FWF Der Wissenschaftsfonds.

Differential Principal Factors (DPF) in Pure Metacyclic Fields

Conference:	3rd International Conference on Algebra and Number Theory (ICANTA Oujda 2019)
Place:	Université Mohammed Premier Faculté des Sciences
Venue:	Oujda, Region Oriental, Morocco
Date:	April 24 – 27, 2019
Author:	Daniel C. Mayer (Graz, Austria)
Affiliation:	Austrian Science Fund (FWF)

A presentation within the frame of the international scientific research project

Towers of p-Class Fields over Algebraic Number Fields

supported by the Austrian Science Fund (FWF): Projects J 0497-PHY and P 26008-N25

1. General Pure Metacyclic Fields

$p \geq 3$ ... odd prime number,
$D \geq 2$ ... pth power free integer,
$\zeta_p = \exp(2\pi\sqrt{-1}/p)$ ... primitive pth root of unity,
$N = \mathbb{Q}(\zeta_p, \sqrt[p]{D})$ ... **pure metacyclic** field, deg. $(p-1)\cdot p$,
$L = \mathbb{Q}(\sqrt[p]{D})$ ... **pure** subfield of degree p of N,
$K = \mathbb{Q}(\zeta_p)$ ... pth cyclotomic field.

1.1. Galois Cohomology

$G = \mathrm{Gal}(N/K) = \langle\sigma\rangle \simeq C_p$, rel. group of Kummer ext.,
$E_{N/K} = U_N \cap \ker(N_{N/K})$, where U_N = unit group of N.

Herbrand Quotient of the G-module U_N:

$$\frac{\#\mathrm{H}^0(G,U_N)}{\#\mathrm{H}^1(G,U_N)} = \frac{(\ker(\Delta):\mathrm{im}(\mathcal{N}))}{(\ker(\mathcal{N}):\mathrm{im}(\Delta))} = \frac{(U_K : N_{N/K}(U_N))}{(E_{N/K} : U_N^{\sigma-1})},$$

$$\#H^1(G,U_N) = \#H^0(G,U_N)\cdot[N:K] = p^{U+1},\ 0 \leq U \leq \frac{p-1}{2}.$$

Iwasawa Isomorphism: $\quad H^1(G,U_N) \simeq \overbrace{\mathcal{P}_{N/K}/\mathcal{P}_K}^{\text{DPF}},$

$$(\mathcal{P}_{N/K} : \mathcal{P}_K) = (E_{N/K} : U_N^{\sigma-1}) = (U_K : N_{N/K}(U_N))\cdot p.$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \underbrace{\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}}_{\text{abs. DPF}} \times \underbrace{\left(\mathcal{P}_{N/K}/\mathcal{P}_K \cap \ker(N_{N/L})\right)}_{\text{rel. DPF (norm kernel)}},\ U+1 = A+R.$$

1.2. **General Norm Kernel**

Theorem.[12] (Primitive ambiguous ideals)
Let $p \in \mathbb{P}$ be a prime, and $q \in \mathbb{N}$ be an integer coprime to p. Suppose F/F_0 is a number field extension of degree p, E_0/F_0 is an extension of degree q, and $E = F \cdot E_0$ is the compositum of F and E_0. (Most important situation: $q = p - 1$.)
(1) The norm map $N_{E/F} : \mathcal{I}_E \to \mathcal{I}_F$ satisfies

$$N_{E/F}(\mathcal{I}_{E/E_0}) \le \mathcal{I}_{F/F_0} \text{ and } N_{E/F}(\mathcal{I}_{E_0}) \le \mathcal{I}_{F_0},$$

and induces an epimorphism

$$N_{E/F} : \mathcal{I}_{E/E_0}/\mathcal{I}_{E_0} \to \mathcal{I}_{F/F_0}/\mathcal{I}_{F_0}.$$

