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Pattern Recognition via Artin Transfers, Applied to p-Class Field Towers

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Towers of p-Class Fields over Algebraic Number Fields

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CHAPTER 0. FOUNDATIONS

(Theoretical and Computational)

§ 0.1. Abelian type invariants of p-class groups

Let $p \in \mathbb{P}$ be a prime number. For an algebraic number field $K/\mathbb{Q}$ and a non-negative integer $n \in \mathbb{N}_0$, let

(1) $\quad \mathrm{Lyr}_n(K) := \{K \le E \le \mathrm{F}_p^1(K) \mid [E:K] = p^n\}$

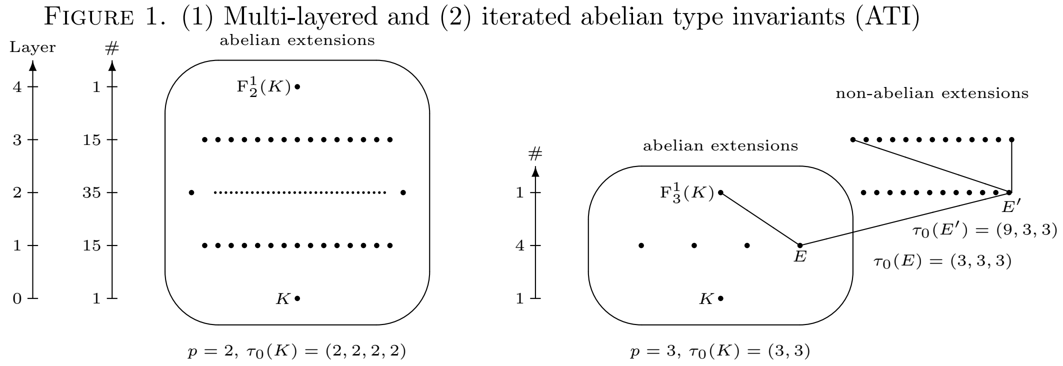
denote the nth *layer* of unramified abelian p-extensions E/K within the first Hilbert p-class field $\mathrm{F}_p^1(K)$ of K. In particular, $\mathrm{Lyr}_0(K) = \{K\}$ consists of the base field K alone.

Now two kinds of *generalized abelian type invariants* (ATI) are declared by means of the following constructions:

multi-layered abelian type invariants are defined by $\tau_n(K) := (\mathrm{Cl}_p(E))_{E\in\mathrm{Lyr}_n(K)}$, in particular $\tau_0(K) = \mathrm{Cl}_p(K)$,

and **iterated abelian type invariants of higher order** can be defined recursively by
$\tau^{(0)}(K) := \tau_0(K)$ and $\tau^{(n+1)}(K) := [\tau_0(K); (\tau^{(n)}(E))_{E\in\mathrm{Lyr}_1(K)}]$,
for $n \ge 0$,
in particular $\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)]$ is the **IPAD** in the sense of Boston, Bush and Hajir (Ref. 8). See Figure 1.

FIGURE 1. (1) Multi-layered and (2) iterated abelian type invariants (ATI)



§ 0.2. Computation of abelian type invariants

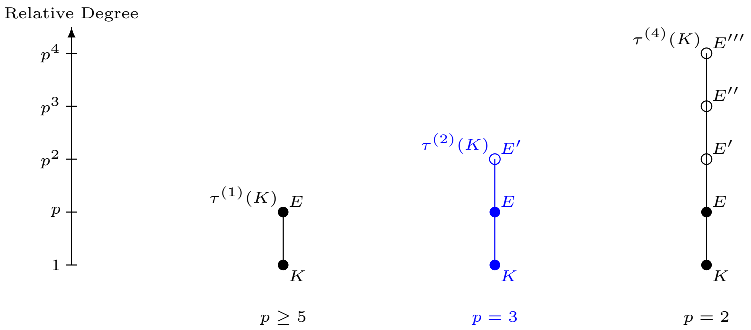
Since the first step of the *pattern recognition via Artin transfers* (§ 1) consists of the computation of p-class groups and capitulation kernels, it must first be analyzed which kinds of generalized abelian type invariants can actually be determined with the aid of computational algebra systems like PARI/GP and MAGMA (Ref. 6, 7, 11) for various primes p and base fields K. See Figure 2.

• For a quadratic base field K and any prime $5 \le p \le 19$, the abelian extensions E of the first layer have absolute degree $10 \le [E:\mathbb{Q}] \le 38$ and the ATI of *first order*, i.e. the IPAD, $\tau^{(1)}(K) = [\mathrm{Cl}_p(K); (\mathrm{Cl}_p(E))_{E\in\mathrm{Lyr}_1(K)}]$, can usually be determined.

• For $p = 3$ and quadratic, resp. biquadratic, base fields K, the ATI of *second order* $\tau^{(2)}(K) = [\tau_0(K); (\tau^{(1)}(E))_{E\in\mathrm{Lyr}_1(K)}] = [\mathrm{Cl}_p(K); (\mathrm{Cl}_p(E); (\mathrm{Cl}_p(E'))_{E'\in\mathrm{Lyr}_1(E)})_{E\in\mathrm{Lyr}_1(K)}]$ involving extensions E, E' of absolute degree $6 = [E:\mathbb{Q}] < [E':\mathbb{Q}] = 18$, respectively $12 = [E:\mathbb{Q}] < [E':\mathbb{Q}] = 36$, which are partially non-abelian, are feasible in general.

• For $p = 2$ and quadratic base fields K, the ATI of up to the *fourth order* $\tau^{(4)}(K) = [\tau_0(K); (\tau^{(3)}(E))_{E\in\mathrm{Lyr}_1(K)}]$ involving extensions $E^{\times}$ of absolute degree $4 \le [E^{\times}:\mathbb{Q}] \le 32$, which are non-abelian for a great deal, should still be within computational reach.

FIGURE 2. Abelian type invariants (ATI) of the first, second and fourth order



CHAPTER 1.
THE STRATEGY

Remark. A Strategy is not just a Theorem. It is rather an outstanding scientific achievement, which provides a general procedure for proving thousands of Theorems, writing hundreds of Papers.

§ 1.1. Strategy of pattern recognition via Artin transfers

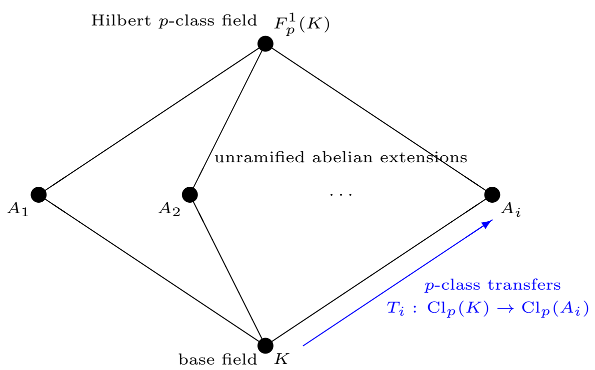
In the sequel, let a fixed prime number p be selected. The *strategy of pattern recognition with the aid of Artin transfers* is a progressive innovation, which I introduced into algebraic number theory during the years 2009 − 2016, for seeking the automorphism group $\mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ of the second Hilbert p-class field $\mathrm{F}_p^2(K)$ over an algebraic number field K (Ref. 13) by a process of pattern recognition, and to identify it uniquely in many cases.

In December 2009, I used the strategy of pattern recognition, under the preliminary title "*Accelerated approach to the target*", for determining the capitulation type $\varkappa(K)$ in unramified abelian extensions only from the structures $\tau(K)$ of their p-class groups (§ 2.1). In September 2011, I presented the corresponding procedure "*Principalization algorithm via class group structure*" (Ref. 14) in a lecture at the Danube University in Krems, Lower Austria.

§ 1.2. Arithmetic and algebraic p-Artin pattern

Definition. For an algebraic number field K and the family $(A_i)_{1\le i\le n}$ of all unramified abelian p-extensions of K, which are contained within the first Hilbert p-class field $\mathrm{F}_p^1(K)$ of K, the pair $\mathrm{AP}(K) = (\tau(K), \varkappa(K))$ of all targets $\tau(K) = (\mathrm{Cl}_p(A_i))_{1\le i\le n}$ (ATI) and kernels $\varkappa(K) = (\ker(T_i))_{1\le i\le n}$ of the transfer homomorphisms $T_i : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(A_i)$ of p-classes from K to A_i is called the *arithmetic Artin pattern* of K with respect to p. See Figure 3.

FIGURE 3. Transfers of p-classes from K to extensions A_i

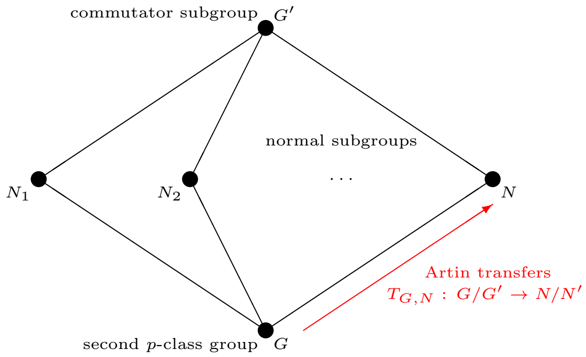


The strategy of *pattern recognition via Artin transfers* consists of three steps:

• First the *number theoretic* Artin pattern $\mathrm{AP}(K)$ of an algebraic number field K is declared (as in the preceding definition) as the collection $(\tau(K), \varkappa(K))$ of all targets $\tau(K) = (\mathrm{Cl}_p(A))_A$ (ATI) and kernels $\varkappa(K) = (\ker(T_{K,A}))_A$ of *class extension* homomorphisms $T_{K,A} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(A)$ from K to unramified abelian extension fields $K \le A \le \mathrm{F}_p^1(K)$ of K within the Hilbert p-class field $\mathrm{F}_p^1(K)$ of K. **Lower layers** of this arithmetic Artin pattern are determined with the aid of class field theoretic routines of the computational algebra system MAGMA (Ref. 6, 7, 11).

• In the second step, the number theoretic Artin pattern $\mathrm{AP}(K)$ of K is interpreted as *group theoretic* Artin pattern $\mathrm{AP}(G)$ of the metabelian Galois group $G = \mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ of the second Hilbert p-class field $\mathrm{F}_p^2(K)$ of K. It consists of all targets $\tau(G) = (N/N')_N$ (abelian quotient invariants, AQI) and kernels $\varkappa(G) = (\ker(T_{G,N}))_N$ of *Artin transfer* homomorphisms $T_{G,N} : G/G' \to N/N'$ from G into normal subgroups $G' \le N \le G$, which contain the commutator subgroup G' of G (Fig. 4). The latter is isomorphic to the p-class group $\mathrm{Cl}_p(\mathrm{F}_p^1(K)) \simeq \mathrm{Gal}(\mathrm{F}_p^2(K)/\mathrm{F}_p^1(K))$ of the Hilbert p-class field $\mathrm{F}_p^1(K)$ of K.

FIGURE 4. Artin transfers from G into normal subgroups N



If the unramified abelian extension A/K is the fixed field $A = \mathrm{Fix}(N)$ of the normal subgroup $N < G$, in the sense of the Galois correspondence, then the isomorphism $N/N' \simeq \mathrm{Gal}(\mathrm{F}_p^2(K)/A)/\mathrm{Gal}(\mathrm{F}_p^2(K)/\mathrm{F}_p^1(A)) \simeq \mathrm{Cl}_p(A)$, is a consequence of the Artin reciprocity law (Ref. 2), and $\ker(T_{G,N}) \simeq \ker(T_{K,A})$, according to the theory of Artin transfers (Ref. 3).

