r_4)$		3739	3187	3259	5647	
$(\alpha_1,\ldots,\alpha_4)$		$(172700,4526)$	$(254996,2714)$	$(92060,3854)$	$(102584,9006)$	
		local matrix				
$(d/47) = -1$	47	0	0	$+1$	0	4
$(d/59) = -1$	59	0	$+1$	0	0	3
$(d/73) = +1$	73	$+1$	0	0	0	1
$(d/79) = +1$	79	0	0	0	$+1$	2
$d \equiv -3 \pmod 9$	3	0	0	0	$+1$	2
$d \equiv -3 \pmod 9$	3^2	0	0	0	0	$-$

Positions of 3-ring spaces of conductors $c \mid 9 \cdot 47 \cdot 59 \cdot 73 \cdot 79$ over $\mathbb{Q}(\sqrt{-8\,751})$:



3.3. **Trichotomy of multiplicity formulas.** A common feature of all multiplicity formulas, except formula (0.0), is the **Trichotomy**

$$m_p(d,c) = U(\varrho) \cdot F(\omega,u) \cdot R(v,\ldots)$$

into a product of three components,

(1) **unramified** contribution $U(\varrho) = p^\varrho$,
(2) **free** contribution $F(\omega,u) = p^\omega \cdot (p-1)^u$, and
(3) **restrictive** contribution (dependent on $0 \le \delta_p(c) \le 3$)

$$R() = \frac{1}{p-1} \qquad (0.1)$$

$$R(v) = \frac{1}{p}[(p-1)^{v-1} - (-1)^{v-1}] \qquad (1.1)$$

$$R(v,n_i) = \frac{1}{p^2}[(p-1)^{v-1} - \sum_{i=1}^{p+1}(-1)^{v-n_i}(p-1)^{n_i}] \qquad (2.1)$$

$$R(v,b_i) = \frac{1}{p^3}[(p-1)^{v-1} - \sum_{i=1}^{p^2+p+1}(-1)^{v-b_i}(p-1)^{b_i}] \qquad (3.1)$$

Table 3. Restrictive multiplicity factor in dependence on positions of 3-ring spaces

v	n_1	n_2	n_3	n_4	$\delta_3(c)$	$R(v, n_1, \ldots, n_4)$
0	0	0	0	0	0	$\frac{1}{2}$
1	1	0	0	0	1	0
2	1	1	0	0	2	0
	2	0	0	0	1	1
3	1	1	1	0	2	1
	2	1	0	0	2	0
	3	0	0	0	1	1
4	1	1	1	1	2	0
	2	1	1	0	2	1
	2	2	0	0	2	2
	3	1	0	0	2	0
	4	0	0	0	1	3
5	2	1	1	1	2	2
	2	2	1	0	2	1
	3	1	1	0	2	3
	3	2	0	0	2	2
	4	1	0	0	2	0
	5	0	0	0	1	5

3.4. **Solution of Taussky's 1970 problem.**

Theorem 3.3. Let p be an odd prime. Assume that G is a p-group of order $|G| = p^5$ having abelianization G/G' of type (p,p).

(1) Then G is metabelian and 2-generated, $G = \langle x, y\rangle$, with commutator subgroup of order $|G'| = p^3$, lower central series generated by $s_2 = [y,x]$, $s_3 = s_2^{x-1}$, $s_4 = s_2^{(x-1)^2}$, $t_3 = s_2^{y-1}$, $t_4 = s_2^{(y-1)^2}$ and
- either $\mathrm{cl}(G) = 4$, $\mathrm{cc}(G) = 1$, and G belongs to the stem of Φ_9 or Φ_{10}
- or $\mathrm{cl}(G) = 3$, $\mathrm{cc}(G) = 2$, and G belongs to the stem of Φ_6.

(2) The following statements are equivalent.
- The annihilator $\mathfrak{A} = \{f(X,Y) \in \mathbb{Z}[X,Y] \mid s_2^{f(x-1,y-1)} = 1\}$ or 'symbolic order' of s_2 is given by $\mathfrak{A} = \langle p, X^2, XY, Y^2\rangle = \mathfrak{L}_2$.
- $s_2^p = 1$ and s_3, t_3 lie in the centre $\zeta_1(G)$ of G.
- G belongs to the stem of Hall's isoclinism family Φ_6.
- G is top vertex (without parent) of coclass graph $\mathcal{G}(5,2)$.