(2) There are isomorphisms of elementary abelian p-groups

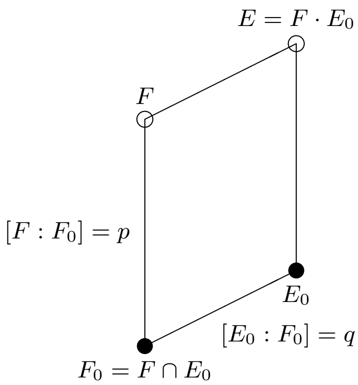
$$\mathcal{I}_{F/F_0}/\mathcal{I}_{F_0} \simeq (\mathcal{I}_{E/E_0}/\mathcal{I}_{E_0})/\ker(N_{E/F})$$

and

$$\mathcal{I}_{E/E_0}/\mathcal{I}_{E_0} \simeq (\mathcal{I}_{F/F_0}/\mathcal{I}_{F_0}) \times \ker(N_{E/F})$$

(Natural decomposition of primitive ambiguous ideal groups).

FIGURE 1. Compositum of extensions F/F_0 and E_0/F_0 with coprime degrees



Classification into coarse types by U and A, $\bullet$ coarse type splits into subtypes, $\circ$ does not split.

FIGURE 2. Classification of simply real dihedral and pure cubic fields, $p = 3$

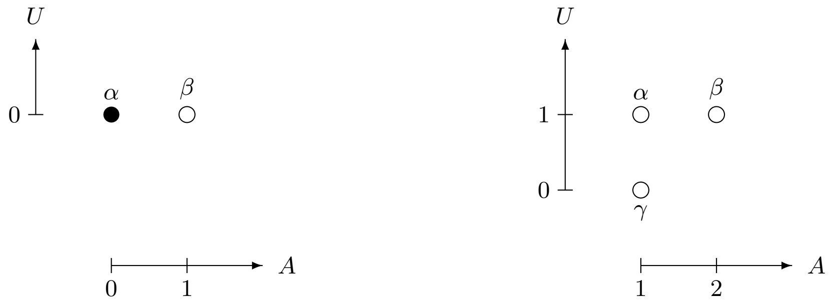


FIGURE 3. Classification of totally real dihedral and pure quintic fields, $p = 5$

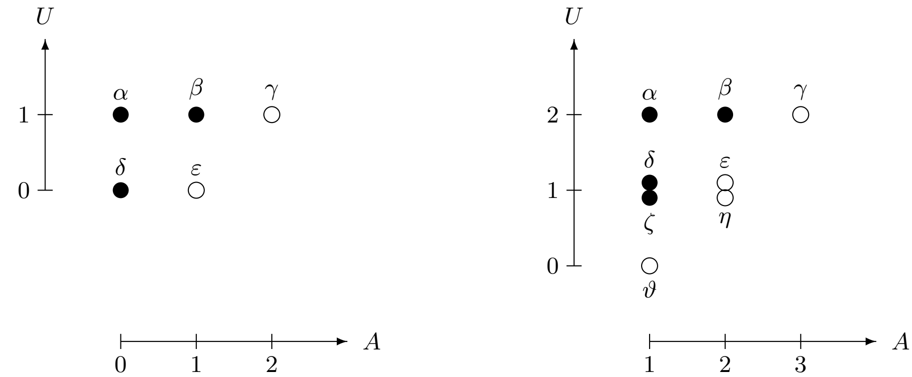
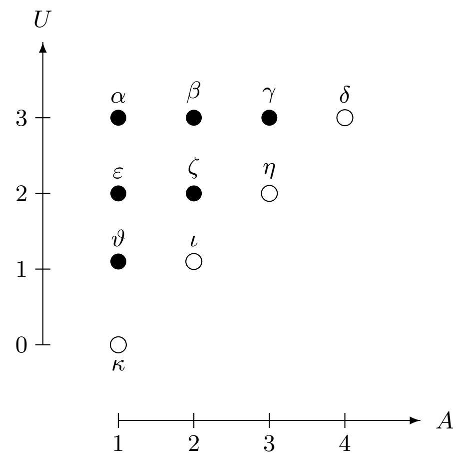


