FWF Der Wissenschaftsfonds.

Deep Transfers of p-Class Tower Groups

Conference:	3rd International Conference on Groups and Algebras (ICGA Sanya 2018)
Place:	International Asia-Pacific Convention Center
Venue:	Sanya, Hainan, China
Date:	January 13 – 15, 2018
Author:	Daniel C. Mayer (Graz, Austria)
Affiliation:	Austrian Science Fund (FWF)

A presentation within the frame of the international scientific research project

Towers of p-Class Fields over Algebraic Number Fields

supported by the Austrian Science Fund (FWF): P 26008-N25

I. GROUP THEORETIC FOUNDATIONS

1. Finite 3-groups G with coclass $\mathrm{cc}(G) = 1$

They possess maximal possible *nilpotency class* $c := \mathrm{cl}(G)$.

Let the lower central series of G be defined by
$\gamma_1 G := G$ and $\gamma_i G := [\gamma_{i-1}G, G]$ for $2 \le i \le c+1$,
such that $\gamma_{c+1}G = 1$ (i.e., $c+1$ is the *index of nilpotency*).
If $|G| = 3^n$ and
the factors $\gamma_i G/\gamma_{i+1}G \simeq C_3$ are cyclic for $2 \le i \le c$
(i.e., of minimal possible order),
then $3^n = (G : \gamma_2 G) \cdot \prod_{i=2}^{c} (\gamma_i G : \gamma_{i+1} G) = (G : \gamma_2 G) \cdot 3^{c-1}$.
If G is non-abelian, then $G/\gamma_2 G$ cannot be cyclic,
whence $(G : \gamma_2 G) \ge 3^2$, $n \ge 2 + c - 1 = c + 1$,
and the maximal possible value of $c \le n - 1$ is $n - 1$.
The difference $\mathrm{cc}(G) := n - \mathrm{cl}(G)$ is called the *coclass* of G.
Here we have $\mathrm{cc}(G) = n - (n-1) = 1$

Since $c = n - 1$ implies $G/\gamma_2 G \simeq C_3 \times C_3$, the group
$G = \langle x, y \rangle$ has two generators, according to W. Burnside.

Proposition. (Parametrized polycyclic pc-presentation.)
Let $s_2 = [y, x]$, $s_i = [s_{i-1}, x]$ for $3 \le i \le n$, $s_{n-1} \ne 1 = s_n$,
then a template for infinitely many presentations
of all finite non-abelian 3-groups G with coclass $\mathrm{cc}(G) = 1$
is due to N. Blackburn[(4)] (1958): G is isomorphic to
$G_a^n(z, w) = \langle x, y \mid x^3 = s_{n-1}^w,\ y^3 s_2^3 s_3 = s_{n-1}^z,\ [y, s_2] = s_{n-1}^a \rangle$,
where the parameters
$a \in \{0, 1\}$, $w, z \in \{-1, 0, 1\}$ are bounded,
but the index of nilpotency $n \ge 3$ is unbounded.

Two-step centralizer of $G' = \gamma_2 G$ (commutator subgroup):

$$\chi_2 G/\gamma_4 G := \mathrm{centralizer}_{G/\gamma_4 G}(\gamma_2 G/\gamma_4 G),$$

that is, $\chi_2 G$ is the biggest subgroup of G such that

$$[\chi_2 G, \gamma_2 G] \le \gamma_4 G,$$

introduced by A. WIMAN[19] (1952), studied by Blackburn.

Lemma. If $G = \langle x, y\rangle \simeq G_a^n(z,w)$ is a finite 3-group with coclass $\mathrm{cc}(G) = 1$ and $n \ge 4$, given in Blackburn presentation, then $\chi_2 G = \langle y, G'\rangle =: H_1$ is a maximal subgroup of G and

$$\gamma_{n-k} G := [H_1, G'] = H_1' = \begin{cases} 1 = \gamma_{n-0} G & \text{if } a = 0, \\ \zeta(G) = \gamma_{n-1} G & \text{if } a = 1. \end{cases}$$

The invariant $k = a$ is called *defect of commutativity* of G.

Corollary. There exist four maximal subgroups of G,

$$H_1,\ H_2 = \langle x, G'\rangle,\ H_3 = \langle xy, G'\rangle,\ H_4 = \langle xy^2, G'\rangle.$$

Whereas the commutator quotient of the two-step centralizer[1] depends on $n \ge 4$ and k and has order $|H_1/H_1'| \ge 27$,

$$H_1/H_1' \simeq \begin{cases} C_3 \times C_3 \times C_3 & \text{if } n = 4,\ z = 1,\ w = 0, \\ A(3, n-1-k) & \text{otherwise } (\textbf{Polarization}), \end{cases}$$

the other abelianizations are constant:

$(\forall\, 2 \le j \le 4) \quad H_j/H_j' \simeq C_3 \times C_3$ (**Stabilization**).

Theorem 1.[13] Let $G \simeq G_a^n(z,w)$ be a finite 3-group with coclass $\mathrm{cc}(G) = 1$ and order $|G| = 3^n$, $n \ge 4$.

Those 3 maximal subgroups H_j, $2 \le j \le 4$, of G, which are distinct from the two-step centralizer $H_1 = \chi_2 G$, are also finite 3-groups with coclass $\mathrm{cc}(H_j) = 1$.

Each of them is isomorphic to either $G_0^{n-1}(0, \mathbf{0})$ or $G_0^{n-1}(0, \mathbf{1})$.

Details are given in the Main Theorem of § 2.

---

1 $A(3, 2u)$, resp. $A(3, 2u+1)$, denotes the nearly homocyclic abelian 3-group $C_{3^u} \times C_{3^u}$, resp. $C_{3^{u+1}} \times C_{3^u}$.

Proof. (of Theorem 1)
In fact, this proof is valid for any metabelian p-group G with coclass $\mathrm{cc}(G) = 1$, where p denotes an arbitrary prime number.

• Let H be a maximal subgroup of $G \simeq G_a^n(z, w)$ different from the two-step centralizer $H_1 = \chi_2 G = \langle y, G' \rangle$, then H has abelian quotient invariants (AQI) (p, p), i.e. $H/H' \simeq C_p \times C_p$.

Since the abelian commutator subgroup G' of G is a maximal subgroup of the metabelian group H, H must necessarily be of coclass $\mathrm{cc}(H) = 1$. (Metabelian p-groups of bigger coclass with AQI (p, p) do not possess any abelian maximal subgroups.)

• Now denote the generators of G, resp. H, in their presentations according to R. J. Miech(15) (1970) by

$$\begin{array}{rcccccc} & x & y & s_2 & s_3 & \ldots & s_c \\ \text{resp.} & \xi & \upsilon & \sigma_2 & \sigma_3 & \ldots & \sigma_\kappa \end{array}$$

then the lower central series of H coincides with the truncated lower central series of G,

$$(\forall\, 2 \le i \le c) \quad \gamma_i H = \gamma_{i+1} G, \text{ and thus}$$

$\upsilon = s_2$, $\sigma_2 = s_3$, $\sigma_3 = s_4$, ..., $\sigma_\kappa = s_c$ with $\kappa = c - 1$, and $\xi \ne y$ is one of the elements x, xy, xy^2, ..., xy^{p-1}. Then the relations $[\upsilon, \sigma_2] = [s_2, s_3] = 1 = \sigma_\kappa^\alpha$ and[2]

$$\upsilon^{\binom{p}{1}} \sigma_2^{\binom{p}{2}} \ldots \sigma_p^{\binom{p}{p}} = s_2^{\binom{p}{1}} s_3^{\binom{p}{2}} \ldots s_{p+1}^{\binom{p}{p}} = 1 = \sigma_\kappa^\zeta$$

show that the parameters in the presentation $H = G_\alpha^\nu(\zeta, \omega)$ are given by $\alpha = 0$, $\nu = n - 1$, $\zeta = 0$ and $\omega \in \{0, 1\}$.[3] □

---

2 For $p = 3$, these power relations are $\upsilon^3 \sigma_2^3 \sigma_3 = s_2^3 s_3^3 s_4 = 1$.

3 All groups $G_0^{n-1}(0, \omega)$ with $\omega \ne 0$ and fixed n are isomorphic.

2. Group theoretic main result

Main Theorem.[13]
Let $G = \langle x, y\rangle \simeq G_a^n(z, w)$ be a finite 3-group of coclass $\mathrm{cc}(G) = 1$ and order $|G| = 3^n$ with generators x, y such that $y \in \chi_2 G$ and $x \in G \setminus \chi_2 G$, given by a polycyclic power commutator presentation with parameters $a \in \{0, 1\}$, $w, z \in \{-1, 0, 1\}$, and index of nilpotency $n \geq 4$ (i.e., class $c = n - 1 \geq 3$).

Then three of the four maximal subgroups, $H_j = \langle xy^{j-2}, G'\rangle < G$, $2 \leq j \leq 4$, are non-abelian 3-groups of coclass $\mathrm{cc}(H_j) = 1$, as listed in Table 1 in dependence on the parameters n, a, z, w.

Table 1. Non-abelian maximal subgroups H_j, $2 \leq j \leq 4$, of 3-groups G of coclass 1

	$G \simeq$	n	a	z	w	$H_2 = \langle x, G'\rangle$	$H_3 = \langle xy, G'\rangle$	$H_4 = \langle xy^2, G'\rangle$	$N(0)$
d	$G_0^n(0,0)$	≥ 4	0	0	0	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,\mathbf{0})$	3
q	$G_0^n(0,\mathbf{1})$	≥ 4	0	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	0
s	$G_0^n(\mathbf{1},0)$	≥ 4	0	1	0	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	1
s	$G_0^n(-\mathbf{1},0)$	≥ 4	0	-1	0	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	1
	$G_1^n(0,-\mathbf{1})$	≥ 5	1	0	-1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,\mathbf{0})$	2
	$G_1^n(0,\mathbf{0})$	≥ 5	1	0	0	$\simeq G_0^{n-1}(0,\mathbf{0})$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	1
	$G_1^n(0,\mathbf{1})$	≥ 5	1	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	0

$N(0)$ denotes the number of groups $G_0^{n-1}(0,\mathbf{0})$ among H_2, H_3, H_4.