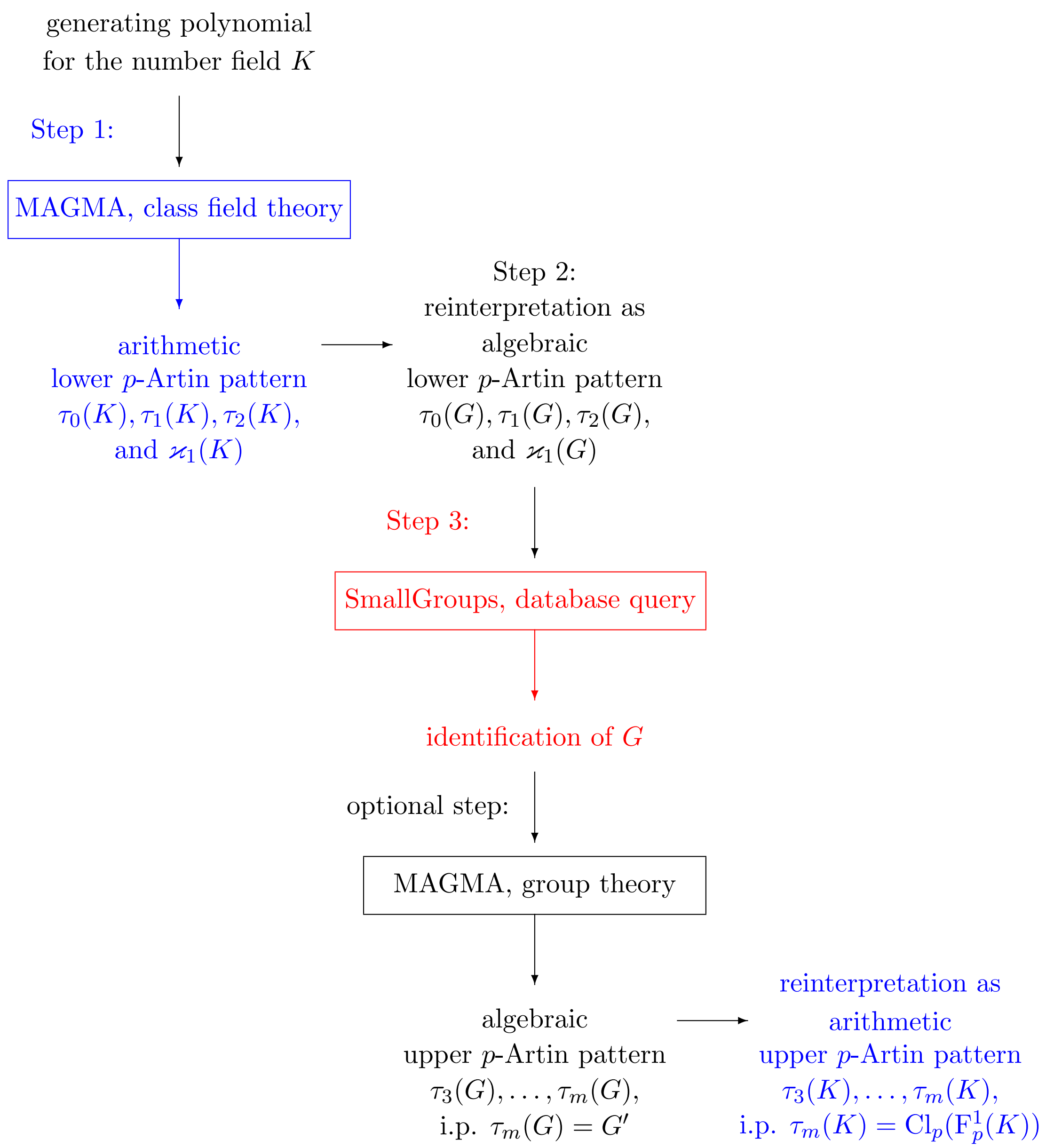
• Finally, the group theoretic Artin pattern $\mathrm{AP}(G)$ is used as a *search pattern* in a *database query*, for instance as an SQL (structured query language) statement,

“SELECT $*$ FROM $\langle Table \rangle$ WHERE $\langle Pattern \rangle$”

from a *table* of finite p-groups G, e.g. from the SmallGroups Library or from an *extended table*, constructed selectively by means of the p-group generation algorithm of Newman (Ref. 22) and O’Brien (Ref. 23). This *process of filtration* frequently yields a unique candidate for the second p-class group $G = \mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ of the field K. Thus, the informations on targets $\tau(G)$ and kernels $\varkappa(G)$ of transfers, collected in the Artin pattern $\mathrm{AP}(G)$, characterize the metabelian approximation $\mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ of the pro-p automorphism group $\mathrm{Gal}(\mathrm{F}_p^\infty(K)/K)$ of the p-class field tower, that is the maximal unramified pro-p extension of K, similar as *unambiguous finger prints*. See the Diagram in Figure 5.

In Figure 5, K denotes an algebraic number field with non-cyclic p-class group $\tau_0(K) = \mathrm{Cl}_p(K)$ for a fixed prime number p, and $G = \mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ denotes the metabelian automorphism group of the second Hilbert p-class field $\mathrm{F}_p^2(K)$ of K.

FIGURE 5. Working Flow Diagram of the Pattern Recognition



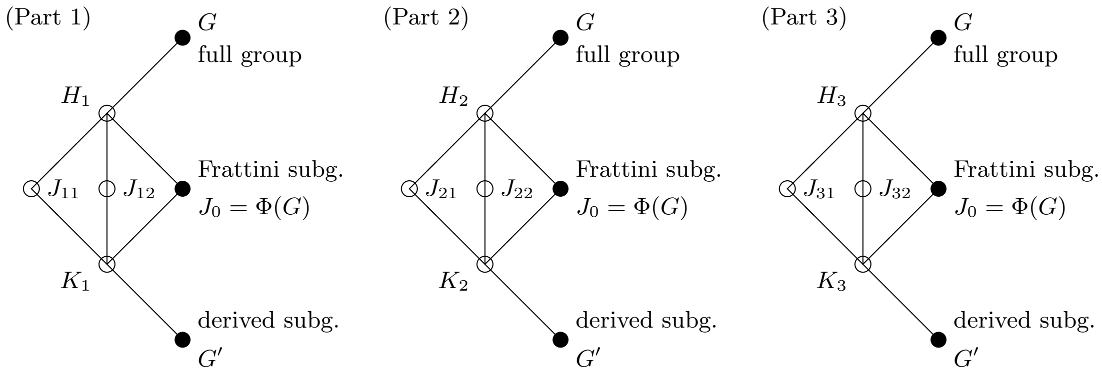
Theorem. Let $p = 2$ and K be a real quadratic field with 2-Artin pattern

$$(2)\quad \begin{aligned} &\tau_0 = (22), \quad \tau_1 = (211, 211, 311), \quad \varkappa_1 = (J_0, J_0, K_1), \\ &\tau_2 = (31, 31, 31, 31, 422, 311; 2111), \quad \varkappa_2 = (H_1, H_1, H_2, H_2, J_{31}, H_3; H_3). \end{aligned}$$

Then the 2-class tower of K has length $\ell_2(K) = 2$ and the metabelian Galois group $G = \mathrm{Gal}(\mathrm{F}_2^\infty(K)/K)$ is isomorphic to $\langle 512, 1465\rangle - \#1; 2$ with order 1024, coclass 4, relation rank $d_2 = 3$, $\tau_3 = (211, 211, 322)$, $\varkappa_3 = (G, G, H_3)$, $\tau_4 = (222) \simeq \mathrm{Cl}_2(\mathrm{F}_2^1(K))$ (Ref. 21).

Figure 6 shows the three parts of the normal lattice of G above the derived subgroup G'.

FIGURE 6. Normal lattice of $G/G' \simeq C_4 \times C_4$



Example. 595 561 is the single known discriminant $0 < d_K < 10^6$ of a real quadratic field K with $\mathrm{Cl}_2(K) \simeq C_4 \times C_4$ and Artin pattern in Formula (2). It is mentioned in Ref. 4, Example 7.4, p. 1192, that PARI failed to determine $\tau_4 = (222) \simeq \mathrm{Cl}_2(\mathrm{F}_2^1(K))$.

§ 1.3. Determining the tower length by means of the relation rank

The relation rank $d_2(G)$ of G then finally admits the decision over the length $\ell_p(K)$ of the p-class field tower of K, based on the *Shafarevich cohomology criterion*

$$\mathrm{rk}_p(K) \le d_2(G) \le \mathrm{rk}_p(K) + r + \theta \tag{3}$$

in dependence on the p-class rank $\mathrm{rk}_p(K)$, the signature (r_1, r_2) and the torsion free Dirichlet unit rank $r = r_1 + r_2 - 1$ of the number field K (Ref. 15, 17). Here, θ denotes the flag

$$\theta = \begin{cases} 1 & \text{if } K \text{ contains the } p\text{th roots of unity,} \\ 0 & \text{else.} \end{cases} \tag{4}$$

§ 1.4. Recently discovered esthetic aspects of mathematics

A particularly esthetic form of the Artin pattern $\mathrm{AP}(G) = (\tau(G), \varkappa(G))$ was discovered by myself for p-groups G with *multiple-layered* maximal abelian quotient G/G'. The outstanding property of such a pattern concerns the transfer kernels $\varkappa(G)$ and will be exposed in detail under the concept of *harmonically balanced capitulation kernels* (§ 4).

CHAPTER 2.
APPLICATIONS to SPORADIC groups

FIGURE 7. 3-Artin pattern of *sporadic* groups outside of coclass trees

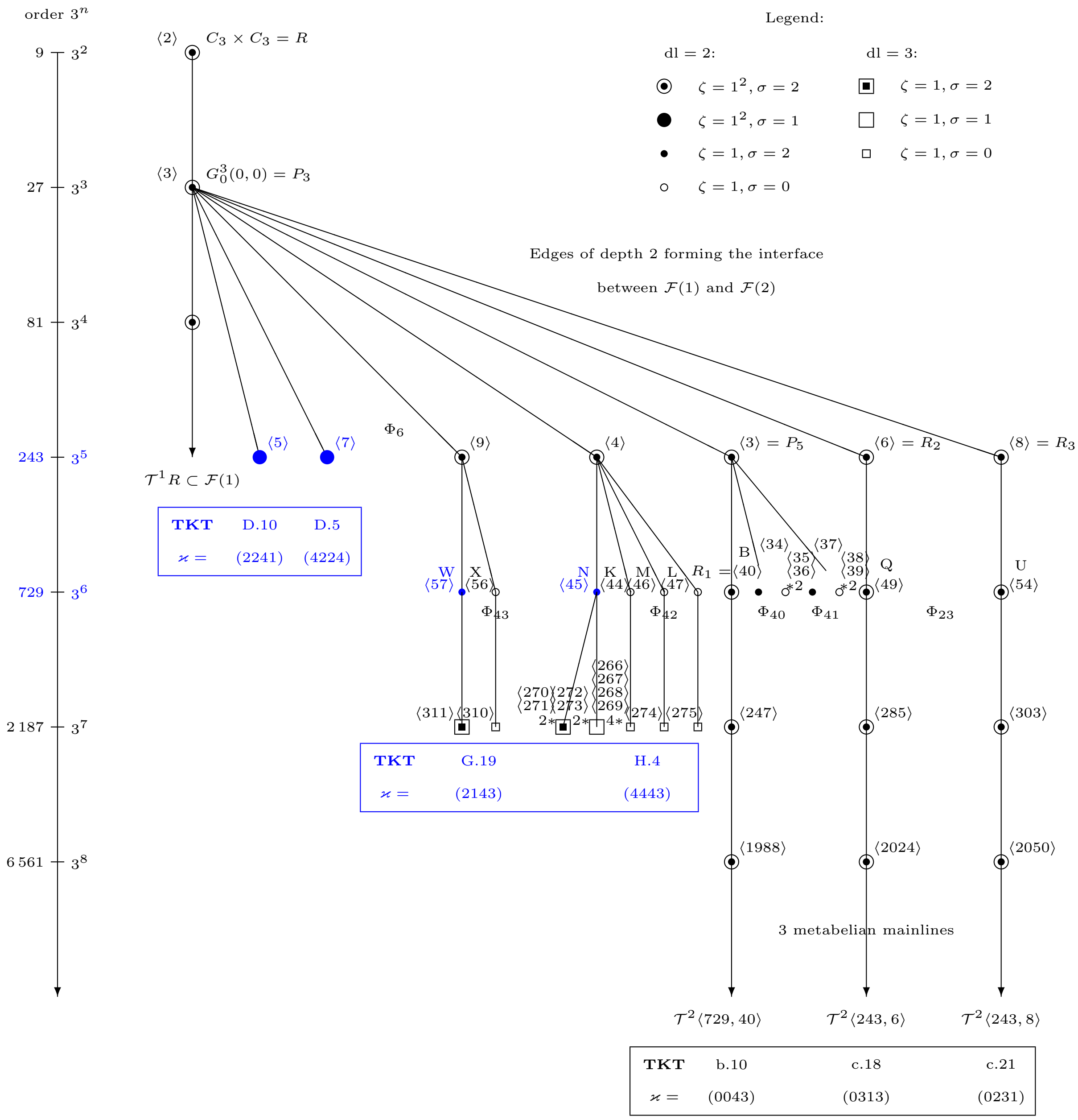


Figure 7 shows the top region of the *forest* $\mathcal{F}(2)$ of finite 3-groups G with coclass $\mathrm{cc}(G) = 2$, which are also called *second maximal class*. The forest consists of two *isolated* groups with SmallGroup identifiers $\langle 243, 5\rangle$ and $\langle 243, 7\rangle$ (Ref. 5), two *finite trees* with roots $\langle 243, 4\rangle$ and $\langle 243, 9\rangle$, and three *infinite coclass trees* with roots $\langle 243, 3\rangle$, $\langle 243, 6\rangle$ and $\langle 243, 8\rangle$, having metabelian mainlines. Groups in the finite components are called *sporadic*, the vertices of the infinite components are called *periodic* (Ref. 16, 20). $\varkappa = \varkappa(G)$ denotes the transfer kernel type (TKT).

§ 2.1. Connections between the components of the Artin pattern admitting considerable computational simplification

In contrast to the so-called *periodic* groups on coclass trees, which will be discussed a little bit later, there occasionally occurs a *bijective correspondence* between the kernels $\varkappa(G)$ and targets $\tau(G)$ of the Artin pattern for *sporadic* groups G outside of coclass trees. Consequently, the targets $\tau(G)$, which can be computed more easily, uniquely determine the kernels $\varkappa(G)$, thus the entire Artin pattern $\mathrm{AP}(G) = (\tau(G), \varkappa(G))$, and eventually also the group G itself.