(3) If $G \in \Phi_6 \cap \mathcal{G}(5,2)$ and $x^p = s_2^{x-1}$ and $y^p = s_2^{y-1}$, then G is isomorphic to James' $\Phi_6(221)_a$.
For $p = 5$, $G \simeq \langle 3125, 14\rangle$ in the SmallGroups library.

Theorem 3.4 (D. C. Mayer, Nov. 08, 2010)**.** The transfer kernel types (TKT) $\varkappa(G)$ of the 12 top vertices of coclass graph $\mathcal{G}(5,2)$ are given by Table 4.

Table 4. TKT of twelve 5-groups of order 5^5 in isoclinism family Φ_6

Identifier of 5-group		TKT		
SmallGroup	R. James	η	$\varkappa$	property
$\langle 3125, 14\rangle *$	$\Phi_6(221)_a$	6	(123456)	identity
$\langle 3125, 11\rangle *$	$\Phi_6(221)_{b_1}$	2	(125364)	4-cycle
$\langle 3125, 7\rangle$	$\Phi_6(221)_{b_2}$	2	(126543)	two transpos.
$\langle 3125, 8\rangle *$	$\Phi_6(221)_{c_1}$	1	(612435)	5-cycle
$\langle 3125, 13\rangle *$	$\Phi_6(221)_{c_2}$	1	(612435)	5-cycle
$\langle 3125, 10\rangle$	$\Phi_6(221)_{d_0}$	0	(214365)	three transpos.
$\langle 3125, 12\rangle *$	$\Phi_6(221)_{d_1}$	0	(512643)	6-cycle
$\langle 3125, 9\rangle *$	$\Phi_6(221)_{d_2}$	0	(312564)	two 3-cycles
$\langle 3125, 4\rangle$	$\Phi_6(21^3)_a$	2	(022222)	nrl.const.with fp.
$\langle 3125, 5\rangle$	$\Phi_6(21^3)_{b_1}$	1	(011111)	nearly constant
$\langle 3125, 6\rangle$	$\Phi_6(21^3)_{b_2}$	1	(011111)	nearly constant
$\langle 3125, 3\rangle$	$\Phi_6(1^5)$	6	(000000)	constant

In Table 4, TKTs $\varkappa$ for 5-groups are given for the first time. The only exception is the group $\langle 3125, 14\rangle$ which was presented in the well-known 1970 paper by Taussky as an example that the coarse TKT $\kappa = (\mathrm{AAAAAA})$ can occur for $p = 5$.
A partial characterization is given by counters of fixed point transfer kernels, resp. abelianizations (transfer targets) of type $(5,5,5)$,

$$\eta = \#\{1 \le i \le 6 \mid \kappa(i) = \mathrm{A}\} = \#\{1 \le i \le 6 \mid \tau(i) = (5,5,5)\}.$$

A star after the SmallGroup identifier denotes a Schur σ-group.

Theorem 3.5 (Bush and Mayer)**.** Let K be an arbitrary number field with 5-class group $\mathrm{Cl}_5(K)$ of type $(5,5)$, whose TKT and TTT are given by

(1) either $\varkappa(K)=(1,2,3,4,5,6)$ (identity) and $\tau(K)=\bigl((5,5,5)^6\bigr)$
(2) or $\varkappa(K)=(1,2,5,3,6,4)$ (4-cycle) and $\tau(K)=\bigl((5,25)^4,(5,5,5)^2\bigr)$
(3) or $\varkappa(K)=(6,1,2,4,3,5)$ (5-cycle) and $\tau(K)=\bigl((5,25)^5,(5,5,5)\bigr)$
(4) or $\varkappa(K)=(5,1,2,6,4,3)$ (6-cycle) and $\tau(K)=\bigl((5,25)^6\bigr)$
(5) or $\varkappa(K)=(3,1,2,5,6,4)$ (two 3-cycles) and $\tau(K)=\bigl((5,25)^6\bigr)$.