Our particular attention will be devoted to the **deficient groups** with positive defect $k = a = 1$ (red color)[4].

4 For a fixed value of $n \geq 5$, these groups are isoclinic in the sense of Hall[7] and Easterfield.[6]

The supplementary Table 2 shows
the abelian maximal subgroups
of the remaining two extra special 3-groups
of coclass $\mathrm{cc}(G) = 1$, class $c = n - 1 = 2$, and order $|G| = 3^3$.

TABLE 2. Abelian maximal subgroups H_j, $1 \le j \le 4$, of extra special 3-groups G

$G \simeq$	n	a	z	w	$H_1 = \langle y, G' \rangle$	$H_2 = \langle x, G' \rangle$	$H_3 = \langle xy, G' \rangle$	$H_4 = \langle xy^2, G' \rangle$
$G_0^3(0,0)$	3	0	0	0	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$
$G_0^3(0,1)$	3	0	0	1	$\simeq C_3 \times C_3$	$\simeq C_9$	$\simeq C_9$	$\simeq C_9$

Result: According to the Main Theorem, the number $N(0)$ of groups $G_0^{n-1}(0,\mathbf{0})$ among H_2, H_3, H_4 is able to characterize the groups $G_1^n(0,w)$ with positive defect of commutativity $k = a = 1$ and any $n \ge 5$ unambiguously:

$$N(0) = \begin{cases} 2 & \text{if } w = -1, \\ 1 & \text{if } w = 0, \\ 0 & \text{if } w = 1. \end{cases}$$

Question: Can we distinguish the
two possible maximal subgroups,
the dihedron-related group $G_0^{n-1}(0,\mathbf{0})$ and
the quaternion-related group $G_0^{n-1}(0,\mathbf{1})$,
by some invariant which is accessible to direct computation?

3. Shallow and deep Schur transfers

These homomorphisms were introduced by **I. Schur**(17) (1902).

Let G be a finite 3-group with $\mathrm{cc}(G) = 1$. Then G is metabelian with $G'' = 1$ and abelian commutator subgroup G', according to Blackburn.(3)

Shallow transfers[5] to the maximal subgroups ($1 \le j \le 4$):

$$T_{G,H_j} : G/G' \to H_j/H_j',\ g{\cdot}G' \mapsto \begin{cases} g^3 & \text{if } g \in G \setminus H_j, \\ g^{1+h+h^2} & \text{if } g \in H_j, \end{cases}$$

where $h \in G \setminus H_j$ is arbitrary.

Proposition. If $n \ge 5$ and $G \simeq G_1^n(0,w)$, then $\ker(T_{G,H_j}) = G/G'$ for $1 \le j \le 4$, independently of w.
$\Longrightarrow$ Shallow transfers do not identify G unambiguously.

Deep transfers ($1 \le j \le 4$) to the commutator subgroup:

$$T_{H_j,G'} : H_j/H_j' \to G',\ g \cdot H_j' \mapsto \begin{cases} g^3 & \text{if } g \in H_j \setminus G', \\ g^{1+h+h^2} & \text{if } g \in G', \end{cases}$$

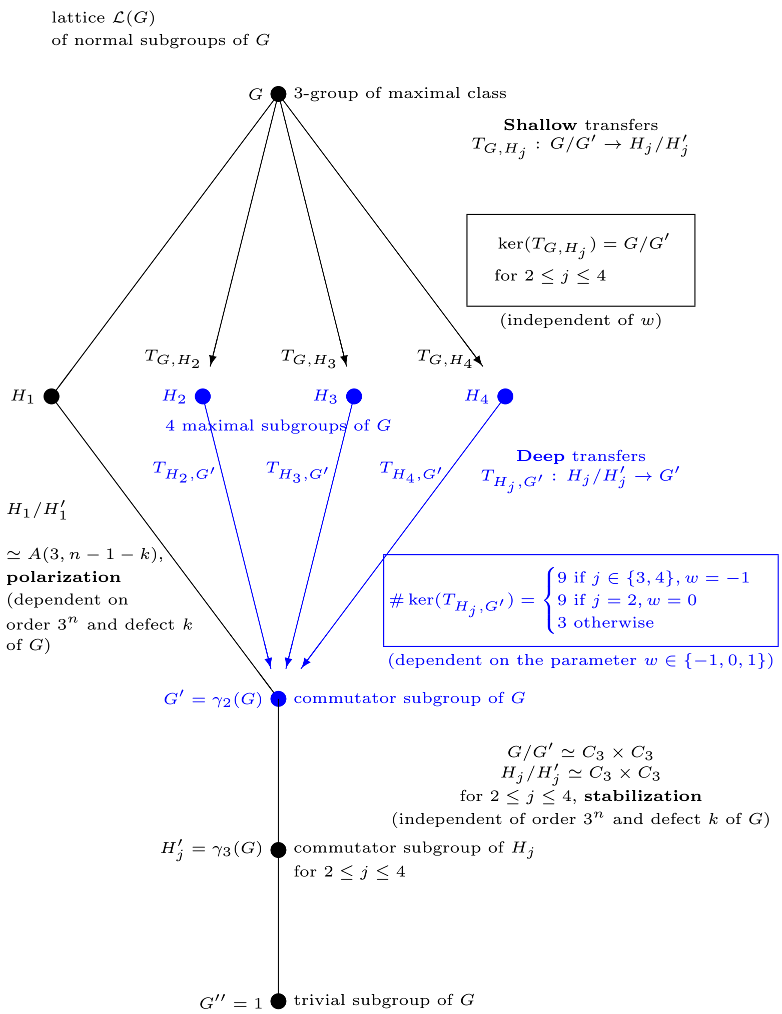
where $h \in H_j \setminus G'$ is arbitrary.

Theorem 2.(14) If $n \ge 5$ and $G \simeq G_1^n(0,w)$, then

$$\#\ker(T_{H_j,G'}) = \begin{cases} 9 \text{ if } j \in \{3,4\},\ w = -1, \\ 9 \text{ if } j = 2,\ w = 0, \\ 3 \text{ otherwise,} \end{cases}$$

in dependence on the parameter $w \in \{-1,0,1\}$.
$\Longrightarrow$
Deep transfers admit the unambiguous identification of G.

5 Schur did not distinguish between shallow and deep transfers.

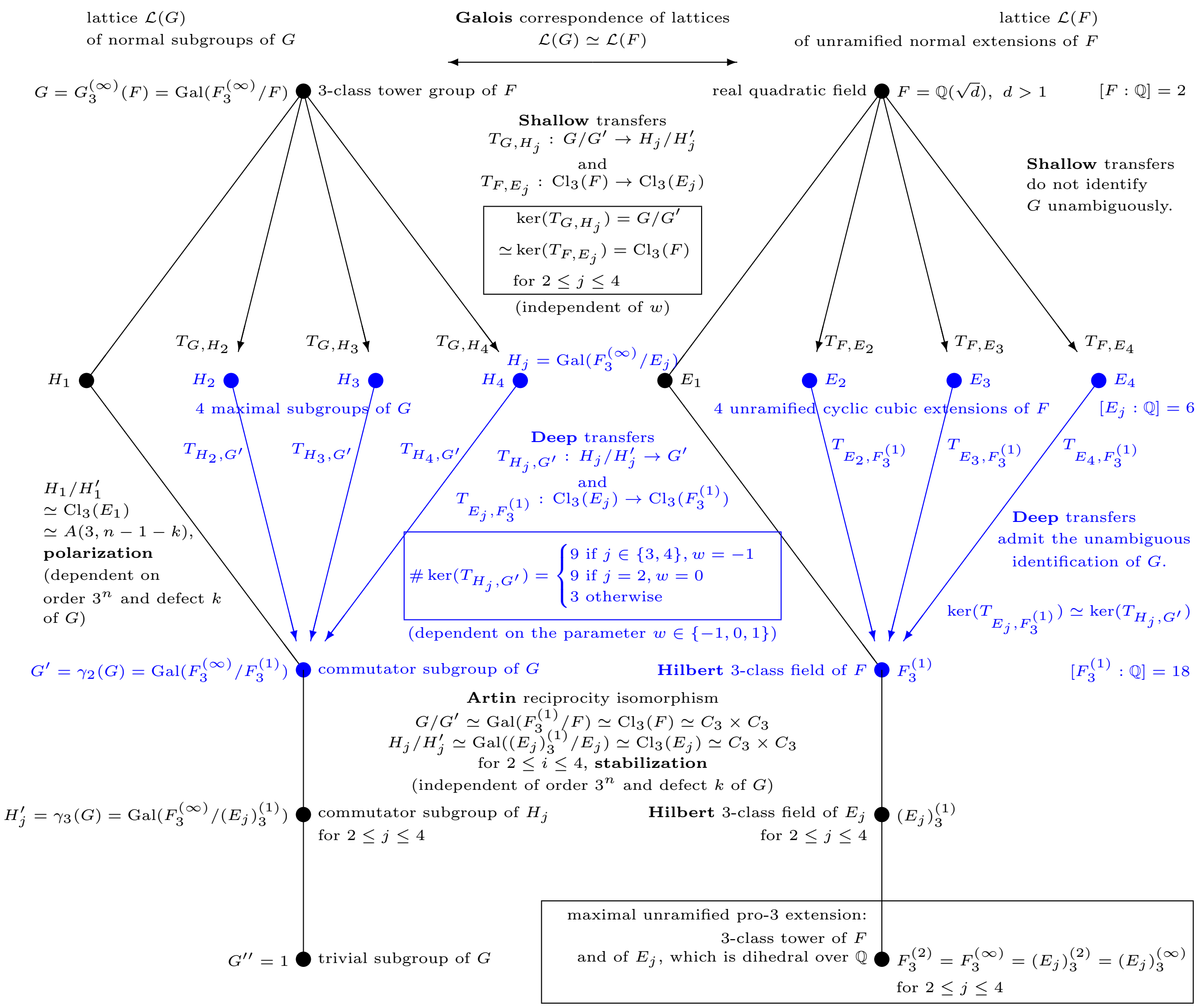


Result: According to Theorem 2 in § 3, the number $N(0)$ of total deep Schur transfer kernels $\# \ker(T_{H_j,G'}) = 9$ among the maximal subgroups H_j, $2 \le j \le 4$, of the stabilization is able to characterize the deficient groups $G_1^n(0, w)$ with positive defect $k = a = 1$ and any $n \ge 5$ unambiguously:

$$N(0) = \begin{cases} 2 & \text{if } w = -1, \\ 1 & \text{if } w = 0, \\ 0 & \text{if } w = 1. \end{cases}$$

Question: Can we translate the kernels of
deep Schur transfers from group theory to number theory?