Example. Under the key word "*Accelerated approach to the target*", this lucky circumstance permitted the determination of the capitulation type $\varkappa(K)$ merely from the structures $\tau(K)$ of the 3-class groups for real quadratic number fields K, $d_K < 10^6$, with 3-class group $\mathrm{Cl}_3(K) \simeq (3, 3)$, which I investigated in December 2009. In June 2010, I studied imaginary quadratic fields K of the same type with discriminants $-10^6 < d_K$. It turned out that this considerable computational simplification can be employed for nearly two thirds $\approx 66\%$ of all actual realizations (Ref. 13, 14). For $-10^7 < d_K$, computed by myself in 2016, and for $-10^8 < d_K$, computed by Boston, Bush, Hajir (Ref. 8), the percentage of simplifications decreases slightly to $\approx 64\%$. See the summary in Table 1.

Since only two isomorphism types occur among the constituents of $\tau(K)$, namely $(3, 3, 3)\hat{=}1^3$ and $(9, 3)\hat{=}21$, the invariant $\varepsilon = \#\{1 \le i \le 4 \mid \mathrm{Cl}_3(N_i) \simeq 1^3\}$ suffices for the unambiguous description. (Here we use logarithmic abelian type invariants and symbolic exponents indicating iteration.) The metabelian 3-group G can be characterized by its SmallGroup identifier (Ref. 5) containing the order and a counter.

TABLE 1. Imaginary quadratic fields of type $(3, 3)$, according to Ref. 13, 14 and 8

Discriminant	Total#	$\varepsilon = 1$	$\varepsilon = 2$	$\varepsilon = 3$	$\varepsilon = 0$	$\Sigma\%$
$-10^6 < d_K < 0$	2 020	33.0%	13.3%	14.7%	4.7%	65.7%
$-10^7 < d_K < 0$	24 476	31.14%	14.81%	14.79%	4.163%	64.903%
$-10^8 < d_K < 0$	276 375	30.159%	14.979%	14.823%	3.7724%	63.7334%
$\tau(G)$		$(1^3, (21)^3)$	$((1^3)^2, (21)^2)$	$((1^3)^3, 21)$	$((21)^4)$	
$\varkappa(G)$		(2241)	(4224)	(4443)	(2143)	
TKT		D.10	D.5	H.4	G.19	
$G \simeq$		$\langle 243, 5\rangle$	$\langle 243, 7\rangle$	$\langle 729, 45\rangle$	$\langle 729, 57\rangle$	

§ 2.2. Rigorous theorems concerning sporadic groups of type $(3, 3)$

Now we express the results of the preceding section § 2.1 in strictly proven statements.

As before, let $G = \mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ be the second p-class group of the number field K (Ref. 13). In Theorem 1, the abelian type invariants $\tau_1(K)$ *alone* determine G completely for the groups in Table 1, in particular, the structure of the 3-class group of the Hilbert 3-class field $\mathrm{F}_3^1(K)$, which alone forms the second layer, can be determined without number theoretic computations, since it is isomorphic to the derived subgroup of G, $\tau_2(K) = \mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq G'$. Theorem 1 (2010, Ref. 14) and Corollary 1 (2015) are due to myself.

Theorem 1. (**3-class groups alone** in the **sporadic** case $p = 3$. See Figure 7.)
First let K be an algebraic number field of type $\mathrm{Cl}_3(K) \simeq (3, 3)$ without total capitulation.

(1) $\tau_1(K) = (21, 21, 111, 21) \Longrightarrow G \simeq \langle 243, 5\rangle$, $\varkappa_1(K) = (2241)$, $\mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (111)$.
(2) $\tau_1(K) = (111, 21, 111, 21) \Longrightarrow G \simeq \langle 243, 7\rangle$, $\varkappa_1(K) = (4224)$, $\mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (111)$.

Now let K be a (real or imaginary) quadratic number field of type $\mathrm{Cl}_3(K) \simeq (3, 3)$.

(3) $\tau_1(K) = (111, 111, 21, 111) \Longrightarrow G \simeq \langle 729, 45\rangle$, $\varkappa_1(K) = (4443)$, $\mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (211)$.
(4) $\tau_1(K) = (21, 21, 21, 21) \Longrightarrow G \simeq \langle 729, 57\rangle$, $\varkappa_1(K) = (2143)$, $\mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (1111)$.

Since Theorem 1 does not specify the full 3-class tower, a supplement must be added:

Corollary 1. The former two groups are Schur σ-groups (Ref. 1, 10), so the 3-class field tower of K has length $\ell_3(K) = 2$. The groups $\langle 243, 4\rangle$ with $\tau_1 = (111, 111, 21, 111)$ and $\langle 243, 9\rangle$ with $\tau_1(K) = (21, 21, 21, 21)$ cannot occur as G for any quadratic field K, since they do not satisfy the Schoof condition (Ref. 25), and the latter two groups have too big relation rank $d_2(G) = 4$ for a two-stage tower, whence K must have $\ell_3(K) \geq 3$. See § 5.1.

Example 1. Among the 2020 imaginary quadratic fields K of type $\mathrm{Cl}_3(K) \simeq (3, 3)$ with fundamental discriminants in the range $-10^6 < d_K < 0$ (Ref. 13, 14),
$G \simeq \langle 243, 5\rangle$ occurs for 667 fields (33.0%), for instance $d_K = -4027$,
$G \simeq \langle 243, 7\rangle$ occurs for 269 fields (13.3%),
$G \simeq \langle 729, 45\rangle$ occurs for 297 fields (14.7%), for instance $d_K = -3896$,
$G \simeq \langle 729, 57\rangle$ occurs for 94 fields (4.7%).
Together, the four groups occur *dominantly* for 1327 fields, i.e. nearly *two thirds* (65.7%).

CHAPTER 3.
APPLICATIONS to PERIODIC groups

FIGURE 8. 3-Artin pattern of *periodic* groups on a coclass tree

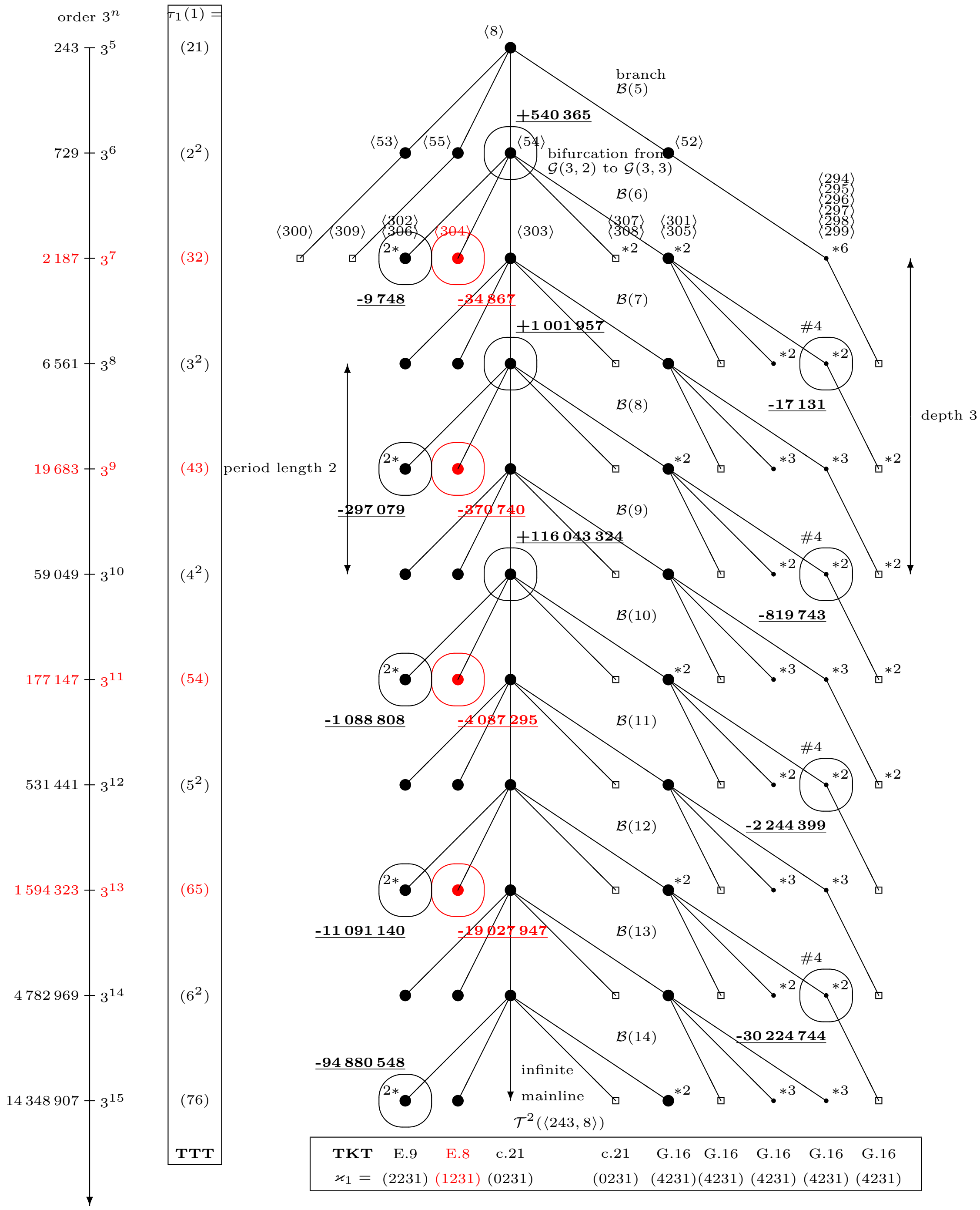


Figure 8 shows all details of the last coclass tree $\mathcal{T}^2\langle 243, 8\rangle$ in Figure 7. The tree contains the groups $\langle 2187, 302\rangle$, $\langle 2187, 304\rangle$ and $\langle 2187, 306\rangle$ in Theorem 2. The other groups $\langle 2187, 288\rangle$, $\langle 2187, 289\rangle$ and $\langle 2187, 290\rangle$ belong to the closely related and rather similar coclass tree $\mathcal{T}^2\langle 243, 6\rangle$ in Figure 7.

Generally, for the *periodic* groups G on a coclass tree, if there is given a set of assigned kernels $\varkappa(G)$, then in general there exists a *countable infinitude* of distinct abelian quotient invariants $\tau(G)$, which can be described in the form of a *ground state* and indefinitely increasing *excited states*.

On the other hand, if there is given a set of assigned targets $\tau(G)$, then in general there also exist several, but necessarily only finitely many, kernels $\varkappa(G)$, the *kernel variants* or *capitulation types*.

In both cases, the group G can only be characterized uniquely by the entire Artin pattern $\mathrm{AP}(G) = (\tau(G), \varkappa(G))$, since a single component alone does not suffice for the unambiguous identification.

§ 3.1. Preliminary theorems concerning periodic groups of type $(3, 3)$

As before, let $G = \mathrm{Gal}(\mathrm{F}_p^2(K)/K)$ be the second p-class group of the number field K (Ref. 13). In Theorem 2, the *full* 3-Artin pattern $\mathrm{AP}(K) = (\tau_1(K), \varkappa_1(K))$ is required for finding G. Theorem 2 was proved by myself in 2010 (Ref. 13, 14).

Theorem 2. (**3-class groups and capitulation** in the **periodic** case $p = 3$)
Let K be a (real or imaginary) quadratic number field of type $\mathrm{Cl}_3(K) \simeq (3, 3)$.

(1) $\tau_1(K) = (32, 21, 111, 21), \varkappa_1(K) = (1313) \Longrightarrow G \simeq \langle 2187, 288\rangle, \mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (221)$.
(2) $\tau_1(K) = (32, 21, 111, 21)$, $\varkappa_1(K) = (2313) \Longrightarrow \mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (221)$ and either $G \simeq \langle 2187, 289\rangle$ or $G \simeq \langle 2187, 290\rangle$.
(3) $\tau_1(K) = (32, 21, 21, 21), \varkappa_1(K) = (1231) \Longrightarrow G \simeq \langle 2187, 304\rangle, \mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (221)$.
(4) $\tau_1(K) = (32, 21, 21, 21)$, $\varkappa_1(K) = (2231) \Longrightarrow \mathrm{Cl}_3(\mathrm{F}_3^1(K)) \simeq (221)$ and either $G \simeq \langle 2187, 302\rangle$ or $G \simeq \langle 2187, 306\rangle$. See Figure 8 for (3) and (4).

Example 2. Among the 2020 imaginary quadratic fields K of type $\mathrm{Cl}_3(K) \simeq (3, 3)$ with fundamental discriminants in the range $-10^6 < d_K < 0$ (Ref. 13, 14),
$G \simeq \langle 2187, 288\rangle$ occurs for 66 fields (3.3%),
$G \simeq \langle 2187, 289\rangle$ or $G \simeq \langle 2187, 290\rangle$ occurs for 120 fields (6.0%),
$G \simeq \langle 2187, 304\rangle$ occurs for 75 fields (3.7%), for instance $d_K = -34867$,
$G \simeq \langle 2187, 302\rangle$ or $G \simeq \langle 2187, 306\rangle$ occurs for 122 fields (6.0%), for instance $d_K = -9748$.
Together, these six groups occur *significantly* for 383 fields, i.e. nearly *one fifth* (19.0%).