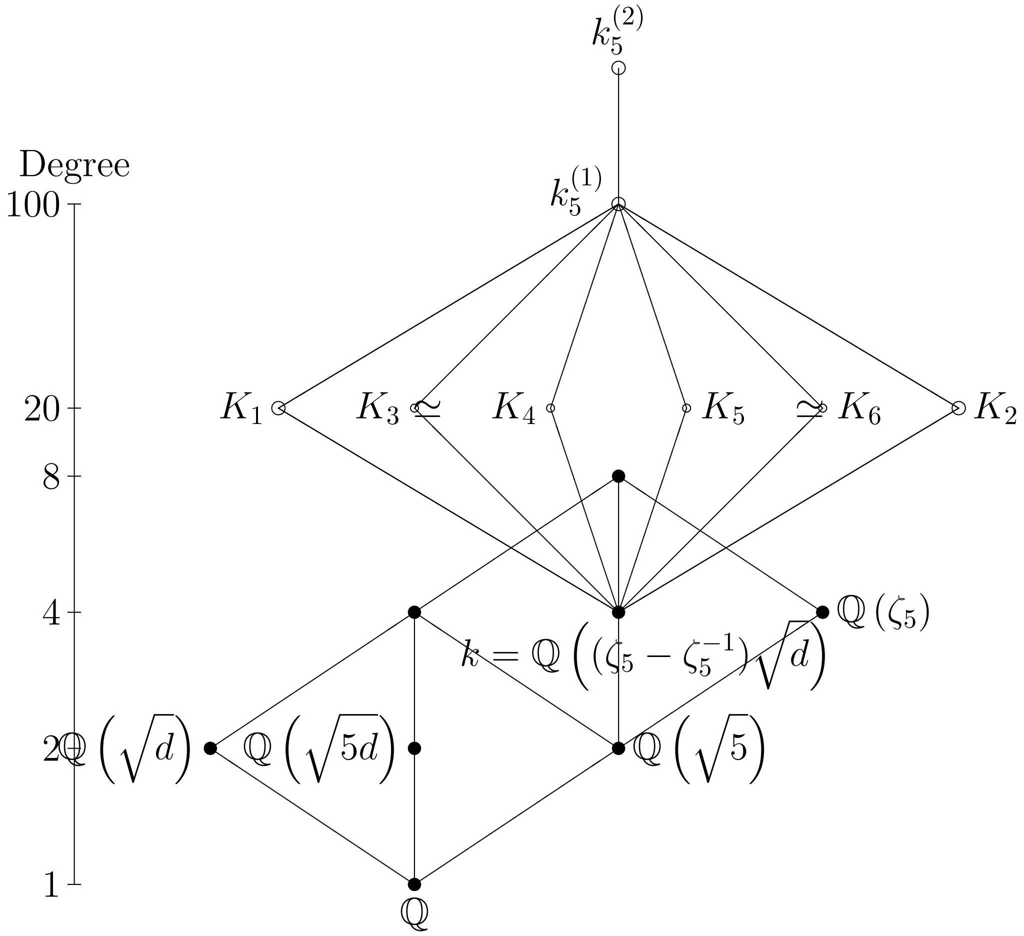
Then K has a 5-class field tower of exact length $\ell_5(K)=2$.

Table 5. Ten variants of $G = \mathrm{G}_5^2(K)$ for 31 fields $K = \mathbb{Q}(\sqrt{D})$, $-89\,751 \le D < 0$