II. NUMBER THEORETIC APPLICATIONS


lattice $\mathcal{L}(G)$
of normal subgroups of G
Galois correspondence of lattices
$\mathcal{L}(G) \simeq \mathcal{L}(F)$
lattice $\mathcal{L}(F)$
of unramified normal extensions of F
$G = G_3^{(\infty)}(F) = \mathrm{Gal}(F_3^{(\infty)}/F)$
3-class tower group of F
real quadratic field
$F = \mathbb{Q}(\sqrt{d}),\ d > 1$
$[F : \mathbb{Q}] = 2$
Shallow transfers
$T_{G,H_j} : G/G' \to H_j/H_j'$
and
$T_{F,E_j} : \mathrm{Cl}_3(F) \to \mathrm{Cl}_3(E_j)$
$\ker(T_{G,H_j}) = G/G'$
$\simeq \ker(T_{F,E_j}) = \mathrm{Cl}_3(F)$
for $2 \le j \le 4$
(independent of w)
Shallow transfers
do not identify
G unambiguously.
T_{G,H_2}
T_{G,H_3}
T_{G,H_4}
$H_j = \mathrm{Gal}(F_3^{(\infty)}/E_j)$
T_{F,E_2}
T_{F,E_3}
T_{F,E_4}
H_1
H_2
H_3
H_4
E_1
E_2
E_3
E_4
4 maximal subgroups of G
4 unramified cyclic cubic extensions of F
$[E_j : \mathbb{Q}] = 6$
$T_{H_2,G'}$
$T_{H_3,G'}$
$T_{H_4,G'}$
Deep transfers
$T_{H_j,G'} : H_j/H_j' \to G'$
and
$T_{E_j,F_3^{(1)}} : \mathrm{Cl}_3(E_j) \to \mathrm{Cl}_3(F_3^{(1)})$
$T_{E_2,F_3^{(1)}}$
$T_{E_3,F_3^{(1)}}$
$T_{E_4,F_3^{(1)}}$
H_1/H_1'
$\simeq \mathrm{Cl}_3(E_1)$
$\simeq A(3, n-1-k)$,
polarization
(dependent on
order 3^n and defect k
of G)
$\# \ker(T_{H_j,G'}) = \begin{cases} 9 \text{ if } j \in \{3,4\}, w = -1 \\ 9 \text{ if } j = 2, w = 0 \\ 3 \text{ otherwise} \end{cases}$
(dependent on the parameter $w \in \{-1, 0, 1\}$)
Deep transfers
admit the unambiguous
identification of G.
$\ker(T_{E_j,F_3^{(1)}}) \simeq \ker(T_{H_j,G'})$
$G' = \gamma_2(G) = \mathrm{Gal}(F_3^{(\infty)}/F_3^{(1)})$
commutator subgroup of G
Hilbert 3-class field of F
$F_3^{(1)}$
$[F_3^{(1)} : \mathbb{Q}] = 18$
Artin reciprocity isomorphism
$G/G' \simeq \mathrm{Gal}(F_3^{(1)}/F) \simeq \mathrm{Cl}_3(F) \simeq C_3 \times C_3$
$H_j/H_j' \simeq \mathrm{Gal}((E_j)_3^{(1)}/E_j) \simeq \mathrm{Cl}_3(E_j) \simeq C_3 \times C_3$
for $2 \le i \le 4$, **stabilization**
(independent of order 3^n and defect k of G)
$H_j' = \gamma_3(G) = \mathrm{Gal}(F_3^{(\infty)}/(E_j)_3^{(1)})$
commutator subgroup of H_j
for $2 \le j \le 4$
Hilbert 3-class field of E_j
for $2 \le j \le 4$
$(E_j)_3^{(1)}$
maximal unramified pro-3 extension:
3-class tower of F
and of E_j, which is dihedral over $\mathbb{Q}$
$G'' = 1$
trivial subgroup of G
$F_3^{(2)} = F_3^{(\infty)} = (E_j)_3^{(2)} = (E_j)_3^{(\infty)}$
for $2 \le j \le 4$

4. Shallow and deep Artin transfers

We consider a **quadratic** field with discriminant $d \neq 1$,

$$F = \mathbb{Q}(\sqrt{d}), \quad [F : \mathbb{Q}] = 2,$$

having an elementary bicyclic 3-class group of rank two,

$$\mathrm{Cl}_3(F) \simeq C_3 \times C_3.$$

By Artin's reciprocity law of class field theory,
F possesses four unramified cyclic cubic extensions,

$$E_j/F, \quad [E_j : F] = 3, \quad 1 \leq j \leq 4,$$

lying within the Hilbert 3-class field $F_3^{(1)}$ of F.

Each E_j is **dihedral** over $\mathbb{Q}$ with degree $[E_j : \mathbb{Q}] = 6$,

$$\mathrm{Gal}(E_j/\mathbb{Q}) \simeq D(6), \quad 1 \leq j \leq 4.$$

We apply Chapter I (on group theoretic foundations)
to the 3-class tower group of F,

$$G = G_3^{(\infty)}(F) = \mathrm{Gal}(F_3^{(\infty)}/F),$$

with four maximal subgroups,

$$H_j = \mathrm{Gal}(F_3^{(\infty)}/E_j), \quad 1 \leq j \leq 4,$$

and we gain information about E. Artin's[(1)]
shallow 3-class transfers $T_{F,E_j} : \mathrm{Cl}_3(F) \to \mathrm{Cl}_3(E_j)$, and
the **deep** 3-class transfers $T_{E_j,F_3^{(1)}} : \mathrm{Cl}_3(E_j) \to \mathrm{Cl}_3(F_3^{(1)})$,
whose kernels coincide with those of the Schur transfers,

$$\ker(T_{F,E_j}) \simeq \ker(T_{G,H_j}), \quad \ker(T_{E_j,F_3^{(1)}}) \simeq \ker(T_{H_j,G'}).$$

According to the following Lemma, $\mathrm{cc}(G) = 1$ implies

$$F_3^{(2)} = F_3^{(\infty)} = (E_j)_3^{(2)} = (E_j)_3^{(\infty)}, \quad 1 \leq j \leq 4.$$

Suppose that p is a prime,
F is an algebraic number field
with non-trivial p-class group $\mathrm{Cl}_p F$,
and E is one of the unramified abelian p-extensions of F.

Then E is a subfield of the Hilbert p-class field $F_p^{(1)}$,
and we show that, even in this general situation,
a finite p-class tower of F
exerts a very severe restriction on the p-class tower of E.

Lemma.[13]
Assume that F possesses a p-class tower $F_p^{(\infty)} = F_p^{(n)}$ of exact length $\ell_p F = n$ for an integer $n \geq 1$. Then $E_p^{(\infty)} = F_p^{(\infty)}$ and $\mathrm{Gal}(E_p^{(\infty)}/E)$ is a subgroup of index $[E : F]$ in $\mathrm{Gal}(F_p^{(\infty)}/F)$.

In particular, if $[E : F] = p$,
then $\mathrm{Gal}(E_p^{(\infty)}/E)$ is a maximal subgroup of $\mathrm{Gal}(F_p^{(\infty)}/F)$.

Proof. According to the assumptions,
and to the ordering theorem of class field theory,
there exists a tower of field extensions

$$F \leq E \leq F_p^{(1)} \leq E_p^{(1)} \leq F_p^{(2)} \leq E_p^{(2)} \leq \ldots$$
$$\ldots \leq F_p^{(n)} \leq E_p^{(n)} \leq F_p^{(n+1)} \leq E_p^{(n+1)} \leq F_p^{(n+2)},$$

where $\ell_p F = n$ enforces the coincidence

$$F_p^{(n)} = E_p^{(n)} = F_p^{(n+1)} = E_p^{(n+1)} = F_p^{(n+2)}.$$

Since $\mathrm{Gal}(F_p^{(n)}/F)/\mathrm{Gal}(F_p^{(n)}/E) \simeq \mathrm{Gal}(E/F)$,
the group index of $\mathrm{Gal}(E_p^{(n)}/E) = \mathrm{Gal}(F_p^{(n)}/E)$ in $\mathrm{Gal}(F_p^{(n)}/F)$
is equal to the field degree $[E : F]$, and
$\mathrm{Gal}(E_p^{(\infty)}/E) = \mathrm{Gal}(E_p^{(n)}/E)$ is a subgroup of index $[E : F]$
in $\mathrm{Gal}(F_p^{(n)}/F) = \mathrm{Gal}(F_p^{(\infty)}/F)$. $\square$

5. Identifiers of finite p-groups

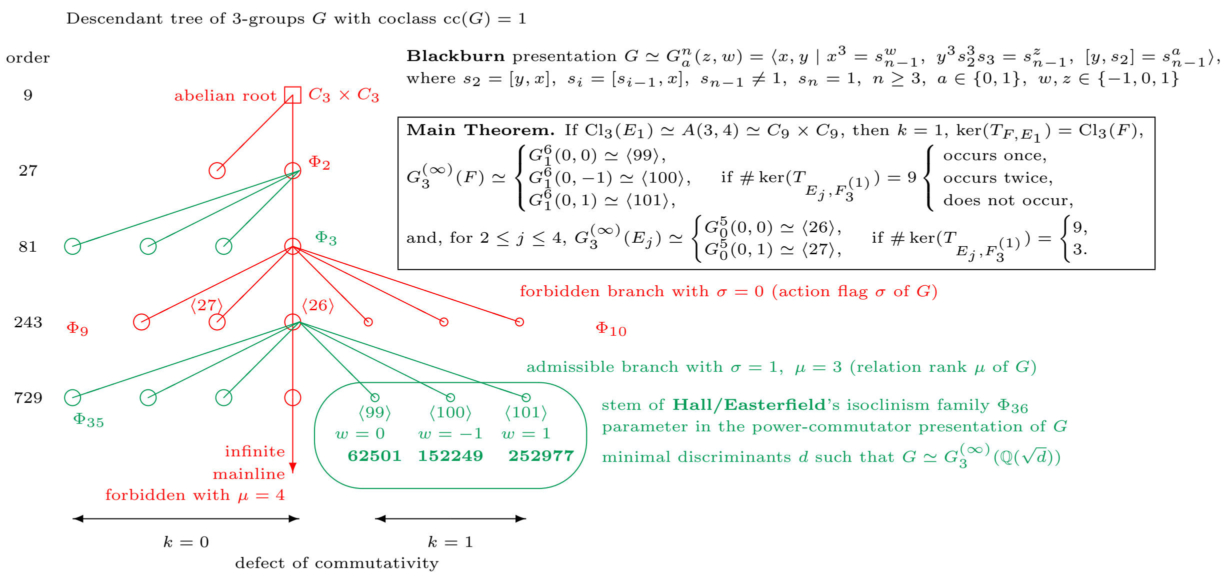
p ... a prime number,
e ... a positive integer (for instance, $e \le 8$ if $p = 3$),
N ... the number of all p-groups G of order $|G| = p^e$.