FIGURE 4. Classification of pure septic fields, $p = 7$



1.3. **General DPF Types**

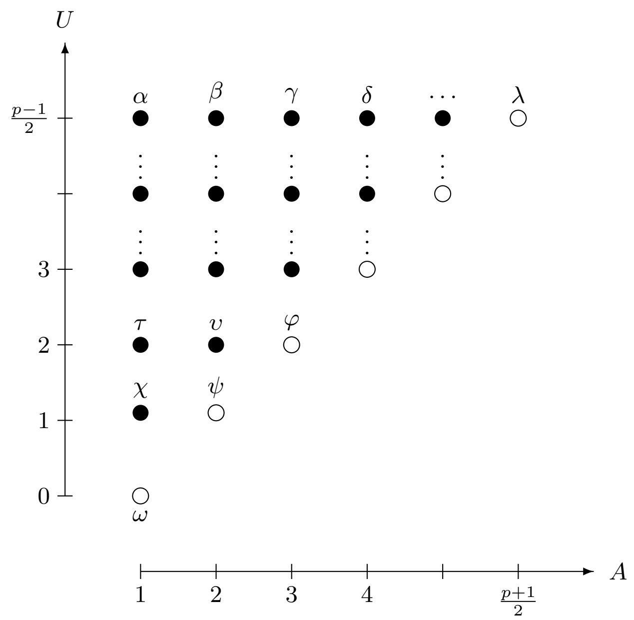
• **Total** $\mathbb{F}_p$-Vector Space of All Differential Principal Factors with Dimension:

$$\dim_{\mathbb{F}_p}(\mathcal{P}_{N/K}/\mathcal{P}_K) = U + 1,$$

• Subspace of **Absolute** Differential Principal Factors with Dimension:

$$A := \dim_{\mathbb{F}_p}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}),\ 1 \leq A \leq U + 1$$

FIGURE 5. Classification of general pure fields of odd prime degree $p \geq 11$



1.4. **General Homogeneous Multiplets**

Theorem. Let $(N_1,\ldots,N_m)$ be a multiplet of nonisomorphic pure metacyclic fields $N_i=\mathbb{Q}(\zeta_p,\sqrt[p]{D_i})$, with a primitive pth root of unity ζ_p and distinct normalized pth power free radicands $D_i>1$, which share a common conductor f over the cyclotomic field $K=\mathbb{Q}(\zeta_p)$.

(1) If either $f^{p-1}=p^{p+1}$ or $f=q$ with a prime $q\in\mathbb{P}$ such that $q^{p-1}\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=1$ and the **singulet** N_1 is of species 2 for $f=q$, of species 1a otherwise, and of the unique DPF type ω with $U=R=0$ and $A=1$.

(2) If $f^{p-1}=p^{p+1}\cdot q^{p-1}$ with a prime $q\in\mathbb{P}\setminus\{p\}$ such that $q^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=p-1$ and the **multiplet** $(N_1,\ldots,N_{p-1})$ is of species 1a and of the **homogeneous** DPF type $(\psi,\ldots,\psi)$ with $U=1$, $R=0$ and $A=2$.

(3) If $f^{p-1}=p^2\cdot q^{p-1}$ with a prime $q\in\mathbb{P}\setminus\{p\}$ such that $q^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=1$ and the **singulet** N_1 is of species 1b and of the unique DPF type ψ with $U=1$, $R=0$ and $A=2$.

(4) If $f=q_1\cdot q_2$ with distinct primes $q_j\in\mathbb{P}\setminus\{p\}$ such that $q_j^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q_j does not split in K for $1\le j\le 2$, then $m=1$ and the **singulet** N_1 is of species 2 and of the unique DPF type ψ with $U=1$, $R=0$ and $A=2$.