D	$\tau(K)$	$\tau(0)$	$\varkappa(K)$	G	$\mathrm{cc}(G)$	#
$-11\,199$	$(5,5^2)^6$	$(5,5,5)$	(512643)	$\langle 3125,12\rangle *$	2	7
$-17\,944$	$(5,5^2)^6$	$(5,5,5)$	(312564)	$\langle 3125,9\rangle *$	2	2
$-42\,871$	$(5,5^2)^6$	$(5,5,5)$	(214365)	$\langle 3125,10\rangle$	2	3
				or $\langle 15625,680\rangle$		
$-12\,451$	$(5,5,5),(5,5^2)^5$	$(5,5,5)$	(612435)	$\langle 3125,8\rangle *$	2	5
				or $\langle 3125,13\rangle *$		
$-30\,263$	$(5,5,5)^2,(5,5^2)^4$	$(5,5,5)$	(126543)	$\langle 3125,7\rangle$	2	4
				or $\langle 15625,647\rangle$		
$-37\,363$	$(5,5,5)^2,(5,5^2)^4$	$(5,5,5)$	(125364)	$\langle 3125,11\rangle *$	2	2
$-89\,751$	$(5,5,5)^6$	$(5,5,5)$	(123456)	$\langle 3125,14\rangle *$	2	5
$-62\,632$	$(5,5,5,5^2),(5,5,5),(5,5^2)^4$		(322222)	$\langle 78125,\#\rangle$	2	1
$-67\,031$	$(5,5,5,5,5),(5,5^2)^5$		(211111)	$\langle 78125,\#\rangle$	2	1
$-67\,063$	$(5,5,5,5,5),(5,5,5),(5,5^2)^4$		(322222)	$\langle 78125,\#\rangle$	2	2

The transfer target type (TTT) $\tau(G)$ of second 5-class groups $\mathrm{G}_5^2(K)$ has been computed for all 1336 quadratic number fields $K = \mathbb{Q}(\sqrt{D})$, having discriminant $-2\,270\,831 \le D \le 26\,695\,193$ and 5-class group $\mathrm{Cl}_5(K)$ of type $(5,5)$, with the aid of MAGMA. As a refinement, we calculated the transfer kernel type (TKT) $\varkappa(G)$ for 31 fields $K = \mathbb{Q}(\sqrt{D})$, $-89\,751 \le D < 0$, as given in Table 5.
A star after the SmallGroup identifier of G denotes a Schur σ-group.

FIGURE 7. Hilbert 5-class field $k_5^{(1)}$ of quartic field $k = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$



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The group $\mathrm{Gal}(k_5^{(2)}|k)$ for $k = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ of type $(5, 5)$

Table 6. 7 variants of $G=\mathrm{G}_5^2(K)$ for 41 fields $K=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{D}\right)$

D	$\tau(K)$	$\tau(0)$	$\varkappa(K)$	G	$\mathrm{cc}(G)$	#	%
$-12\,883$	$(5,5)^6$		(000000)	$\langle 125,3\rangle$	1	4	100%
257	$(5,5,5)^2,(5,5^2)^4$	$(5,5,5)$	(022222)	$\langle 3125,4\rangle$	2	9	24%
457	$(5,5,5)^2,(5,5^2)^4$	$(5,5,5)$	(125364)	$\langle 3125,11\rangle*$	2		
508	$(5,5,5)^2,(5,5^2)^4$	$(5,5,5)$	(126543)	$\langle 3125,7\rangle$	2		
				or $\langle 15625,647\rangle$			
581	$(5,5,5)^6$	$(5,5,5)$	(123456)	$\langle 3125,14\rangle*$	2	4	11%
$1\,137$	$(5,5,5,5^2),(5,5,5),(5,5^2)^4$		(111111)	$\langle 78125,\#\rangle$	2	3	8%
$4\,357$	$(5)^6$		(000000)	$\langle 25,2\rangle$	1	3	8%

In cooperation with A. Azizi and M. Talbi, and based on the most recent view of the quintic reflection theorem by Y. Kishi, we have computed the isomorphism type of the second 5-class group $G=\mathrm{G}_5^2(K)$ of 41 cyclic quartic fields $K=\mathbb{Q}\left((\zeta_5-\zeta_5^{-1})\sqrt{D}\right)$, $\zeta_5=\exp(\frac{2\pi i}{5})$, $-15\,419\le D<5\,000$, $5\nmid D$, of type $(5,5)$. Such a field is the 5-dual 'mirror image' of the quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{5D})$.
Isomorphisms among the extensions $L_i|K$, $1\le i\le 6$, cause severe constraints on the group G.
A star after the SmallGroup identifier of G denotes a Schur σ-group.

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