Absolute Identifiers: For $1 \le i \le N$,
the symbol

$$\langle p^e, i \rangle \quad \text{or briefly} \quad \langle i \rangle$$

denotes the
ith p-group of order p^e,
according to the SmallGroups Database[2] of MAGMA[9]
(returned by the command `IdentifyGroup()`,
provided that $e \le 6$ if $p = 3$).

The symbols $\Phi_2, \Phi_3, \Phi_9, \Phi_{10}$, resp. Φ_{35}, Φ_{36},
denote isoclinism families by Hall[7], resp. Easterfield[6].



6. Coarse information on the groups $G_{\mathbf{1}}^{n}(0,\mathbf{w})$

Throughout the sections §§ 6 to 10,
we assume that $F=\mathbb{Q}(\sqrt{d})$ is a **real** quadratic field with discriminant $d>1$ and 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$.

Furthermore, we denote by E_j/F, $1\le j\le 4$,
the four unramified cyclic cubic extensions of F
within the Hilbert 3-class field $F_3^{(1)}$ of F,
which give rise to **totally real** dihedral absolute extensions $E_j/\mathbb{Q}$ of degree six with torsion free Dirichlet unit rank $r=r_1+r_2-1=6+0-1=5$.

Lemma.[13] (**Coarse** structure:)
If the 3-class group Cl_3E_j is of type $C_3\times C_3$
for some $1\le j\le 4$, then (after a suitable renumeration)

$$(\forall\, 2\le j\le 4)\quad \mathrm{Cl}_3E_j\simeq C_3\times C_3\quad \textbf{(stabilization)}$$

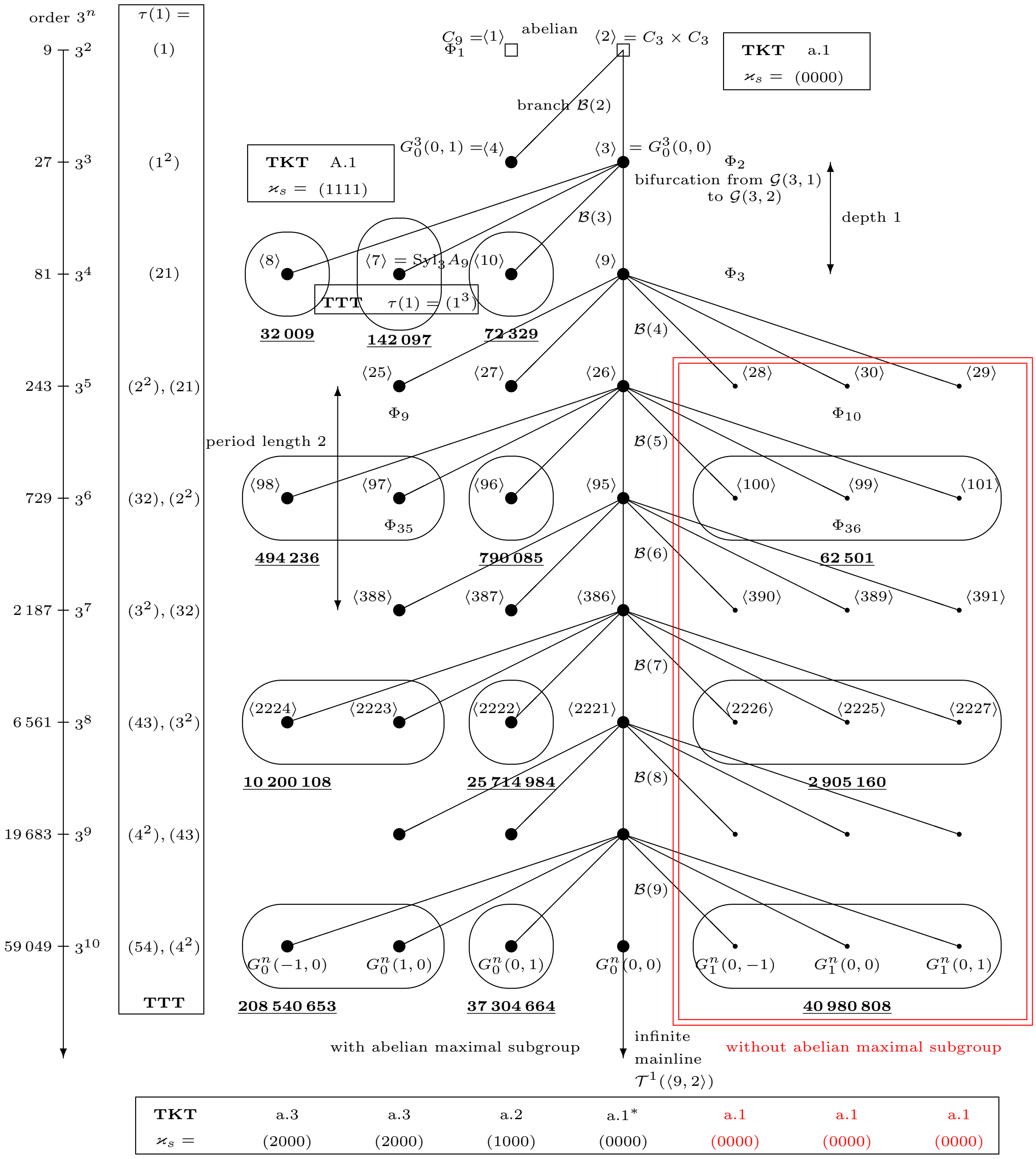
and the 3-tower group $G=G_3^{(\infty)}F$ is of coclass $\mathrm{cc}(G)=1$.

(**Fine** structure:)
If the 3-class group Cl_3E_1 of the distinguished extension E_1,
which corresponds to the two-step centralizer $H_1=\chi_2G$ of G
and is called the **polarization**,
has even logarithmic order $\mathrm{Cl}_3E_1\simeq A(3,2u)$ for some $u\ge 2$,
then $G\simeq G_{\mathbf{1}}^{2u+2}(0,\mathbf{w})$ for some $\mathbf{w}\in\{-1,0,1\}$ and $k=1$.

At the 2nd ICGA 2016 Suzhou, we[11],[12] have presented
the following distribution of minimal discriminants d
on the coclass-1 tree with abelian root $C_3\times C_3$.

For positive defect $k=1$, we were unable to distinguish
between the three possible values of the parameter $\mathbf{w}$.
(See the red double contour rectangle on the right hand side.)

Distribution[6] of 3-tower groups $G_3^\infty F$ of real quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with fundamental discriminants $32\,009 \le d \le 208\,540\,653$



---

6 $\mathrm{Cl}_3\mathbb{Q}(\sqrt{d})$, resp. $\ker(T_{\mathbb{Q}(\sqrt{d}),E_j})$, for $d = 62\,501$ was found by Pall[(16)], resp. Heider and Schmithals[(8)].

7. Separating the ground state $G_{\mathbf{1}}^6(0,\mathbf{w})$

Theorem. (Ground state of type a.1.)

If $\mathrm{Cl}_3E_1 \simeq A(3,4) \simeq C_9 \times C_9$, then

1. $\ker(T_{F,E_1}) = \mathrm{Cl}_3F$ (total shallow transfer).