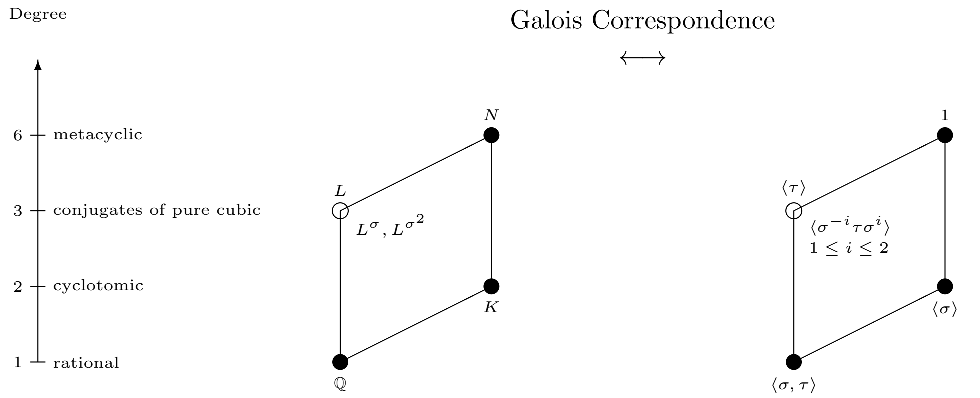
The conditions for the unit norm index can be expressed equivalently by $U=0\Longleftrightarrow N_{N/K}(U_N)=U_K$,
and $U=1\Longleftrightarrow N_{N/K}(U_N)\simeq U_K/\langle\zeta_p\rangle$.

P. Barrucand and H. Cohn, 1971:

3. **Pure Cubic Fields,** $p = 3$

$D \geq 2$ ... third power free integer,
$\zeta_3 = \exp(2\pi\sqrt{-1}/3)$ ... primitive third root of unity,
$N = \mathbb{Q}(\zeta_3, \sqrt[3]{D})$ ... **pure metacyclic** field, degree 6,
$L = \mathbb{Q}(\sqrt[3]{D})$ ... **pure cubic** subfield of N,
$K = \mathbb{Q}(\zeta_3)$ ... third cyclotomic field, imaginary quadratic.

FIGURE 6. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



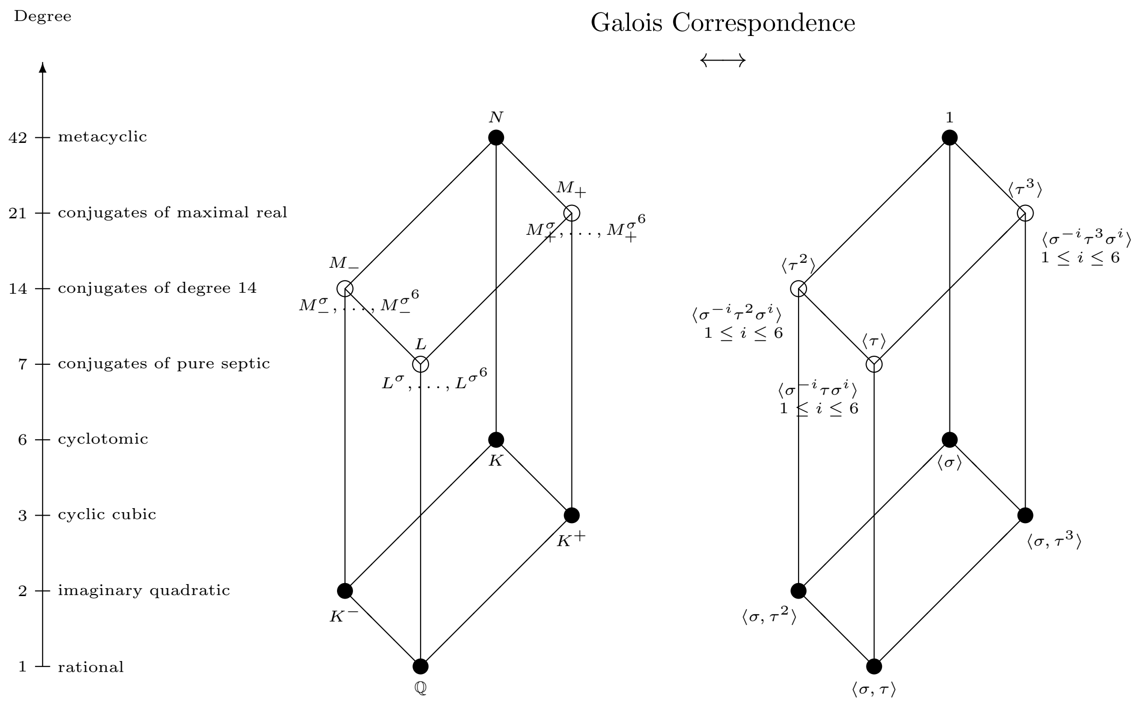
D. C. Mayer, 2018:

7. **Pure Septic Fields,** $p = 7$

$D \geq 2$ ... seventh power free integer,
$\zeta_7 = \exp(2\pi\sqrt{-1}/7)$ ... primitive seventh root of unity,
$N = \mathbb{Q}(\zeta_7, \sqrt[7]{D})$ ... **pure metacyclic** field, degree 42,
$M_+ = \mathbb{Q}(\varrho, \sqrt[7]{D})$ ... maximal real subfield of N, deg. 21,
$M_- = \mathbb{Q}(\sqrt{-7}, \sqrt[7]{D})$ ... complex degree 14 subfield of N,
$L = \mathbb{Q}(\sqrt[7]{D})$ ... **pure septic** subfield of N,
$K = \mathbb{Q}(\zeta_7)$ ... seventh cyclotomic field, cyclic sextic,
$K^+ = \mathbb{Q}(\varrho)$ ... maximal real subfield of K, cyclic cubic,
$K^- = \mathbb{Q}(\sqrt{-7})$ ... imaginary quadratic subfield of K.

Remark. The cubic irrationality which generates K^+ is a solution of $\varrho^3 - \varrho^2 - 2\varrho + 1 = 0$.

FIGURE 7. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$

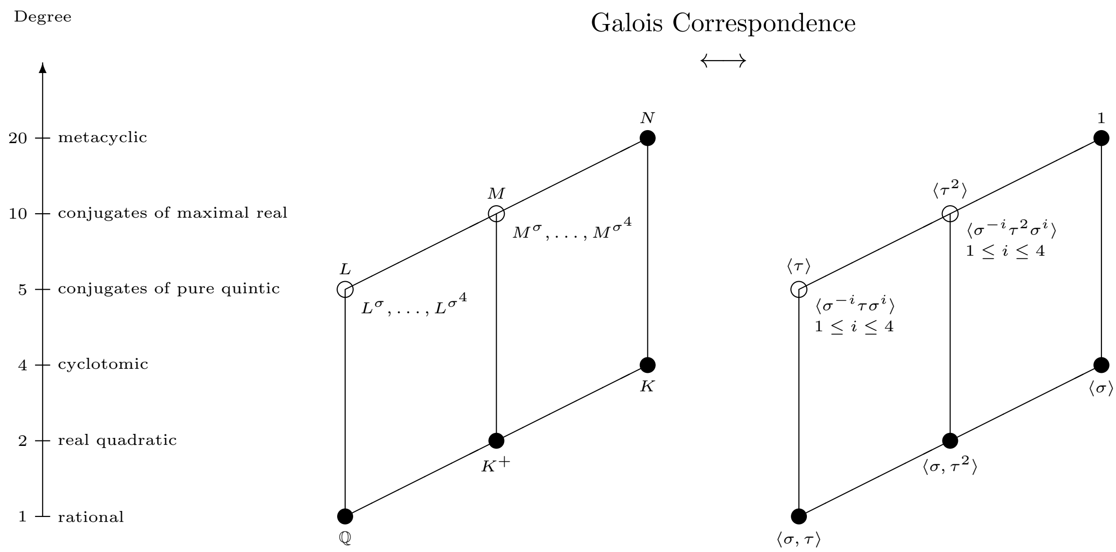


D. C. Mayer, 2013:

5. **Pure Quintic Fields,** $p = 5$

$D \geq 2$ … fifth power free integer,
$\zeta_5 = \exp(2\pi\sqrt{-1}/5)$ … primitive fifth root of unity,
$N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$ … **pure metacyclic** field, degree 20,
$M = \mathbb{Q}(\sqrt{5}, \sqrt[5]{D})$ … maximal real subfield of N, deg. 10,
$L = \mathbb{Q}(\sqrt[5]{D})$ … **pure quintic** subfield of N,
$K = \mathbb{Q}(\zeta_5)$ … fifth cyclotomic field, cyclic quartic,
$K^+ = \mathbb{Q}(\sqrt{5})$ … maximal real subfield of K, real quadratic.