Theorem 2 does not specify the full 3-class tower, which will be done a little bit later in Corollary 2 and Theorem 3. But first, I want to emphasize an important remark:

Historical Remark. It might not be sufficiently well known that the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ in Example 2 caused an intolerable uncertainty with respect to the length of its 3-class field tower for nearly 80 years, and that, in cooperation with Michael R. Bush, I succeeded, on 22-24 August 2012, in disproving the erroneous claim $\ell_3(K) = 2$ of the famous authors Arnold Scholz and Olga Taussky-Todd, published in the Crelle Journal in 1934 (Ref. 24., pp. 20, 41).

The error was spread by Heider and Schmithals in 1982 (p. 20, Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen, J. Reine Angew. Math. **336** (1982), 1–25) and by myself in 1991 (p. 84, Principalization in complex S_3-fields, Congressus Numerantium **80** (1991), 73–87) simply by referring to Ref. 24.

In 1992, Franz Lemmermeyer read my 1991 paper and drew my attention to the paper by Brink and Gold (Class field towers of imaginary quadratic fields, manuscr. math. **57** (1987) 425–450), where doubts are expressed concerning the claim $\ell_3(K) = 2$ of Scholz and Taussky, and a non-metabelian 3-group with derived length 3 is constructed which could be the 3-tower group of K, thus weakening the claim to the inequality $\ell_3(K) \geq 2$, but not definitely excluding $\ell_3(K) = 2$. Together with Mike F. Newman, we identified the group constructed by Brink and Gold as $\langle 6561, 621\rangle$, which is very close to the correct groups $\langle 6561, 620\rangle$ and $\langle 6561, 624\rangle$.

However, the ultimate clarification of the problem by Bush and myself (Ref. 9) requires (1) the Shafarevich criterion $d_2(G) = 2$ for a Schur σ-group, which discourages $\ell_3(K) = 2$, (2) the monotony principle for τ and $\varkappa$ on descendant trees (for uniqueness) and (3) the fact that an epimorphism onto a Schur σ-group is an isomorphism, which prohibits $\ell_3(K) \geq 4$.

§ 3.2. Main theorems concerning periodic groups of type $(3, 3)$

We complete the information in Theorem 2 by a statement concerning the length of the 3-class field tower (Ref. 9). See the first of the following three tree diagrams, where $\langle 729, 54\rangle - \#1; 4 = \langle 2187, 304\rangle$ and $\langle 729, 54\rangle - \#2; 4 = \langle 6561, 622\rangle$.
(The other two tree diagrams concern excited states and are given only to illuminate the broad range of applications.)

Corollary 2 was proved by Boston, Bush and myself in 2012. This was the first proof of precise $\ell_p(K) = 3$ for odd prime p.

Corollary 2. Any **imaginary** quadratic field K with one of the four 3-Artin patterns $\mathrm{AP}(K) = (\tau_1(K), \varkappa_1(K))$ in Theorem 2 has a non-metabelian 3-class field tower of exact length $\ell_3(K) = 3$. The Galois group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of the maximal unramified pro-3 extension $\mathrm{F}_3^\infty(K)$ of K is given by its SmallGroup identifier (Ref. 5, 12)

$$\mathfrak{G} \simeq \begin{cases} \langle 6561, 622\rangle & \text{if } \mathrm{AP}(K) = ((32, 21, 21, 21), (1231)), \\ \langle 6561, 620\rangle \text{ or } \langle 6561, 624\rangle & \text{if } \mathrm{AP}(K) = ((32, 21, 21, 21), (2231)). \end{cases}$$

In 1934 Scholz and Taussky (Ref. 24) proved that the identity permutation $\varkappa = (1234)$ with *four fixed points* is *impossible* as capitulation type of any number field K with $\mathrm{Cl}_3(K) \simeq (3, 3)$. However, there is a unique capitulation type with *three fixed points*, namely the type $\varkappa = (1231)$ in Theorem 2.(3), which was called the *unknown type* by Heider and Schmithals in 1982, since no actual realization by a number field was known at that time.

In 2003, I discovered the minimal discriminant $d_K = -34867$ of an *imaginary* quadratic field K with capitulation type $\varkappa_1(K) = (1231)$.

No *real* quadratic number field without total capitulation was known yet in 2003. It was even unknown whether such a real quadratic field can exist at all. With great surprise, I decided this problem affirmatively, when I found the minimal discriminant $d_K = 214712$ without total capitulation in 2006, in fact its capitulation type is the permutation $\varkappa = (2143)$ in Theorem 1.(4). But it required several further years until 2010 before I found three real quadratic fields K with capitulation type $\varkappa_1(K) = (1231)$ as in Theorem 2.(3). Now we turn to these discoveries in Theorem 3 and Example 3.

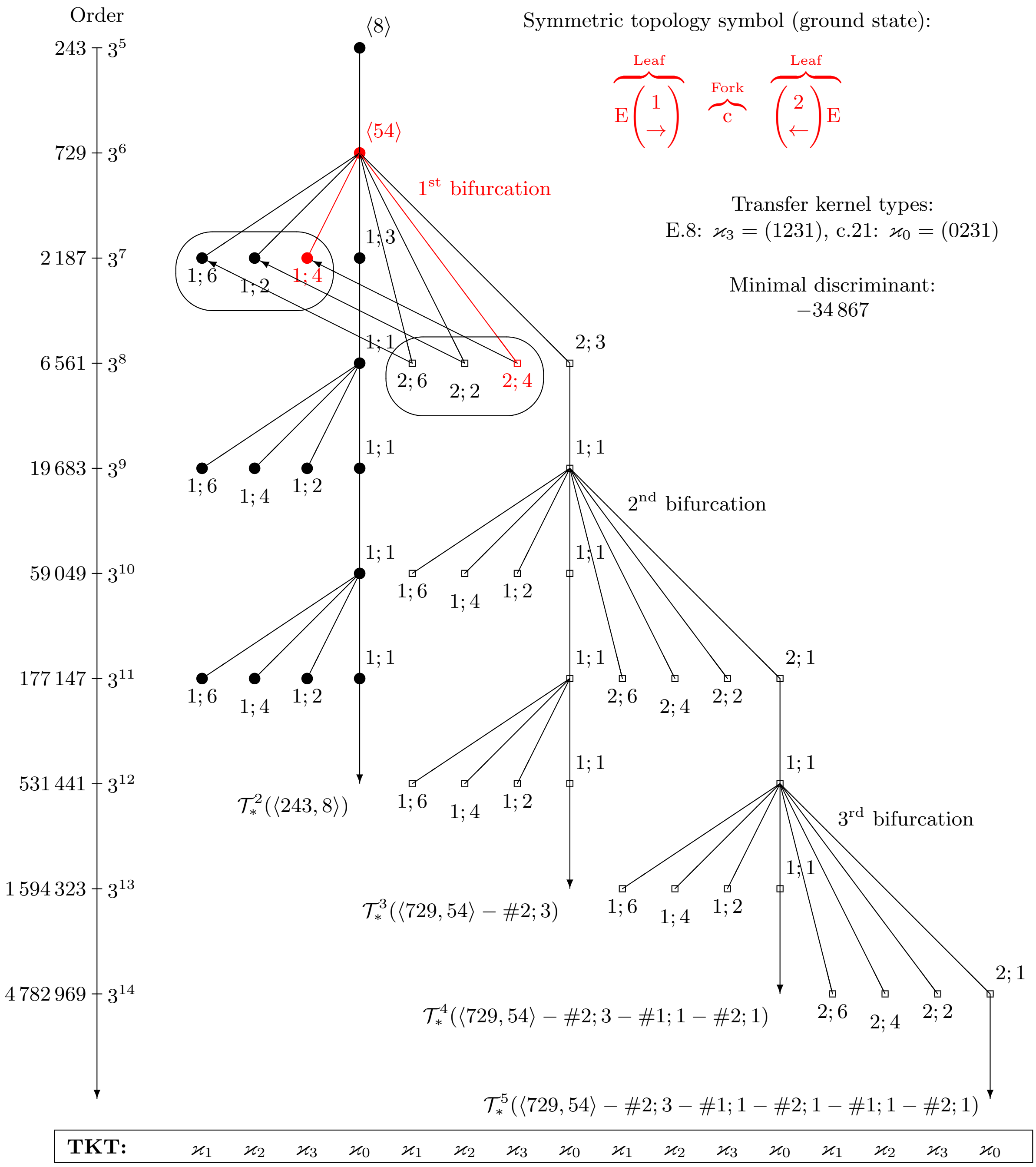
Iterated abelian type invariants of second order are still computable and enable the decision between 3-class field towers of precise length 2 and 3 for real quadratic fields with one of the four 3-Artin patterns in Theorem 2. Such precise statements of necessary and sufficient criteria were unknown until 2015. In this year, I proved Theorem 3 (Ref. 18).

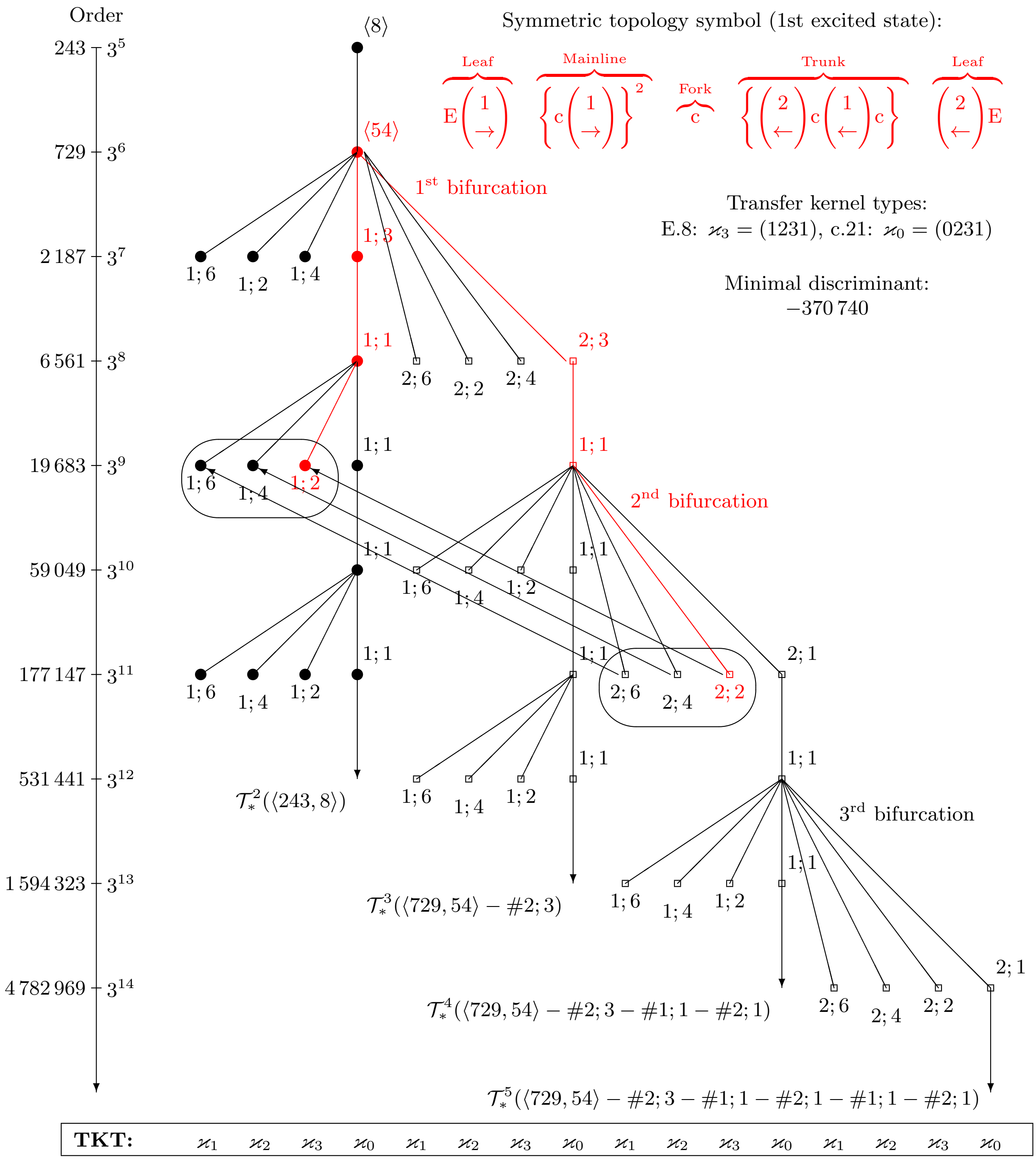
Theorem 3. Let $E_1, \ldots, E_4$ be the four unramified cyclic cubic extensions of the real quadratic field K with 3-Artin pattern $\mathrm{AP}(K) = ((32, 21, 21, 21), (1231))$. Then the 3-class field tower group of K is given by