2. In dependence on the deep transfer kernels, the 3-class tower group of the real quadratic field F is

$$G_3^{(\infty)}F \simeq \begin{cases} G_{\mathbf{1}}^6(0,\mathbf{0}) \simeq \langle 729,99\rangle \\ G_{\mathbf{1}}^6(0,-\mathbf{1}) \simeq \langle 729,100\rangle \\ G_{\mathbf{1}}^6(0,\mathbf{1}) \simeq \langle 729,101\rangle \end{cases}$$

$$\Longleftrightarrow \text{among } 2 \le i \le 4,\ \#\ker(T_{E_i,F_3^{(1)}}) = 9 \begin{cases} \text{occurs once,} \\ \text{occurs twice,} \\ \text{does not occur.} \end{cases}$$

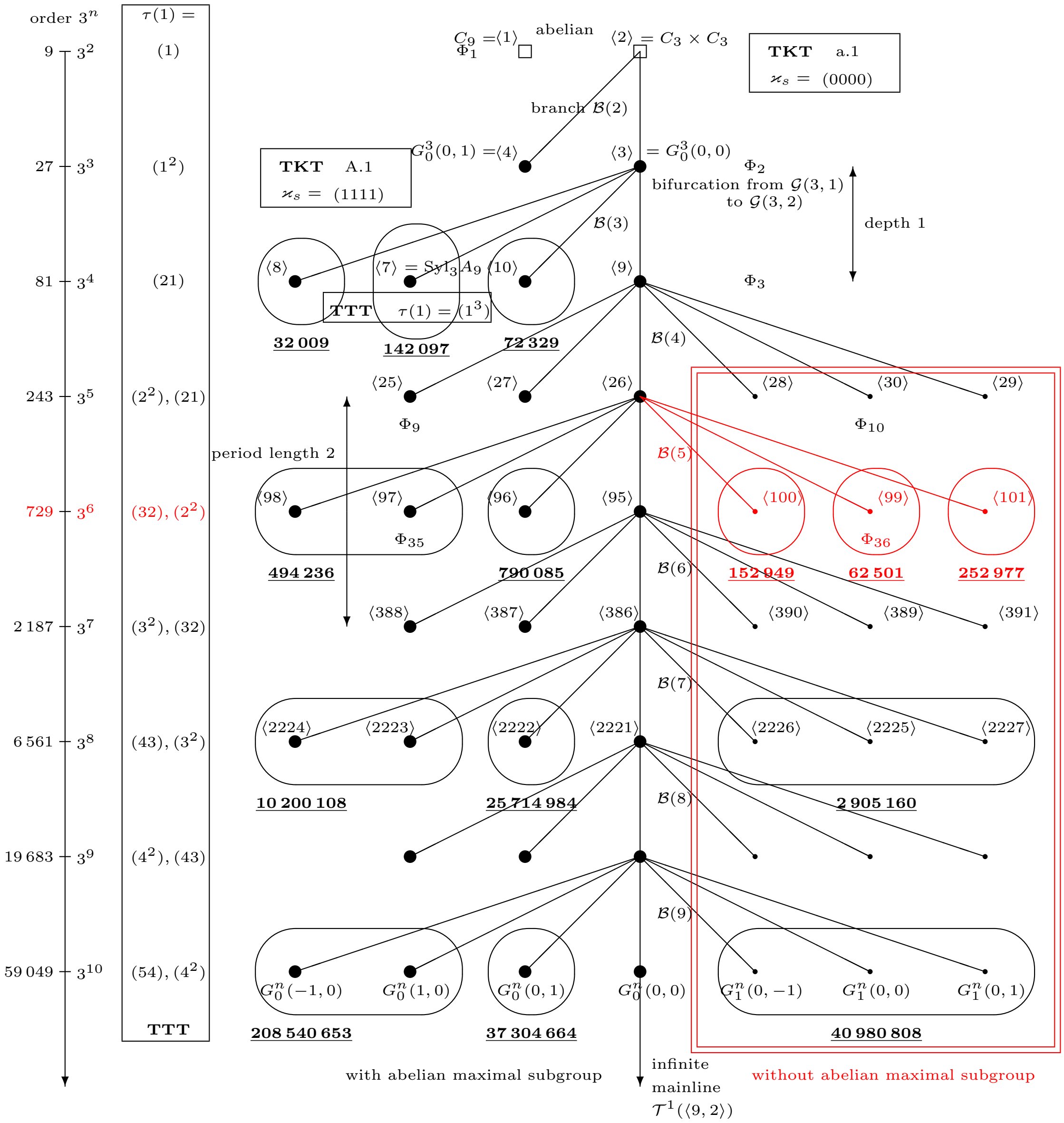
3. In dependence on the deep transfer kernels, the 3-class tower groups of the dihedral fields E_j are

$$(\forall\, 2 \le j \le 4) \quad G_3^{(\infty)}E_j \simeq \begin{cases} G_{\mathbf{0}}^5(0,\mathbf{0}) \simeq \langle 243,26\rangle \\ G_{\mathbf{0}}^5(0,\mathbf{1}) \simeq \langle 243,27\rangle \end{cases}$$

$$\Longleftrightarrow \#\ker(T_{E_j,F_3^{(1)}}) = \begin{cases} 9 \text{ (total deep transfer),} \\ 3 \text{ (partial deep transfer).} \end{cases}$$

Example. The next figure visualizes the minimal discriminants of the ground state adjacent to the red ovals.

Ground state[7] of 3-tower groups $G_3^\infty F \simeq G_1^6(0,w)$ of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with discriminants $62\,501 \le d \le 252\,977$



TKT	a.3	a.3	a.2	a.1*	a.1	a.1	a.1
$\varkappa_s =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

7 $\mathrm{Cl}_3\mathbb{Q}(\sqrt{d})$ and $G_3^2\mathbb{Q}(\sqrt{d})$ for $d = 152\,949$ were found by ourselves[(10)].

8. Separating the first excited state $G^8_{\mathbf{1}}(0,\mathbf{w})$

Theorem. (First excited state of type a.1.)

If $\mathrm{Cl}_3 E_1 \simeq A(3,6) \simeq C_{27} \times C_{27}$, then

1. $\ker(T_{F,E_1}) = \mathrm{Cl}_3 F$ (total shallow transfer).

2. In dependence on the deep transfer kernels, the 3-class tower group of the real quadratic field F is

$$G_3^{(\infty)} F \simeq \begin{cases} G^8_{\mathbf{1}}(0,\mathbf{0}) \simeq \langle 6561, 2225 \rangle \\ G^8_{\mathbf{1}}(0,-\mathbf{1}) \simeq \langle 6561, 2226 \rangle \\ G^8_{\mathbf{1}}(0,\mathbf{1}) \simeq \langle 6561, 2227 \rangle \end{cases}$$

$$\iff \text{among } 2 \le i \le 4,\ \#\ker(T_{E_i,F_3^{(1)}}) = 9 \begin{cases} \text{occurs once,} \\ \text{occurs twice,} \\ \text{does not occur.} \end{cases}$$

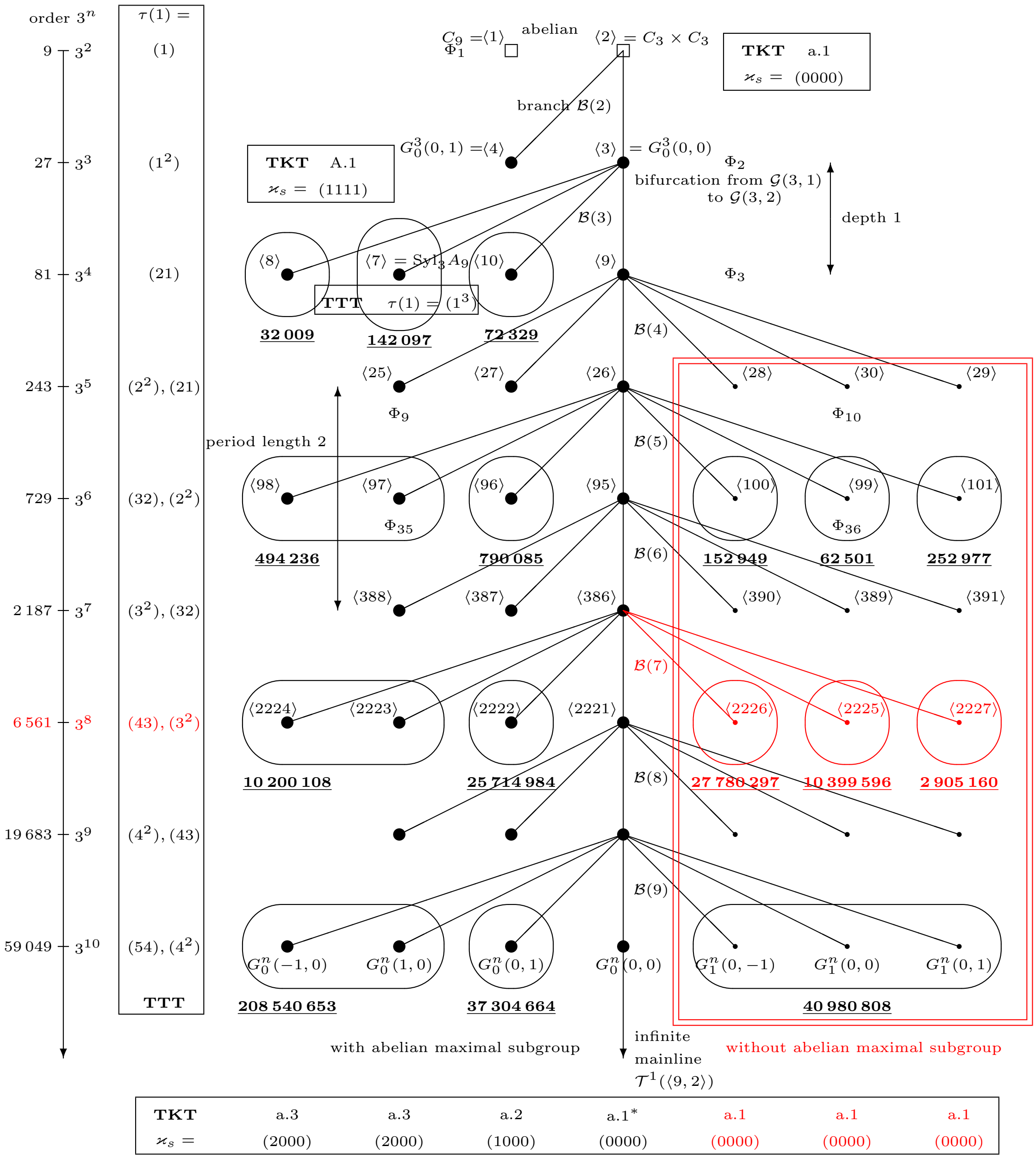
3. In dependence on the deep transfer kernels, the 3-class tower groups of the dihedral fields E_j are

$$(\forall\, 2 \le j \le 4) \quad G_3^{(\infty)} E_j \simeq \begin{cases} G^7_{\mathbf{0}}(0,\mathbf{0}) \simeq \langle 2187, 386 \rangle \\ G^7_{\mathbf{0}}(0,\mathbf{1}) \simeq \langle 2187, 387 \rangle \end{cases}$$

$$\iff \#\ker(T_{E_j,F_3^{(1)}}) = \begin{cases} 9 \text{ (total deep transfer),} \\ 3 \text{ (partial deep transfer).} \end{cases}$$

Example. The next figure visualizes the minimal discriminants of the first excited state adjacent to the red ovals.