FIGURE 8. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



Relative different of N with respect to K: $\mathfrak{D}_{N/K} =$

$$\underbrace{\mathfrak{P}^e \cdot \prod_{i=1}^{n} \mathfrak{Q}_i^4}_{\text{absolute}} \cdot \underbrace{\prod_{i=1}^{s_2} (\mathfrak{L}_i \cdot \mathfrak{L}_i^\tau)^4}_{\text{intermediate}} \cdot \underbrace{\prod_{i=s_2+1}^{s_4} (\mathfrak{L}_i \cdot \mathfrak{L}_i^{\tau^2} \cdot \mathfrak{L}_i^\tau \cdot \mathfrak{L}_i^{\tau^3})^4}_{\text{relative}}$$

5.1. **Absolute** Differential Principal Factors

T ... number of primes q_i dividing the conductor f of N/K,
$q_i\mathcal{O}_L = \mathfrak{q}_i^5$ ... ramification of q_i in L.

• Space of Absolute Differential Factors in $L/\mathbb{Q}$
Dimension:

$$\dim_{\mathbb{F}_5}(\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}) = T(= n + s_2 + s_4), \tag{1}$$

Basis:

$$\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}} \simeq \bigoplus_{i=1}^{T} \mathbb{F}_5\, \mathfrak{q}_i\,. \tag{2}$$

• Subspace of Absolute Differential **Principal** Factors
Dimension:

$$A := \dim_{\mathbb{F}_5}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}), \tag{3}$$

Bounds for the dimension (cohomology: ≤ 3, subspace: $\le T$):

$$1 \le A \le \min(3, T). \tag{4}$$

5.2. **Intermediate** Differential Principal Factors

s_2 ... number of primes $\ell_i \equiv -1 \pmod 5$ dividing f,
s_4 ... number of primes $\ell_{s_2+i} \equiv +1 \pmod 5$ dividing f,
$\ell_i \mathcal{O}_M = (\mathcal{L}_i \cdot \mathcal{L}_i^\tau)^5$ ... 2-splitting of ℓ_i in M.

• Space of Intermediate Differential Factors in M/K^+

Dimension:

$$\dim_{\mathbb{F}_5}\Big((\mathcal{I}_{M/K^+}/\mathcal{I}_{K^+}) \bigcap \ker(N_{M/L})\Big) = s_2 + s_4 \tag{5}$$

Basis with $\mathcal{K}_{(\ell_i)} := \mathcal{L}_i \cdot (\mathcal{L}_i^\tau)^4$:

$$(\mathcal{I}_{M/K^+}/\mathcal{I}_{K^+}) \bigcap \ker(N_{M/L}) \simeq \bigoplus_{i=1}^{s_2+s_4} \mathbb{F}_5\, \mathcal{K}_{(\ell_i)}\,, \tag{6}$$

• Subspace of Intermediate Differential **Principal** Factors
Dimension:

$$I := \dim_{\mathbb{F}_5}\Big((\mathcal{P}_{M/K^+}/\mathcal{P}_{K^+}) \bigcap \ker(N_{M/L})\Big)\,, \tag{7}$$

Bounds (cohomology: ≤ 2, subspace: $\le s_2 + s_4$):

$$0 \le I \le \min(2, s_2 + s_4). \tag{8}$$

5.3. **Relative** Differential Principal Factors

s_4 ... number of primes $\ell_{s_2+i} \equiv +1 \pmod 5$ dividing f,
$\ell_i\mathcal{O}_N = (\mathfrak{L}_i \cdot \mathfrak{L}_i^{\tau^2} \cdot \mathfrak{L}_i^{\tau} \cdot \mathfrak{L}_i^{\tau^3})^5$ ... 4-splitting of ℓ_i in N.