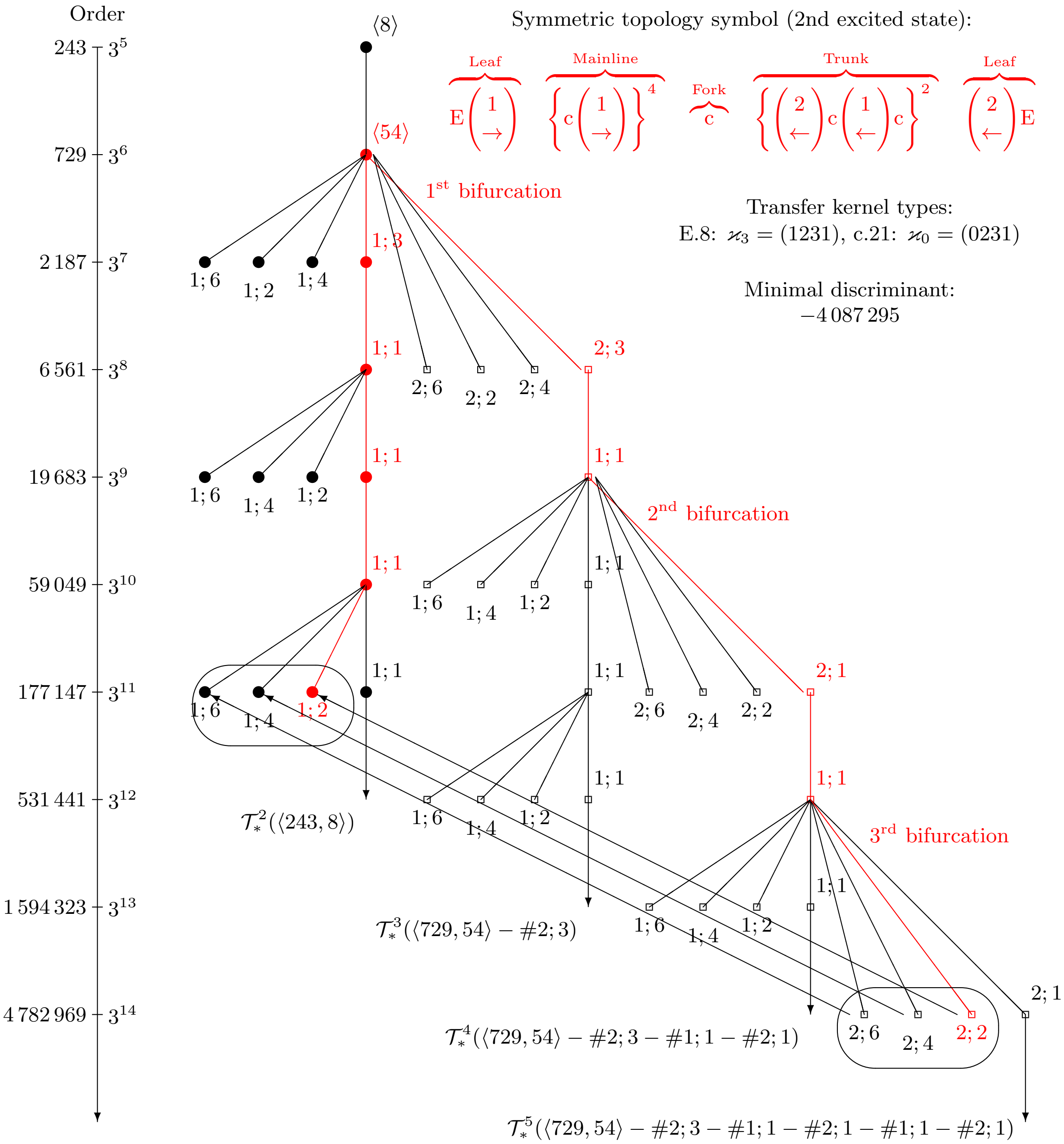
$$\mathfrak{G} \simeq \begin{cases} \langle 2187, 304 \rangle & \text{if } \tau^{(1)}(E_i) = [(21); (221, 21, 21, 21)] \text{ for } 2 \le i \le 4, \\ \langle 6561, 622 \rangle & \text{if } \tau^{(1)}(E_i) = [(21); (221, 31, 31, 31)] \text{ for } 2 \le i \le 4. \end{cases}$$

Example 3. Among the 2576 real quadratic fields K of type $\mathrm{Cl}_3(K) \simeq (3, 3)$ having fundamental discriminants in the range $0 < d_K < 10^7$ (Ref. 13, 14, 18),
$\mathfrak{G} \simeq \langle 6561, 622 \rangle$ with $\ell_3(K) = 3$ occurs for $d_K \in \{6098360, 7100889\}$,
$\mathfrak{G} \simeq \langle 2187, 304 \rangle$ with $\ell_3(K) = 2$ occurs for $d_K = 8632716$.

See the first of the following three tree diagrams, where $\langle 729, 54 \rangle - \#1; 4 = \langle 2187, 304 \rangle$ and $\langle 729, 54 \rangle - \#2; 4 = \langle 6561, 622 \rangle$ concern the *ground* state.
(The other two tree diagrams concern *excited* states and are given only to illuminate the broad range of applications. For the tree topology symbols, see Ref. 17.)


Order
243 – 3^5
729 – 3^6
2 187 – 3^7
6 561 – 3^8
19 683 – 3^9
59 049 – 3^10
177 147 – 3^11
531 441 – 3^12
1 594 323 – 3^13
4 782 969 – 3^14
⟨8⟩
⟨54⟩
1st bifurcation
2nd bifurcation
3rd bifurcation
1; 6
1; 2
1; 4
1; 3
1; 1
2; 6
2; 2
2; 4
2; 3
2; 1
T_*^2(⟨243, 8⟩)
T_*^3(⟨729, 54⟩ − #2; 3)
T_*^4(⟨729, 54⟩ − #2; 3 − #1; 1 − #2; 1)
T_*^5(⟨729, 54⟩ − #2; 3 − #1; 1 − #2; 1 − #1; 1 − #2; 1)
Symmetric topology symbol (ground state):
Leaf
Fork
Leaf
E (1 →) c (2 ←) E
Transfer kernel types:
E.8: ϰ3 = (1231), c.21: ϰ0 = (0231)
Minimal discriminant:
−34 867
TKT:
ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0


Order
243
729
2 187
6 561
19 683
59 049
177 147
531 441
1 594 323
4 782 969
⟨8⟩
⟨54⟩
Symmetric topology symbol (1st excited state):
Leaf
Mainline
Fork
Trunk
Leaf
1st bifurcation
2nd bifurcation
3rd bifurcation
Transfer kernel types:
E.8: ϰ3 = (1231), c.21: ϰ0 = (0231)
Minimal discriminant:
−370 740
TKT:



TKT:	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$

CHAPTER 4.
HARMONIC BALANCE

§ 4.0. **Discovery of harmonically balanced capitulation (HBC)**

In the present chapter, I select a fixed prime number p for my investigations.

I begin with a formal and general definition of the concept of *harmonically balanced capitulation*, which, in spite of my long experience with the capitulation problem, suggested itself in a relaxed and natural way not earlier than very recently during the investigation of *situations with several layers of subobjects*, e.g. number fields of type $(9, 9)$ and $(81, 3)$.

Definition. For a pro-p group G, let $U = (U_i)_{0\le i\le n}$ be the family of all subgroups $G' \le U_i \le G$ of G, which contain the commutator subgroup G' and therefore are necessarily self-conjugate (normal).
Then G possesses *harmonically balanced capitulation*, if the family $\varkappa(G) = (\ker(T_i : G \to U_i/U_i'))_{0\le i\le n}$ of transfer kernels coincides with U up to a permutation $\pi \in \mathfrak{S}_n$, that is, when $(\forall\, 0 \le i \le n)\ \ker(T_i) = U_{\pi(i)}$.

Remark. One of the members in the cycle decomposition of the permutation π is necessarily always the transposition $(0, n)$, because,
for the full group $U_0 = G$, the transfer $T_0 : G \to G/G'$ is the canonical projection with kernel $\ker(T_0) = G' = U_n = U_{\pi(0)}$, and for the commutator subgroup $U_n = G'$, the transfer $T_n : G \to G'/G''$ has total kernel $\ker(T_n) = G = U_0 = U_{\pi(n)}$, according to the *principal ideal theorem*.

The phenomenon of *harmonically balanced capitulation* was discovered by myself in July 2019 for an extensive class of metabelian Schur σ-groups G of order $3^{10} = 59\,049$ with maximal abelian quotient $G/G' \simeq C_9 \times C_9$. I begin my investigations with this *non-elementary bihomocyclic* special case of $p = 3$, with three layers of proper intermediate groups $G' < U < G$ and associated Artin patterns $(\tau_1, \varkappa_1)$, $(\tau_2, \varkappa_2)$, $(\tau_3, \varkappa_3)$, as section § 4.2. In August 2019, I posed the question whether the corresponding phenomenon also occurs in the situation $p = 3$ for *biheterocyclic* abelian quotient $G/G' \simeq C_{3^e} \times C_3$ with exponents $e \ge 2$ and *one elementary component*. That is indeed the case, though with a certain delay only for rather huge orders and only for non-metabelian Schur σ-groups, as I am going to show in the subsequent section § 4.3.

§ 4.1. **The role of Herbrand's quotient for HBC**

Let E/F be a **cyclic** number field extension with **odd** prime power degree $[E:F]=p^n$, $p\in\mathbb{P}$, $p\neq 2$, $n\in\mathbb{N}$, and automorphism group $G=\mathrm{Gal}(E/F)=\langle\sigma\rangle$.

Theorem on Herbrand quotient of 1st and 0th Galois cohomology of unit group U_E:

(5) $$\frac{\#\mathrm{H}^1(G,U_E)}{\#\mathrm{H}^0(G,U_E)}=[E:F].$$

Interpretation by **primitive ambiguous principal ideals** and **unit norm index**:

(6) $$\#(\mathcal{P}_E^G/\mathcal{P}_F)=[E:F]\cdot(U_F:\mathrm{N}_{E/F}(U_E)).$$

Special case of **capitulation kernel**, when E/F is **unramified**:

(7) $$\#\ker(T_{F,E})=[E:F]\cdot(U_F:\mathrm{N}_{E/F}(U_E)),$$

where $T_{F,E}:\ \mathrm{Cl}_p(F)\to\mathrm{Cl}_p(E)$ denotes transfer of p-classes from base field to extension.

HBC over **imaginary quadratic** field $F=\mathbb{Q}(\sqrt{d})$, $d<-3$:
$\mathrm{N}_{E/F}(U_E)=U_F=\{-1,+1\}$, and thus

(8) $$\#\ker(T_{F,E})=[E:F]=p^n.$$

HBC shows that Herbrand's Theorem is obviously true for non-cyclic **abelian** extensions.

§ 4.2. **Non-elementary bi-homo-cyclic case**

In this situation of a finite non-abelian 3-group G with abelian quotient invariants G/G' of type $(9,9)$, the permutation of normal subgroups $G' \le U_i \le G$ defining a harmonically balanced capitulation can be described in the following manner.

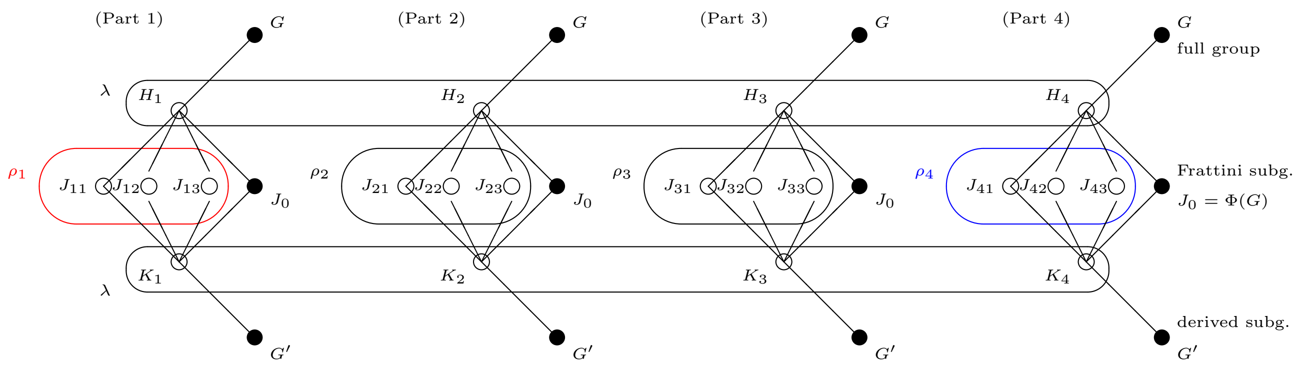
Proposition. G possesses **harmonically balanced capitulation** (briefly HBC) if $\ker(T_{G,J_0}) = J_0$ and there exists a permutation $\lambda \in \mathfrak{S}_4$ such that for all $1 \le i \le 4$

(1) $\ker(T_{G,H_i}) = K_{\lambda(i)}$,

(2) $(\exists\, \rho_i \in \mathfrak{S}_3)\ (\forall\, 1 \le k \le 3)\ \ker(T_{G,J_{ik}}) = J_{\lambda(i),\rho_i(k)}$,

(3) $\ker(T_{G,K_i}) = H_{\lambda(i)}$.

FIGURE 9. Permutations λ, ρ_i defining a harmonically balanced capitulation



The lattice of normal subgroups containing G' in Figure 9 consists of 4 Parts, which are glued together at G, J_0 and G'. Basically the H_i and the K_i are twisted and subject to a permutation λ, whereas the J_{ik} are mapped onto each other with four local permutations ρ_i in each Part.

Among the 287 imaginary quadratic fields K with 3-class group $\mathrm{Cl}_3(K) \simeq (9,9)$ in the range $-10^7 < d_K < 0$ of negative fundamental discriminants, there are 183 fields, i.e. 64%, with HBC, uniquely characterized by uniform 3-class rank three of the four unramified cyclic cubic extensions E_i/K.