First excited state of 3-tower groups $G_3^\infty F \simeq G_1^8(0,w)$ of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with $2\,905\,160 \le d \le 27\,786\,297$



9. Separating the second excited state $G_{\mathbf{1}}^{10}(0,\mathbf{w})$

Theorem. (Second excited state of type a.1.)

If $\mathrm{Cl}_3 E_1 \simeq A(3,8) \simeq C_{81} \times C_{81}$, then

1. $\ker(T_{F,E_1}) = \mathrm{Cl}_3 F$ (total shallow transfer).

2. In dependence on the deep transfer kernels,
the 3-class tower group of the real quadratic field F is

$$G_3^{(\infty)} F \simeq \begin{cases} G_{\mathbf{1}}^{10}(0,\mathbf{0}) \\ G_{\mathbf{1}}^{10}(0,-\mathbf{1}) \\ G_{\mathbf{1}}^{10}(0,\mathbf{1}) \end{cases}$$

$$\Longleftrightarrow \text{among } 2 \le i \le 4,\ \#\ker(T_{E_i,F_3^{(1)}}) = 9 \begin{cases} \text{occurs once,} \\ \text{occurs twice,} \\ \text{does not occur.} \end{cases}$$

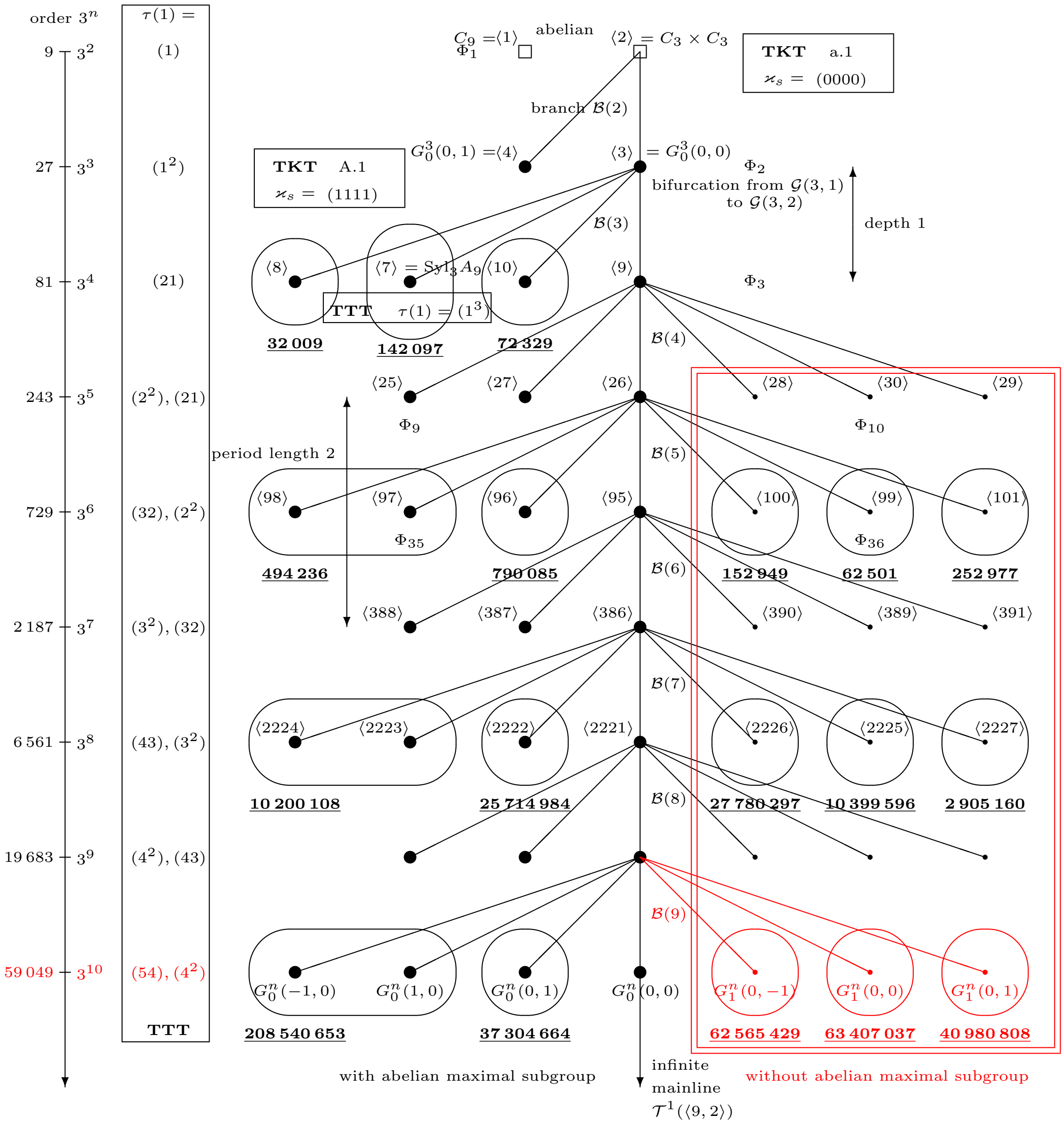
3. In dependence on the deep transfer kernels,
the 3-class tower groups of the dihedral fields E_j are

$$(\forall\, 2 \le j \le 4) \quad G_3^{(\infty)} E_j \simeq \begin{cases} G_{\mathbf{0}}^{9}(0,\mathbf{0}) \\ G_{\mathbf{0}}^{9}(0,\mathbf{1}) \end{cases}$$

$$\Longleftrightarrow \#\ker(T_{E_j,F_3^{(1)}}) = \begin{cases} 9 \text{ (total deep transfer),} \\ 3 \text{ (partial deep transfer).} \end{cases}$$

Example. The next figure visualizes the minimal discriminants of the second excited state adjacent to the red ovals.

Second excited state of 3-tower groups $G_3^\infty F \simeq G_1^{10}(0,w)$ of real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with $40\,980\,808 \le d \le 63\,407\,037$



TKT	a.3	a.3	a.2	a.1*	a.1	a.1	a.1
$\varkappa_s =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

10. Full separation of all states $G_{\mathbf{1}}^n(0,\mathbf{w})$

General Theorem.[13]
(**Hyper fine** structure of an arbitrary state of type a.1.)

The case $\mathrm{Cl}_3 E_1 \simeq A(3,2) \simeq C_3 \times C_3$ cannot occur, but if $\mathrm{Cl}_3 E_1 \simeq A(3,2u) \simeq C_{3^u} \times C_{3^u}$, for some $u \ge 2$, then

1. $(\forall\, 1 \le j \le 4) \quad \ker(T_{F,E_j}) = \mathrm{Cl}_3 F$
(four total shallow transfer kernels).

2. In dependence on the deep transfer kernels of the **stabilization**, the 3-class tower group of the real quadratic field F is

$$G_3^{(\infty)}F \simeq \begin{cases} G_{\mathbf{1}}^{2u+2}(0,-\mathbf{1}) \\ G_{\mathbf{1}}^{2u+2}(0,\mathbf{0}) \\ G_{\mathbf{1}}^{2u+2}(0,\mathbf{1}) \end{cases}$$

$$\Longleftrightarrow \text{among } 2 \le j \le 4,\ \#\ker(T_{E_j,F_3^{(1)}}) = 9 \begin{cases} \text{occurs twice,} \\ \text{occurs once,} \\ \text{does not occur.} \end{cases}$$

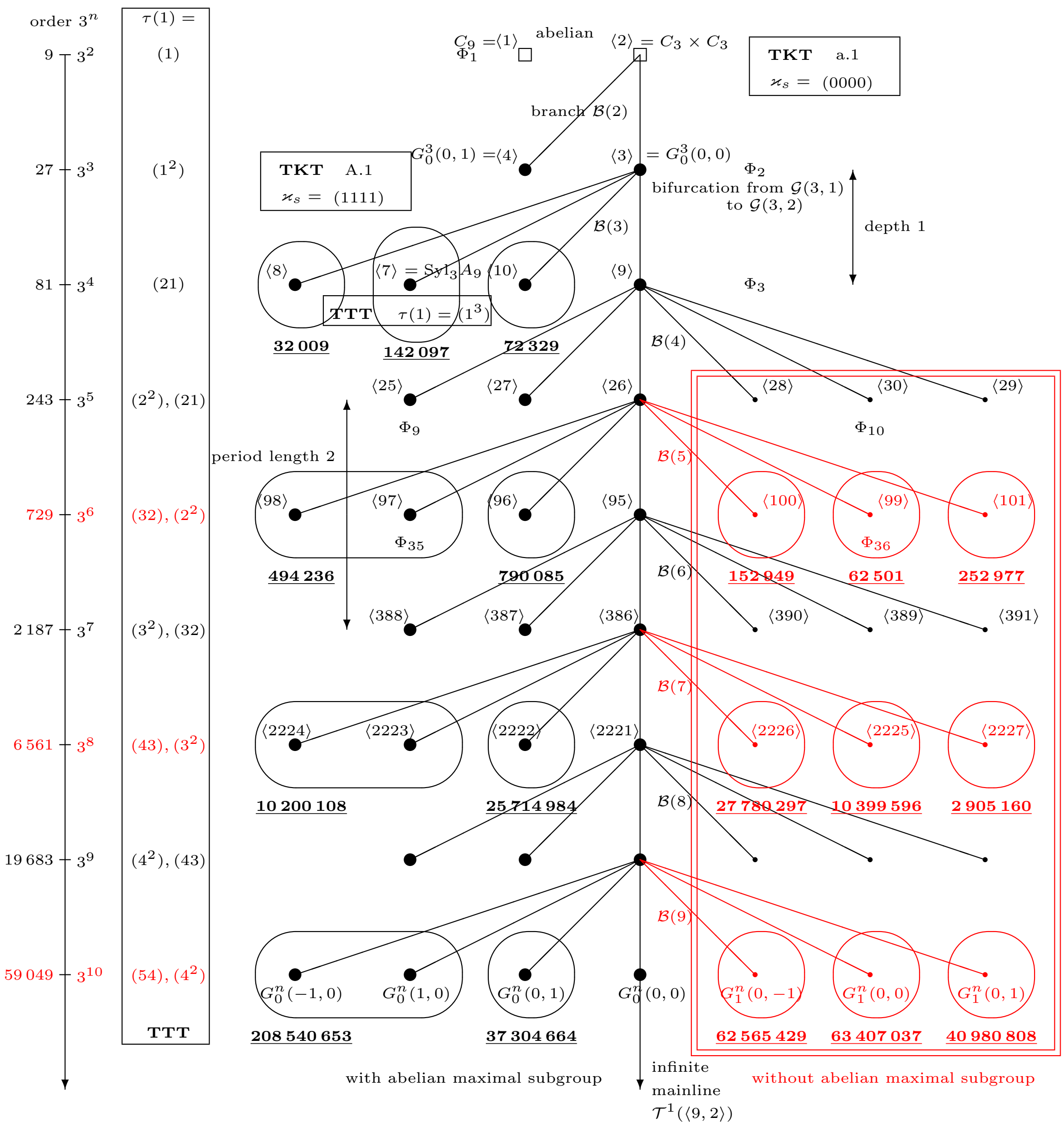
3. In dependence on the deep transfer kernels, the 3-class tower groups of the dihedral fields E_j are

