3-class field towers of exact length 3

Daniel C. Mayer[⋆] (Naglergasse 53, 8010 Graz)
Michael R. Bush (Washington and Lee University, Lexington, Virginia, USA)
Mike F. Newman (Australian National University, Canberra, ACT)

For an algebraic number field K, let P be the Galois group of the maximal unramified pro-3 extension $\mathrm{F}_3^\infty(K)$, and $G \simeq P/P''$ be the Galois group of the second Hilbert 3-class field $\mathrm{F}_3^2(K)$. If K has 3-class group $\mathrm{Cl}_3(K) \simeq P/P'$ of type $(3,3)$ and the capitulation of K in its unramified cyclic cubic extensions $L_1,\ldots,L_4$ is of type $\varkappa(K) = (2,2,3,1)$, resp. $(1,2,3,1)$, then the 3-class numbers of $L_1,\ldots,L_4$ are $(3^c,27,27,27)$, for some $c \geq 4$, and G is isomorphic to $G_0^{c+1,c+2}(0,0,\pm 1,1)$, resp. $G_0^{c+1,c+2}(1,0,-1,1)$, [2, Thm. 1.3, p. 405], with presentation defined in [2, § 3.3.3, p. 430]. G is a metabelian 3-group of order 3^{c+2} and class c. If K is a complex quadratic field then $c \geq 5$ must be odd [2, § 3.4.2, p. 436] and P must have relation rank $r(P)$ equal to the generator rank $d(P) = 2$ [4, p. 146, eqn. (28)]. Such a group is called *balanced*. Since G has $r(G) > d(G) = 2$, the 3-class field tower of K cannot stop at the second stage. This is the first faultless disproof of the statement $P \simeq G$, claimed by Heider and Schmithals [1, p. 20] in full generality, independently of the value of c, and claimed by Scholz and Taussky [3, p. 41] at least for the particular instance $K = \mathbb{Q}(\sqrt{-9748})$, where $\varkappa(K) = (2,2,3,1)$ and $c = 5$. Furthermore, the 3-tower of K is actually of exact length 3, since the 3-tower group $P = P_{r,c}$ is given as the class-c quotient of $L/S_{r,c}$, where L is an infinite topological group with five generators, whose presentation is known explicitly,
$\langle t,u,y,z,a \mid [[u,t],t] = 1, [[u,t],u] = 1, y^3 = 1, z^3 = 1, [y,z] = 1, [t,y] = 1, [u,y] = 1, [t,z] = 1, [u,z] = 1, t^a = u, u^a tuy[u,t]^{-1} = 1, [y,a] = 1, a^3[[t,a],t] = z\rangle$
and $S_{r,c}$ is a closed subgroup generated by two elements,
$y([t,a]^{(c-1)})^r[[t,a]^{(c-3)},[t,a]]$ and $z[t,a]^{(c-1)}$,
involving the parameter $r \in \{-1,0,1\}$ and the odd exponent $c \geq 5$ in $h_3(L_1) = 3^c$. The 3-group $P_{r,c}$ is balanced and non-metabelian of derived length 3, order $3^{(3c+1)/2}$ and class c. The value of r is determined as $r = \pm 1$ for $\varkappa(K) = (2,2,3,1)$, and uniquely as $r = 0$ for $\varkappa(K) = (1,2,3,1)$.

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