Obviously there occur three principal scenarios of capitulation, as the following Table 2 shows. The transfer kernels are either

- *bi-polarized* (BPC), e.g. $\varkappa_1 = (K_4, K_1, K_1, K_1)$, with $\ker(T_{G,J_0}) = J_0$, or
- *uni-polarized* (UPC), e.g. $\varkappa_1 = (K_1, K_1, K_1, K_1)$, with $\ker(T_{G,J_0}) = J_{11}$, or
- *harmonically balanced* (HBC) (the most frequent scenario).

See also Table 3 and Figure 10.

TABLE 2. 20 absolutely smallest discriminants $d < 0$ of $K = \mathbb{Q}(\sqrt{d})$, $\mathrm{Cl}_3(K) \simeq (9,9)$

No.	$-d$	Scenario	Artin Pattern		Relevant Group(s)
			$\varkappa_1$	τ_1	
1	$134\,059 \in \mathbb{P}$	HBC	3241	$(311)^4$	$\langle 6561, 24/25\rangle$
2	$208\,084 = 2^2 \cdot 52\,021$	UPC	2222	$(311)^2, 221, 2221$	$\langle 6561, 20\rangle$
3	$298\,483 \in \mathbb{P}$	HBC	3241	$(311)^4$	$\langle 6561, 24/25\rangle$
4	$426\,291 = 3 \cdot 142\,097$	HBC	2143	$(221)^4$	$\langle 6561, 28\rangle$
5	$430\,411 \in \mathbb{P}$	HBC	3241	$(311)^4$	$\langle 6561, 24/25\rangle$
6	$460\,523 = 7 \cdot 65\,789$	BPC	1411	$(311)^3, 3211$	$\langle 6561, 21/22\rangle$
7	$475\,355 = 5 \cdot 95\,071$	HBC	2143	$(311)^2, (221)^2$	$\langle 6561, 23\rangle$
8	$552\,767 = 19 \cdot 47 \cdot 619$	BPC	3332	$(311)^3, 3211$	$\langle 6561, 21/22\rangle$
9	$647\,359 \in \mathbb{P}$	HBC	1324	$(311)^3, 221$	$\langle 6561, 26/27\rangle$
10	$654\,468 = 2^2 \cdot 3 \cdot 54\,539$	HBC	2341	$(311)^3, 221$	$\langle 6561, 26/27\rangle$
11	$660\,356 = 2^2 \cdot 165\,089$	HBC	4312	$(311)^4$	$\langle 6561, 24/25\rangle$
12	$682\,811 \in \mathbb{P}$	HBC	1342	$(311)^3, 221$	$\langle 6561, 26/27\rangle$
13	$727\,087 = 37 \cdot 43 \cdot 457$	UPC	1111	$(311)^3, 3211$	$\langle 6561, 21/22\rangle$
14	$947\,028 = 2^2 \cdot 3 \cdot 78\,919$	BPC	4111	$(311)^2, 221, 3211$	$\langle 6561, 20\rangle$
15	$962\,351 = 13 \cdot 74\,027$	BPC	3331	$(311)^3, 3211$	$\langle 6561, 21/22\rangle$
16	$1\,023\,347 = 13 \cdot 223 \cdot 353$	BPC	4144	$(311)^2, 221, 3211$	$\langle 6561, 20\rangle$
17	$1\,201\,272 = 2^3 \cdot 3 \cdot 50\,053$	HBC	2143	$(311)^2, (221)^2$	$\langle 6561, 23\rangle$
18	$1\,229\,752 = 2^3 \cdot 153\,719$	UPC	1111	$(311)^3, 3211$	$\langle 6561, 21/22\rangle$
19	$1\,231\,060 = 2^2 \cdot 5 \cdot 61\,553$	HBC	1423	$(311)^4$	$\langle 6561, 24/25\rangle$
20	$1\,287\,544 = 2^3 \cdot 227 \cdot 709$	HBC	4123	$(311)^3, 221$	$\langle 6561, 26/27\rangle$

There occur situations with various numbers of stages $\ell_3(K)$ of the 3-class field tower $K \le \mathrm{F}_3^1(K) \le \ldots \le \mathrm{F}_3^\infty(K)$.
The most simple case is a *metabelian* tower with $\ell_3(K) = 2$.

Theorem. (Two stage tower.)
If K is an imaginary quadratic field with Artin pattern

$$\tau_0 = (22), \quad \tau_1 = ((221)^4), \quad \tau_2 = ((321)^{12}; 222),$$

then the 3-class field tower of K has length $\ell_3(K) = 2$ and its Galois group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ is isomorphic to one of two **metabelian** Schur σ-groups

$$\langle 6561, 28\rangle - \#2; i \text{ (Table 3, Figure 10)}$$

with $i = 4$, resp. $i = 5$, order $3^{10} = 59\,049$, coclass 7,

$$\tau_3 = ((222)^4) \text{ and } \tau_4 = (222).$$

Both groups possess harmonically balanced capitulation:
$\varkappa_1 = (K_1, K_2, K_3, K_4)$,
$\varkappa_2 = (J_{13}, J_{11}, J_{12}; J_{21}, J_{22}, J_{23}; J_{31}, J_{32}, J_{33}; J_{41}, J_{42}, J_{43}; J_0)$,
bzw.
$\varkappa_2 = (J_{13}, J_{11}, J_{12}; J_{23}, J_{21}, J_{22}; J_{33}, J_{31}, J_{32}; J_{42}, J_{43}, J_{41}; J_0)$,
$\varkappa_3 = (H_1, H_2, H_3, H_4)$,
i.e. $\lambda = 1$, $\rho_1 = (132)$, $\rho_2 = \rho_3 = \rho_4 = 1$, resp. $\rho_2 = \rho_3 = (132)$, $\rho_4 = (123)$. (See the permutations in Figure 9.)

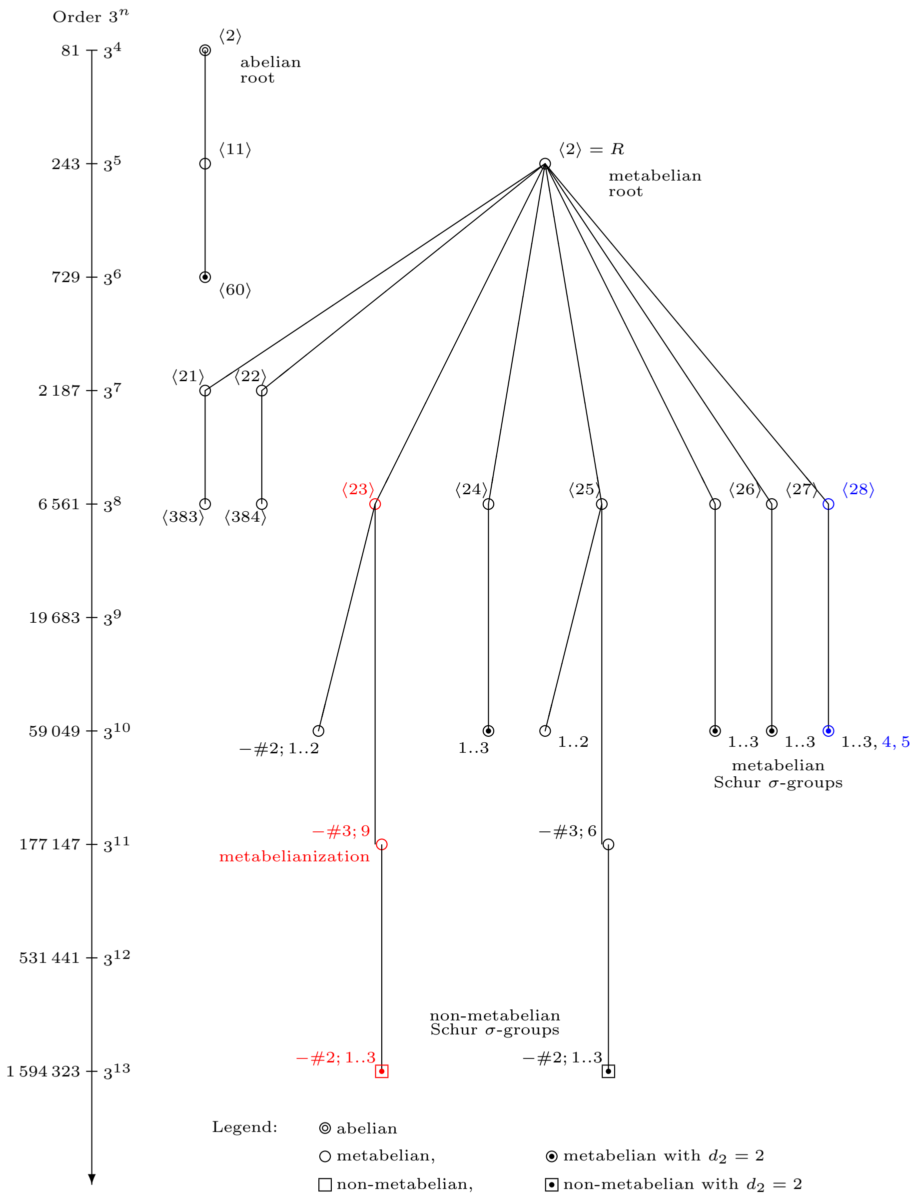
Example.
The absolutely smallest discriminant d_K with this pattern is

$$-426\,291 = -3 \cdot 142\,097.$$

TABLE 3. Descendants $R - \#3; j$ with step size 3 of $R = \langle 243, 2\rangle$

j	SmallGroup	ν	μ	$(N_i, C_i)_{1\le i\le \nu}$
22	$\langle 6561, 23\rangle$	3	5	$(10, 0; 22, 7; 9, 9)$
23	$\langle 6561, 24\rangle$	2	4	$(2, 0; 3, 0)$
24	$\langle 6561, 25\rangle$	3	5	$(7, 0; 14, 5; 6, 6)$
25	$\langle 6561, 26\rangle$	2	4	$(4, 0; 3, 0)$
26	$\langle 6561, 27\rangle$	2	4	$(4, 0; 3, 0)$
27	$\langle 6561, 28\rangle$	2	4	$(3, 0; 5, 0)$

FIGURE 10. Selected descendants of the metabelian root $R = \langle 243, 2\rangle$



However, there also exist *non-metabelian* towers of 3-class fields with $\ell_3(K) \geq 3$.

Theorem. (Tower with three or more stages.)
If K is an imaginary quadratic field with Artin pattern
$\tau_0 = (22), \tau_1 = ((311)^2, (221)^2), \tau_2 = ((42)^6, (222)^2, (321)^4; 222)$,
then the 3-class field tower of K has length $\ell_3(K) \geq 3$
and its Galois group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ is isomorphic to one of the **non-metabelian** Schur σ-groups

$$T_{i,k} = S_i - \#2; k, \quad i \geq 1, \quad 1 \leq k \leq 3,$$

with order 3^{10+3i}, nilpotency class $3+i$, coclass $7+2i$,

$$\tau_3 = ((331)^2, (322)^2) \text{ and } \tau_4 = (2+i, 2, 2),$$

in the descendant tree of the root $V_0 = \langle 6561, 23 \rangle$ (Fig. 11).
All groups have harmonically balanced capitulation:
$\varkappa_1 = (K_4, K_2, K_3, K_1)$,
$\varkappa_2 = (J_{42}, J_{43}, J_{41}; J_{23}, J_{22}, J_{21}; J_{31}, J_{33}, J_{32}; J_{11}, J_{12}, J_{13}; J_0)$,
$\varkappa_3 = (H_4, H_2, H_3, H_1)$,
i.e. $\lambda = (14)$, $\rho_1 = (123)$, $\rho_2 = (13)$, $\rho_3 = (23)$, $\rho_4 = 1$.
(See the permutations in Figure 9.)
The metabelianization $M = G/G''$ of G is isomorphic to

$$\begin{cases} S_1 = V_0 - \#3; 9, \ \#M = 3^{11}, & \text{if } i = 1, \\ V_1 - \#1; 4 \text{ with } V_1 = V_0 - \#3; 7, \ \#M = 3^{12}, & \text{if } i \geq 2. \end{cases}$$

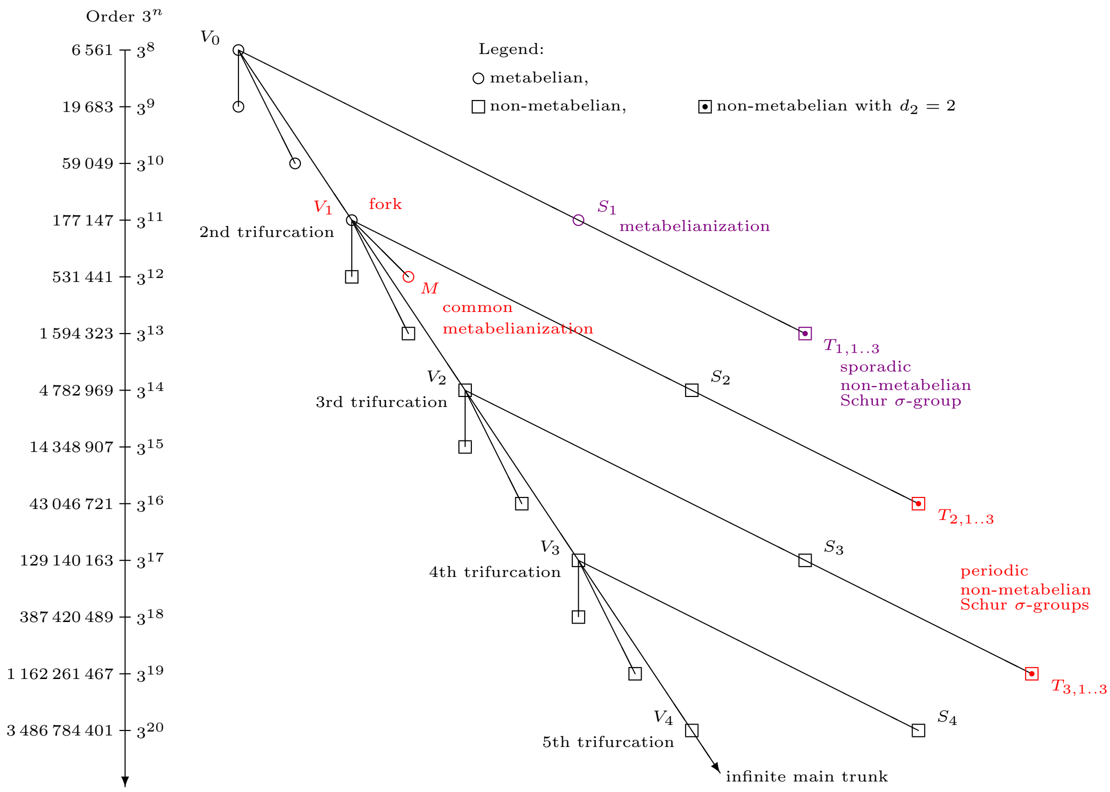
The relation rank of M is always $d_2(M) = 4$ (too big for G).