$$(\forall\, 2 \le j \le 4) \quad G_3^{(\infty)}E_j \simeq \begin{cases} G_{\mathbf{0}}^{2u+1}(0,\mathbf{0}) \\ G_{\mathbf{0}}^{2u+1}(0,\mathbf{1}) \end{cases}$$

$$\Longleftrightarrow \#\ker(T_{E_j,F_3^{(1)}}) = \begin{cases} 9 \text{ (total deep transfer kernel),} \\ 3 \text{ (partial d.t.k. with fixed point).} \end{cases}$$

Example. The next figure visualizes the minimal discriminants of known states of type a.1 adjacent to the red ovals.

Distribution[8] of 3-tower groups $G_3^\infty F \simeq G_1^n(0,w)$ of real quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with discriminants $62\,501 \le d \le 63\,407\,037$



TKT	a.3	a.3	a.2	a.1*	a.1	a.1	a.1
$\varkappa_s =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

8 $d = 40\,980\,808$ was found by Bush[(5)], the full separation was determined by ourselves[(14)].

Proof. (of the Lemma in § 6 and the General Theorem in § 10)

(1) Firstly, the structure of the 3-class groups

$$\mathrm{Cl}_3 E_j \simeq C_3 \times C_3, \text{ for } 2 \le j \le 4, \text{ (stabilization)}$$

enforces *coclass one* for the second 3-class group $G_3^{(2)}F$.

Together with $p = 3$, the coclass one of $G_3^{(2)}F$ implies the *termination* $F_3^{(2)} = F_3^{(\infty)}$ of the 3-class tower of F *at the second stage*[9].

The 3-class group of the distinguished extension E_1/F (polarization),

$$\mathrm{Cl}_3 E_1 \simeq A(3, n-k-1) = A(3, 2u)$$

with even logarithmic order $n - k - 1 = 2u$, $u \ge 1$, has consequences for the invariants $n = \log_3(|G|)$ and k of the 3-class tower group $G = G_3^{(\infty)}F = G_3^{(2)}F$.

Either $k = 0$, $n = 2u + 1$ odd with $u \ge 1$, and $G = G_0^n(0,0)$ must be a mainline vertex, due to the required GI-action[10]. But then the relation rank $\mu(G) = 4$ is too big.

Or $k = 1$, $n = 2u + 2$ even with $u \ge 2$, and $G = G_1^n(0,w)$ is a vertex of depth 1 with defect $k = 1$ having both, the *required relation rank*[11] $\mu(G) = 3$, and a GI-*action* with flag $\sigma = 1$. The positive defect $k = 1$ implies *four total shallow transfer kernels*

$$(\forall\, 1 \le j \le 4) \quad \ker(T_{F,E_j}) = \mathrm{Cl}_3 F \simeq C_3 \times C_3.$$

(2) Secondly, the parameter $w \in \{-1, 0, 1\}$ of $G = G_1^n(0,w)$ is determined by the *number* $N(0)$ *of* occurrences of $\#\ker(T_{E_j,F_3^{(1)}}) = 9$ that is, *total deep transfer kernels*, among $j \in \{2,3,4\}$, according to Theorem 2 in § 3.

(3) Next, the Lemma in § 4 implies the coincidence

$$F_3^{(2)} = (E_j)_3^{(2)} = F_3^{(3)} = (E_j)_3^{(3)} = F_3^{(4)}.$$

Therefore, $(E_j)_3^{(2)} = (E_j)_3^{(\infty)}$ also *terminates at length* $\ell_3 E_j = 2$, and

$$G_3^{(\infty)}E_j = \mathrm{Gal}((E_j)_3^{(\infty)}/E_j) = \mathrm{Gal}(F_3^{(\infty)}/E_j)$$

is a *maximal subgroup* of $G = G_3^{(\infty)}F = \mathrm{Gal}(F_3^{(\infty)}/F)$.

Thus, $G_3^{(\infty)}E_j \simeq G_0^{n-1}(0,w)$ with $w \in \{0,1\}$. Finally, if we put

$$\mathfrak{H}_1 := G' = \mathrm{Gal}(F_3^{(\infty)}/F_3^{(1)}) < \mathfrak{G} := H_j = \mathrm{Gal}(F_3^{(\infty)}/E_j) = G_3^{(\infty)}E_j,$$

then we see that the deep transfer kernel $\ker(T_{E_j,F_3^{(1)}}) = \ker(T_{H_j,G'})$ of G coincides with the shallow transfer kernel $\ker(T_{\mathfrak{G},\mathfrak{H}_1})$ of $\mathfrak{G} \simeq G_0^{n-1}(0,w)$. (Observe that $\mathfrak{H}_1 = G'$ is the polarization of $\mathfrak{G} = H_j$.) □

---

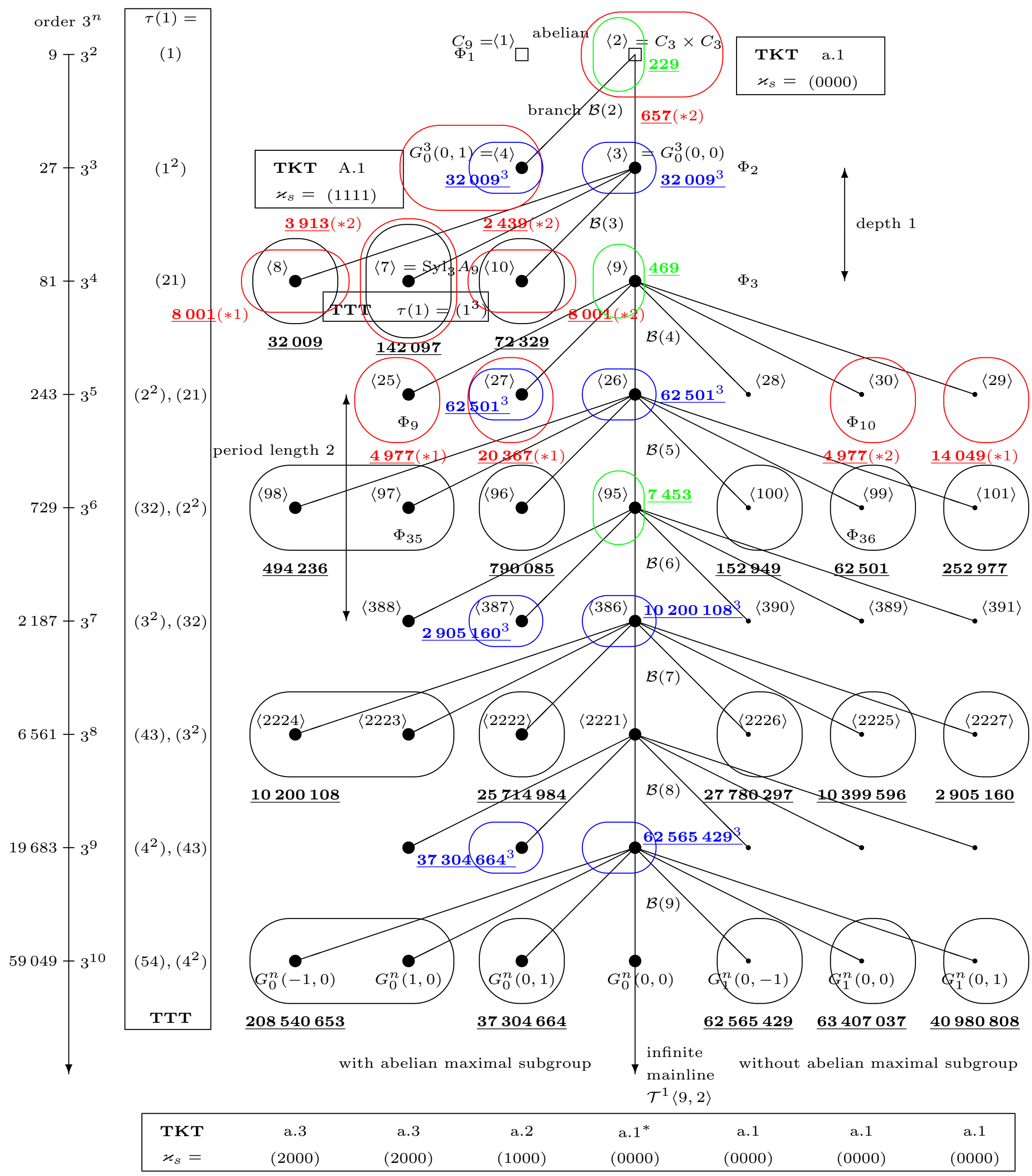
9 According to Blackburn(3).

10 Existence of a generator inverting automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(x) = x^{-1}$ for $x \in G/G'$.