Example.
The absolutely smallest discriminant d_K with this pattern is

$$-475\,355 = -5 \cdot 95\,071.$$

The following diagramm in Figure 11 shows the descendant tree of the metabelian root V_0. The characteristic property of this tree is the occurrence of *periodic trifurcations* at the vertices V_i, $i \geq 0$, of the infinite main trunk and of *periodic sequences* $T_{i,k}$, $i \geq 0$, $1 \leq k \leq 3$, of non-metabelian *Schur σ-groups*. Except $V_0 = \langle 6561, 23 \rangle$ also $V_0 = \langle 6561, 25 \rangle$ can be selected as root, since both descendant trees are isomorphic as directed graphs.

FIGURE 11. Periodic Trifurcations and Schur σ-Groups for $\tau_0 = (22)$



Remark. It will be rewarding to compare for which orders the *smallest metabelian* Schur σ-groups G with relation rank $d_2(G) = 2$ set in for various types of bicyclic abelian quotient invariants G/G':

- For $G/G' \simeq (3,3)$ at $\#G = 3^5 = 243$,
- for $G/G' \simeq (9,3)$ at $\#G = 3^6 = 729$,
- for $G/G' \simeq (27,3)$ at $\#G = 3^7 = 2\,187$,
- for $G/G' \simeq (81,3)$ at $\#G = 3^8 = 6\,561$, and finally, according to Figure 10,
- for $G/G' \simeq (9,9)$ at $\#G = 3^{10} = 59\,049$.

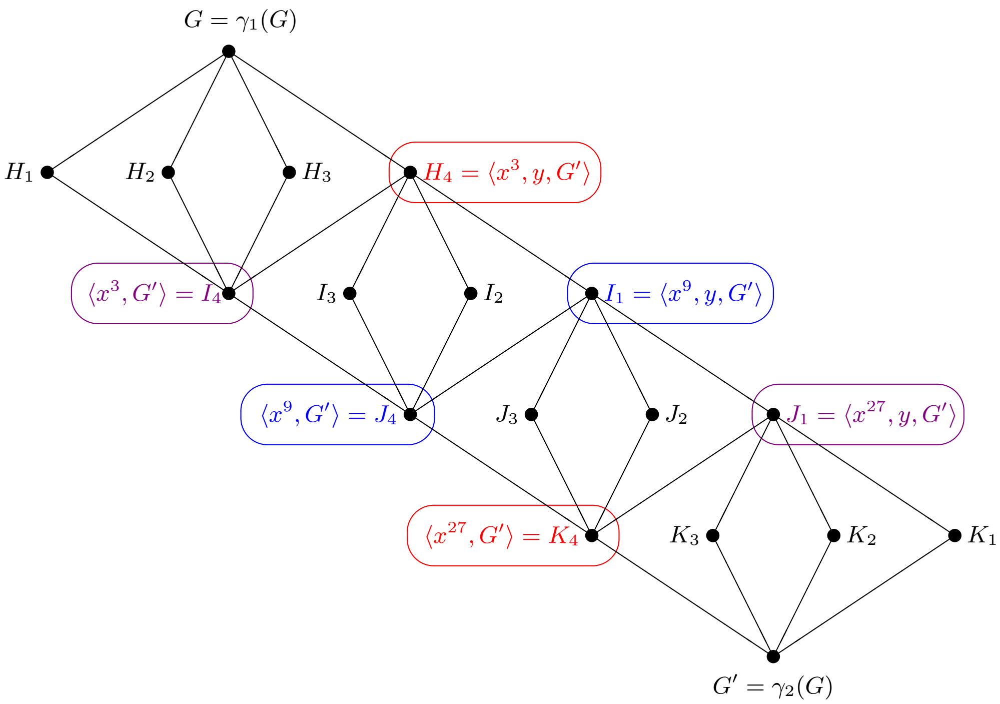
§ 4.3. Bi-hetero-cyclic situation with elementary component

In this situation of a finite non-abelian 3-group G with abelian quotient invariants G/G' of exemplary type $(81,3)\hat{=}(41)$, the permutation of normal subgroups $G' \le U_i \le G$ associated with harmonically balanced capitulation (briefly HBC) is described in the following way.

Proposition. G possesses **harmonically balanced capitulation** if and only if

(1) $(\exists\,\rho_1 \in \mathfrak{S}_3)\ (\forall\,1 \le i \le 3)\ \ker(T_{G,H_i}) = K_{\rho_1(i)}$,
(2) $\ker(T_{G,H_4}) = K_4$,
(3) $(\exists\,\lambda_2 \in \mathfrak{S}_2)\ (\forall\,2 \le k \le 3)\ \ker(T_{G,I_k}) = J_{\lambda_2(k)}$,
(4) $\ker(T_{G,I_1}) = J_4$ and $\ker(T_{G,J_4}) = I_1$,
(5) $(\exists\,\lambda_3 \in \mathfrak{S}_2)\ (\forall\,2 \le k \le 3)\ \ker(T_{G,J_k}) = I_{\lambda_3(k)}$,
(6) $\ker(T_{G,J_1}) = I_4$ and $\ker(T_{G,I_4}) = J_1$,
(7) $(\exists\,\rho_4 \in \mathfrak{S}_3)\ (\forall\,1 \le i \le 3)\ \ker(T_{G,K_i}) = H_{\rho_4(i)}$,
(8) $\ker(T_{G,K_4}) = H_4$.

FIGURE 12. Abelian quotient G/G' of type $\tau_0 = (41)$



As opposed to $\mathrm{Cl}_3(K) \simeq (9,9)$, only 92 cases, i.e. 9.6%, among the 963 imaginary quadratic fields K with 3-class group $\mathrm{Cl}_3(K) \simeq (81,3)$ in the same range $-10^7 < d_K < 0$ of discriminants have HBC at least in the first layer, called capitulation type E.12, $\varkappa_1(K) \sim (1,2,3;4)$. Only 30 of these fields, also 3.1%, possess HBC with respect to all four proper layers.

Obviously there occur three prinzipal scenarios, as the following Table 4 shows. Either

- a *periodic variant* (VAR), or
- a *ground state* (GS) without HBC (except in the first layer), or
- *harmonically balanced capitulation* (HBC) in an excited state.

See also Figure 13.

TABLE 4. 20 absolutely smallest discriminants $d < 0$ of $K = \mathbb{Q}(\sqrt{d})$, $\mathrm{Cl}_3(K) \simeq (81,3)$

No.	$-d$	Scenario	Artin Pattern		Relevant Groups
			$\varkappa_1$	τ_1	
1	$939\,416 = 2^3 \cdot 117\,427$	GS	1342	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
2	$1\,121\,183 = 7 \cdot 160\,169$	GS	1324	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
3	$1\,352\,504 = 2^3 \cdot 169\,063$	HBC	1423	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
4	$1\,702\,455 = 3 \cdot 5 \cdot 113\,497$	HBC	1324	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
5	$1\,833\,635 = 5 \cdot 366\,727$	VAR	1342	$(432)^2, (333)^2$	$\langle 6561, 85/933/934 \rangle$
6	$2\,012\,295 = 3 \cdot 5 \cdot 134\,153$	HBC	1324	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
7	$2\,099\,659 \in \mathbb{P}$	GS	1423	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
8	$2\,483\,204 = 2^2 \cdot 523 \cdot 1\,187$	GS	1234	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
9	$2\,549\,636 = 2^2 \cdot 637\,409$	VAR	1324	$(432)^4$	$\langle 6561, 85/933/934 \rangle$
10	$2\,599\,831 = 13 \cdot 227 \cdot 881$	GS	1342	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
11	$2\,685\,428 = 2^2 \cdot 671\,357$	VAR	1432	$(432)^4$	$\langle 6561, 85/933/934 \rangle$
12	$2\,789\,287 = 31 \cdot 89\,977$	VAR	1243	$(432)^3, 333$	$\langle 6561, 85/933/934 \rangle$
13	$2\,789\,848 = 2^3 \cdot 348\,731$	VAR	1243	$(432)^4$	$\langle 6561, 85/933/934 \rangle$
14	$2\,858\,199 = 3 \cdot 101 \cdot 9\,433$	VAR	1324	$(432)^4$	$\langle 6561, 85/933/934 \rangle$
15	$2\,914\,147 \in \mathbb{P}$	HBC	1432	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
16	$2\,934\,539 = 29 \cdot 47 \cdot 2\,153$	HBC	1243	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
17	$3\,139\,239 = 3 \cdot 1\,046\,413$	HBC	1324	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
18	$3\,145\,255 = 5 \cdot 17 \cdot 37\,003$	GS	1234	$(421)^2, (322)^2$	$\langle 6561, 76/917/918 \rangle$
19	$3\,447\,467 = 439 \cdot 7\,853$	HBC	1423	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$
20	$3\,589\,127 = 23 \cdot 29 \cdot 5\,381$	HBC	1324	$(421)^3, 322$	$\langle 6561, 76/917/918 \rangle$

§ 4.4. **Sporadic 3-groups of type** $\tau_0 = (41)$

Theorem. Among the finite 3-groups with maximal abelian quotient $\tau_0 = (41)$ there are exactly two **metabelian** Schur σ-groups,

$\langle 6561, 975\rangle$ with $\tau_1 = (51, 51, 411; 411)$, $\varkappa_1 = (K_1, K_3, K_4; K_3)$,
$\tau_2 = (311; 41, 41; 411)$, $\varkappa_2 = (J_1; J_1, J_1; J_1)$,
$\tau_3 = (211; 31, 31; 311)$, $\varkappa_3 = (I_1; I_1, I_1; I_1)$,
$\tau_4 = (21; 111, 21, 211)$, $\varkappa_4 = (H_4; H_4, H_4, H_4)$;

$\langle 6561, 976\rangle$ with $\tau_1 = (51, 51, 411; 411)$, $\varkappa_1 = (K_1, K_2, K_4; K_2)$,
$\tau_2 = (311; 41, 41; 411)$, $\varkappa_2 = (J_1; J_1, J_1; J_1)$,
$\tau_3 = (211; 31, 31; 311)$, $\varkappa_3 = (I_1; I_1, I_1; I_1)$,
$\tau_4 = (21; 21, 111, 211)$, $\varkappa_4 = (H_4; H_4, H_4, H_4)$.

None of them possesses harmonically balanced capitulation.

Theorem. The smallest Schur σ-group with harmonically balanced capitulation is **non-metabelian** of derived length $\mathrm{dl} = 3$, has order $3^{17} = 129\,140\,163$, and has the Artin-pattern

$\tau_1 = (51, 51, 51; 321)$, $\varkappa_1 = (K_1, K_2, K_3; K_4)$,
$\tau_2 = (322; 421, 421; 421)$, $\varkappa_2 = (J_4; J_2, J_3; J_1)$,
$\tau_3 = (431; 431, 431; 332)$, $\varkappa_3 = (I_4; I_2, I_3; I_1)$,
$\tau_4 = (54; 54, 54, 432)$, $\varkappa_4 = (H_3; H_1, H_2, H_4)$.

This Theorem will be illuminated by the following detailed statement.

Theorem. In the descendant tree of the root $W = \langle 6561, 76\rangle -2;3$ resp. $W = \langle 6561, 76\rangle - 2;5$ there exists an infinite subtree of σ-groups, having maximal abelian quotient $C_{81} \times C_3$, along a main trunk

$$(9) \qquad \bigl(W - 2;1[-2;1 - 1;1]^n\bigr)_{n\ge 0}$$

with alternating step sizes $s = 2$ and $s = 1$ (i.e. with periodic bifurcations), which contains an infinite periodic sequence of non-metabelian Schur σ-groups

$$(10) \qquad \bigl(G_n = W - 2;1[-2;1 - 1;1]^n - 2;2\bigr)_{n\ge 1}$$

with order $|G_n| = 3^{14+3n}$ and harmonically balanced capitulation $\varkappa(G_n)$. See Figure 13.

The **exception** $G_0 = W - 2;1 - 2;2$ is also a non-metabelian Schur σ-group, but $\varkappa(G_0)$ is **not** a harmonically balanced capitulation.

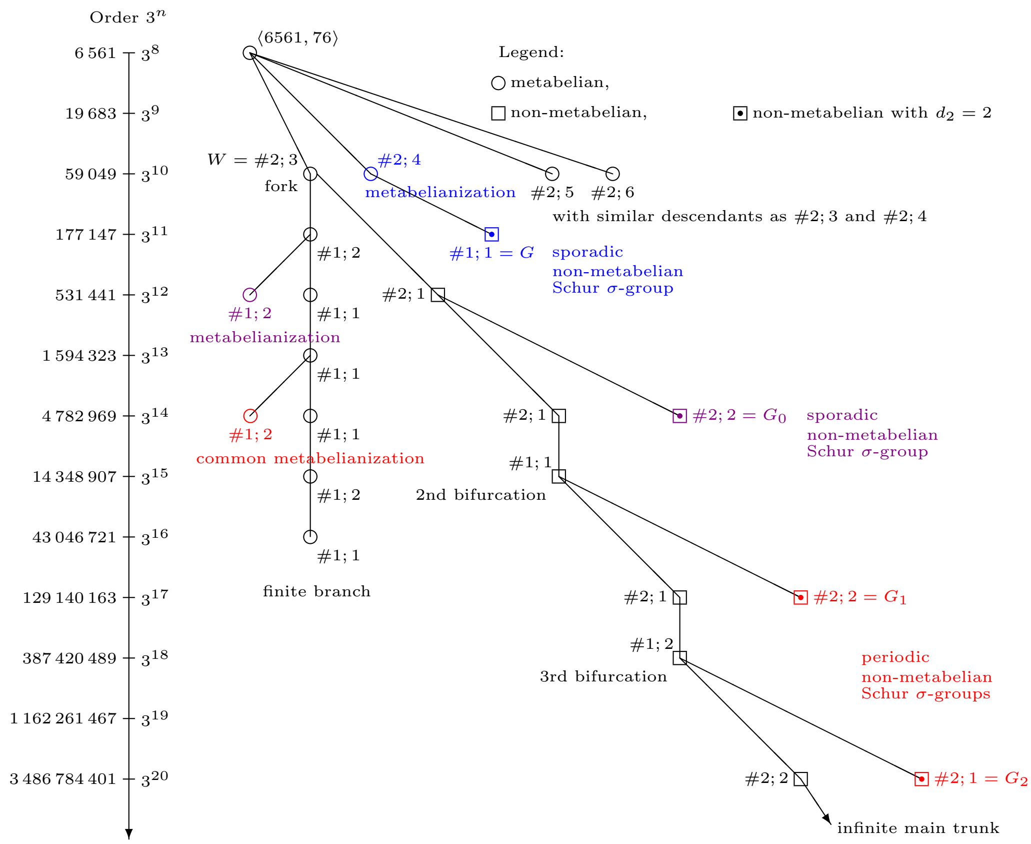
The metabelianization of these Schur σ-groups is

$$M = G_n/G_n'' \simeq \begin{cases} W - 1;2 - 1;2 & \text{for } n = 0, \\ W - 1;2 - 1;1 - 1;1 - 1;2 & \text{for } n \ge 1. \end{cases}$$

The positions of M and G_n are always connected by a **furcation topology** with fork W.

A **further exception** is the smallest Schur σ-group, $G = \langle 6561, 76\rangle - 2;4 - 1;1$ resp. $G = \langle 6561, 76\rangle - 2;6 - 1;1$, which shares at least the capitulation $\varkappa_1(G)$ in the first layer with the mentioned periodic sequence. It is also non-metabelian, with order 3^{11}, $\varkappa(G)$ is **not** a harmonically balanced capitulation, and its metabelianization coincides with its predecessor $M = \langle 6561, 76\rangle - 2;4$ resp. $M = \langle 6561, 76\rangle - 2;6$, which is connected with G by a **descendant topology**.

FIGURE 13. Periodic bifurcations and Schur σ-groups for $\tau_0 = (41)$



Example.

The absolutely smallest discriminant d_K with HBC and $\mathrm{Gal}(\mathrm{F}_3^2(K)/K) \simeq W - 1;2 - 1;1 - 1;1 - 1;2$ of order 3^{14} is

$$-1\,352\,504 = -2^3 \cdot 169\,063.$$

CHAPTER 5.
CURRENT LIMITS

(Close to and Beyond)

§ 5.1. **Close to the Limits**

In 2007, Bartholdi and Bush constructed an infinite sequence of non-metabelian Schur σ-groups $\mathfrak{G}$, having unbounded derived lenght $\mathrm{dl}(\mathfrak{G}) \geq 3$. One of these must arise as Galois group of the entire 3-class field tower for each imaginary quadratic field in the following situation. However, they expressed doubts whether these groups can be distinguished computationally. For the smallest relevant discriminants, I succeeded in identifying the group of lowest order 3^8 and TKT H.4 with the aid of Magma (2015, Ref. 18, see Fig. 7).

Theorem. (Multi-layered ATI of second order for TKT H.4.) Let K be an **imaginary** quadratic number field of type $\mathrm{Cl}_3(K) \simeq (3,3) \simeq 11$, and denote by $L_1,\ldots,L_4$ the four unramified cyclic cubic extensions of K. Then the second 3-class group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ and the full 3-class tower group $\mathfrak{G} = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of K can be identified under the following conditions:

(1) $\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [11; 111, 111, 21, 111] \Longrightarrow G \simeq \langle 729, 45\rangle$ and $\varkappa_1(K) = (4443)$ (that is TKT H.4). Now assume that $G \simeq \langle 729, 45\rangle$, then

(2) $[\tau_0(L_3); \tau_1(L_3); \tau_2(L_3)] = [21; 211, 31, 31, 31; 221, \mathbf{22}, \mathbf{22}, \mathbf{22}] \Longrightarrow \mathfrak{G} \simeq \langle 6561, 606\rangle$ and $\ell_3(K) = 3$ (that is a 3-class field tower with precisely three stages).

(3) $[\tau_0(L_3); \tau_1(L_3); \tau_2(L_3)] = [21; 211, 31, 31, 31; 221, \mathbf{32}, \mathbf{32}, \mathbf{32}] \Longrightarrow$ Any length $\ell_3(K) \geq 3$ may occur and $\mathfrak{G}$ has order at least 3^{11}.

L_1, L_2 and L_4 may also be used for identifying $\mathfrak{G}$ by means of their multi-layered ATI of second order, which, however, contain tridecuplets instead of quartets, and thus are computationally more expensive.

Example. Among the 2020 imaginary quadratic fields K of type $\mathrm{Cl}_3(K) \simeq (3,3)$ with fundamental discriminants in the range $-10^6 < d_K < 0$ (Ref. 13, 14), $G \simeq \langle 729, 45\rangle$ occurs for 297 fields (14.7%), for instance the minimal discriminant $d_K = -3896$, and $\mathfrak{G} \simeq \langle 6561, 606\rangle$, $\ell_3(K) = 3$ has been proved to occur for $d_K = -3896, -25447, -27355$ but definitely disproved for $d_K = -6583, -23428, -27991$.

For the achievements of the preceding results, Magma must be squeezed close to its limits by computing 3-class groups of number fields in the second layer above L_3 and thus of absolute degree $6 \cdot 9 = 54$.

So, for the solution of these tasks, the **class field theoretic** routines of Magma must be employed.

§ 5.2. **Beyond the Limits**

Example 1. For the imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-4447704, -5067967, -8992363\}$, the 3-class group is of type $(3, 3, 3) \simeq 111$. Thus, they necessarily have an infinite 3-class field tower, according to Ref. 10. These fields share the common IPAD $\tau^{(1)}(K) = [111; 3221, (21111)^5, (2211)^7]$. For finding a metabelian 3-group M with $\tau^{(1)}(M) = \tau^{(1)}(K)$, the **group theoretic** routines of Magma must be employed. This leads to the relevant root $R = \langle 729, 122\rangle$, which has nuclear rank $\nu(R) = 8$ and thus possible step sizes $1 \le s \le 8$. When I tried to construct its descendants of step size $s = 3$ on a machine with an unusual amount of 256GB kernel storage, I ended up with a RAM overflow. I asked John Cannon, the leader of the Magma developer group, and he told me that his machines in Sydney fail at the same point. So this problem is generally outside of the current scope of computations.

Example 2. Recently, Magnus Carlson from the Royal Technical Highschool in Stockholm, Sweden, drew my attention to the following imaginary quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with five prime divisors of the fundamental discriminant $d_K = 4 \cdot D$ and 2-class groups of type $(2, 2, 2, 2) \simeq 1111$. The radicands have the shape $D = -273 * p$ with $p \in \{5, 4373, 8741, 157253, 279557\}$. These five fields share a common IPAD $\tau^{(1)}(K) = [1111; (2211)^2, (2111)^2, 222, (221)^8, (321)^2]$ and I tried to find metabelian 2-groups M with $\tau^{(1)}(M) = \tau^{(1)}(K)$. To this end, I started with all finite 2-groups M in the SmallGroups Database of Magma having abelianization of type $M/M' \simeq (2, 2, 2, 2) \simeq 1111$ and orders $16 \le \#(M) \le 512$ and constructed their descendants of step sizes $1 \le s \le 3$ searching for the desired IPAD. I checked more than five million 2-groups of orders from 16 to 1024 for this IPAD without finding a single hit. The systematic and complete construction of orders 2048 and 4096 is impossible, since Magma unfortunately has some intrinsic upper bounds for cardinalities, which terminate $s = 2$ at $\langle 512, 254239\rangle$ and $s = 3$ at $\langle 512, 159023\rangle$. Due to the similarities with the computational barreers in Example 1, I believe that the class field tower of the five fields cannot be finite. In particular, I conjecture that $K = \mathbb{Q}(\sqrt{-1365})$ has an infinite 2-class field tower.

§ 5.3. **Open Problems.**

(1) As indicated in § 0.2, abelian type invariants of order up to 4 are expected to be computable for unramified p-extensions, for the smallest prime $p = 2$. Since no p-class field **towers of exact length four** are known, it would be rewarding and promising to search for such towers over imaginary quadratic fields with 2-class groups of type $(2, 4)$ or $(2, 2, 2)$. (For type $(2, 2)$, all towers are at most of length two.)

(2) The knowledge about the **structure of infinite p-class field towers** is very poor. In the preceding § 5.2, I have explained that there is no chance to find even the second stage by using the p-group generation algorithm, since the relevant roots have unmanageable nuclear ranks and descendant numbers. Here, we have the discrepancy that number theoretic information on the Artin pattern can be obtained without problems, but the group theoretic search does not yield any hit. Maybe it would be possible to try a realization of these Artin patterns by constructing and testing suitable presentations for p-groups outside of the SmallGroups library.

(3) An important project with general group theoretic relevance is the investigation of coclass trees of 3-groups with abelianizations of the types $(3, 3^n)$ with $n \geq 2$. The p-group generation algorithm uses the lower exponent-p central series, which increasingly deviates from the **usual lower central series** for bigger values of $n \geq 2$. So it fails to construct the coclass trees which are fundamental for a deeper structural understanding. Possible work around would be to use either **isoclinism** or **parametrized presentations**.

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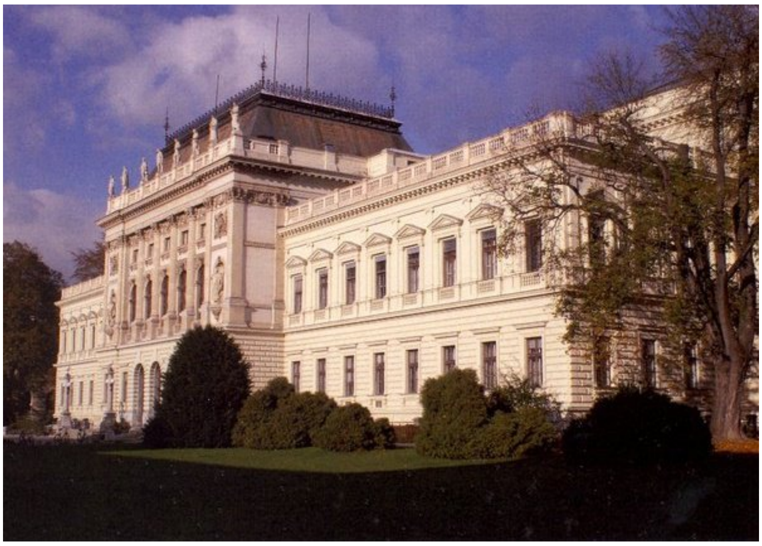
FWF Der Wissenschaftsfonds.



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8 Stone Figures above the magnificent Aula, 37 years ago,
05 July 1983: my Promotion to Philosophiae Doctor, “Spondeo ac Polliceor”

D. C. Mayer, *Harmonically balanced capitulation over quadratic fields of type* $(9, 9)$, arXiv:1908.01982v1, 06 August 2019.