11 According to Shafarevich(18).

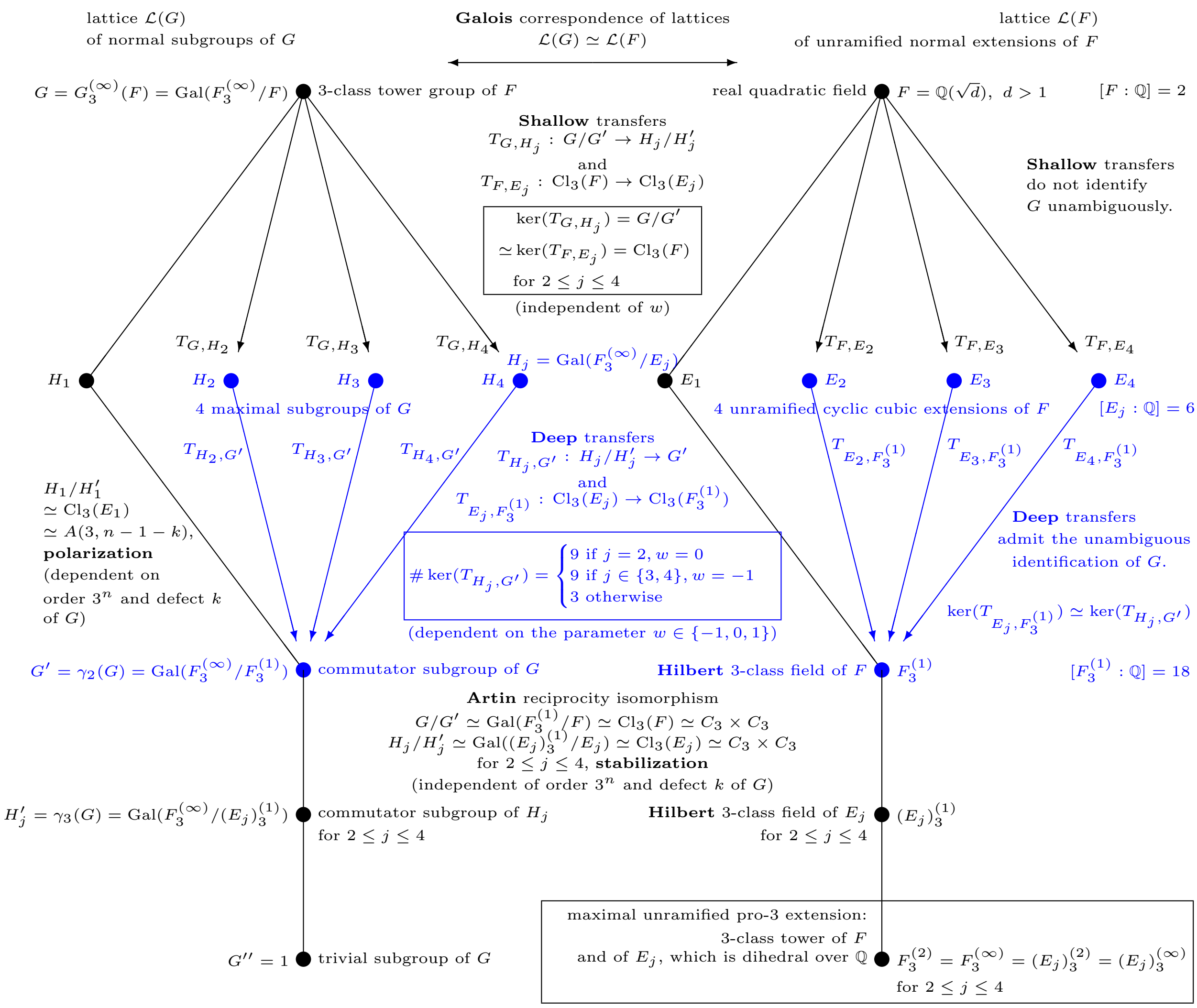
11. All known realizations of 3-groups $G_a^n(z,w)$ as 3-class tower groups $\mathrm{Gal}(F_3^{(\infty)}/F)$

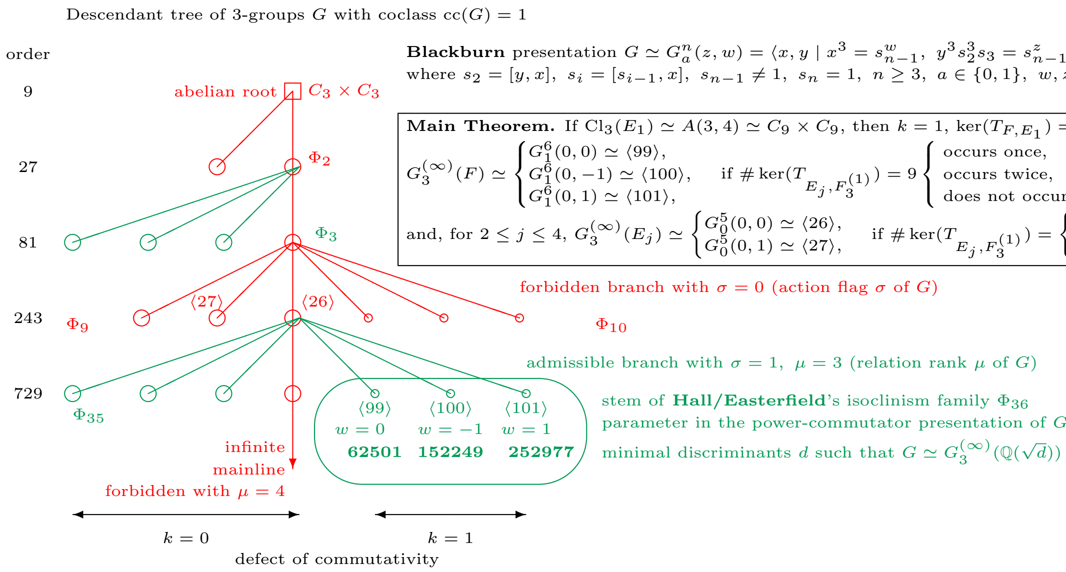
Real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with $d < 10^9$
Cyclic cubic fields F with conductors $c < 10^5$
Totally real dihedral fields F with discriminants $d < 10^{24}$
Bicyclic biquadratic fields $F = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$ with $d < 5 \cdot 10^4$



References.

(1) E. Artin, *Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 46–51, DOI 10.1007/BF02941159.
(2) H. U. Besche, B. Eick and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP package, available also in MAGMA.
(3) N. Blackburn, *On prime-power groups in which the derived group has two generators*, Proc. Camb. Phil. Soc. **53** (1957), 19–27.
(4) N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.
(5) N. Boston, M. R. Bush and F. Hajir, *Heuristics for p-class towers of real quadratic fields*, to appear.
(6) T. E. Easterfield, *A classification of groups of order* p^6, Ph.D. Thesis, Univ. of Cambridge (P. Hall), Cambridge, England, 1940.
(7) P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.
(8) F.-P. Heider und B. Schmithals, *Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen*, J. Reine Angew. Math. **336** (1982), 1–25.
(9) MAGMA Developer Group, MAGMA *Computational Algebra System*, Version 2.23-7, Sydney, 2017, (`http://magma.maths.usyd.edu.au`).
(10) D. C. Mayer, *List of discriminants* $d_L < 200\,000$ *of totally real cubic fields* L*, arranged according to their multiplicities* m *and conductors* f, Computer Centre, Department of Computer Science, University of Manitoba, Winnipeg, Canada, 1991, Austrian Science Fund, Project Nr. J0497-PHY.
(11) D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Math. Phys. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135.
(12) D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, 2nd International Conference on Groups and Algebras (ICGA) 2016, Suzhou, China, presentation delivered on July 26, 2016.
(13) D. C. Mayer, *Successive approximation of p-class towers*, Adv. Pure Math. **7** (2017), no. 12, 660–685, DOI 10.4236/apm.2017.712041, Special Issue on Abstract Algebra, December 2017.
(14) D. C. Mayer, *Deep transfers of p-class tower groups*, to appear in J. Appl. Math. Phys. **6** (2018), no. 1, 36–50, DOI 10.4236/jamp.2018.61005.
(15) R. J. Miech, *Metabelian p-groups of maximal class*, Trans. Amer. Math. Soc. **152** (1970), 331–373.
(16) G. Pall, *Note on irregular determinants*, J. London Math. Soc. **11** (1936), 34–35.
(17) I. Schur, *Neuer Beweis eines Satzes über endliche Gruppen*, Sitzungsberichte der Königlich Preussischen Akad. d. Wissenschaften zu Berlin **42** (1902), 1013–1019.
(18) I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)
(19) A. Wiman, *Über p-Gruppen von maximaler Klasse*, Acta Math. **88** (1952), 317–346.





Blackburn presentation $G \simeq G_a^n(z,w) = \langle x, y \mid x^3 = s_{n-1}^w,\ y^3 s_2^3 s_3 = s_{n-1}^z,\ [y, s_2] = s_{n-1}^a \rangle$,
where $s_2 = [y,x],\ s_i = [s_{i-1}, x],\ s_{n-1} \ne 1,\ s_n = 1,\ n \ge 3,\ a \in \{0,1\},\ w, z \in \{-1,0,1\}$

Main Theorem. If $\mathrm{Cl}_3(E_1) \simeq A(3,4) \simeq C_9 \times C_9$, then $k = 1$, $\ker(T_{F,E_1}) = \mathrm{Cl}_3(F)$,

$$G_3^{(\infty)}(F) \simeq \begin{cases} G_1^6(0,0) \simeq \langle 99 \rangle, \\ G_1^6(0,-1) \simeq \langle 100 \rangle, \\ G_1^6(0,1) \simeq \langle 101 \rangle, \end{cases} \quad \text{if } \# \ker(T_{E_j,F_3^{(1)}}) = 9 \begin{cases} \text{occurs once,} \\ \text{occurs twice,} \\ \text{does not occur,} \end{cases}$$

and, for $2 \le j \le 4$, $G_3^{(\infty)}(E_j) \simeq \begin{cases} G_0^5(0,0) \simeq \langle 26 \rangle, \\ G_0^5(0,1) \simeq \langle 27 \rangle, \end{cases}$ if $\# \ker(T_{E_j,F_3^{(1)}}) = \begin{cases} 9, \\ 3. \end{cases}$