• Space of Relative Differential Factors in N/K
Dimension:

$$\dim_{\mathbb{F}_5}\left((\mathcal{I}_{N/K}/\mathcal{I}_K)\bigcap\ker(N_{N/M})\right) = 2s_4 \tag{9}$$

Basis with $\mathfrak{K}_{(\ell_i),1} := \mathfrak{L}_i \cdot (\mathfrak{L}_i^{\tau^2})^4 \cdot (\mathfrak{L}_i^{\tau})^2 \cdot (\mathfrak{L}_i^{\tau^3})^3$ and
$\mathfrak{K}_{(\ell_i),2} := \mathfrak{L}_i \cdot (\mathfrak{L}_i^{\tau^2})^4 \cdot (\mathfrak{L}_i^{\tau})^3 \cdot (\mathfrak{L}_i^{\tau^3})^2$:

(10)

$$(\mathcal{I}_{N/K}/\mathcal{I}_K)\bigcap\ker(N_{N/M}) \simeq \bigoplus_{i=s_2+1}^{s_2+s_4}\left(\mathbb{F}_5\,\mathfrak{K}_{(\ell_i),1} \oplus \mathbb{F}_5\,\mathfrak{K}_{(\ell_i),2}\right),$$

• Subspace of Relative Differential **Principal** Factors
Dimension:

$$R := \dim_{\mathbb{F}_5}\left((\mathcal{P}_{N/K}/\mathcal{P}_K)\bigcap\ker(N_{N/M})\right), \tag{11}$$

Bounds (cohomology: ≤ 2, subspace: $\le 2s_4$):

$$0 \le R \le \min(2, 2s_4). \tag{12}$$

5.4. **Central Orthogonal Idempotents**

$$\psi_j := \frac{1}{4}\sum_{k=0}^{3} \chi_j(\tau^{-k})\tau^k \quad \text{for } 0 \le j \le 3.$$

$$\psi_0 = -(1+\tau+\tau^2+\tau^3),\ \psi_1 = -(1+2\tau+4\tau^2+3\tau^3),$$

$$\psi_2 = -(1+4\tau+\tau^2+4\tau^3),\ \psi_3 = -(1+3\tau+4\tau^2+2\tau^3).$$

$$\psi_1 \widehat{=} \mathfrak{K}_1 = \mathfrak{L}^{1+4\tau^2+2\tau+3\tau^3} \widehat{=} (1243),\ \psi_3 \widehat{=} \mathfrak{K}_2 = \mathfrak{L}^{1+4\tau^2+3\tau+2\tau^3} \widehat{=} (1342).$$

5.5. **Special Galois Cohomology**

Herbrand Quotient:

$$\frac{\#\mathrm{H}^0(G,U_N)}{\#\mathrm{H}^{-1}(G,U_N)} = \frac{(\ker(\Delta):\mathrm{im}(\mathcal{N}))}{(\ker(\mathcal{N}):\mathrm{im}(\Delta))} = \frac{(U_K : N_{N/K}(U_N))}{(E_{N/K} : U_N^{\sigma-1})},$$

$$\#H^1(G,U_N) = \#H^0(G,U_N)\cdot[N:K] = 5^{U+1},\ 0 \le U \le 2.$$

Iwasawa Isomorphism: $H^1(G,U_N) \simeq \mathcal{P}_{N/K}/\mathcal{P}_K$.

$$(\mathcal{P}_{N/K}:\mathcal{P}_K) = (E_{N/K} : U_N^{\sigma-1}) = (U_K : N_{N/K}(U_N))\cdot 5 \in \{5,25,125\}.$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}} \times \left(\mathcal{P}_{N/K}/\mathcal{P}_K \cap \ker(N_{N/L})\right),\ U+1 = A+R_0,$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq$$
$$\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}} \times \left(\mathcal{P}_{M/K^+}/\mathcal{P}_{K^+} \cap \ker(N_{M/L})\right) \times \left(\mathcal{P}_{N/K}/\mathcal{P}_K \cap \ker(N_{N/M})\right),$$
respectively $U+1 = A+I+R$ ($R_0 = I+R$ splits further).

5.6. **Quintic DPF Types**

Type	U	$U+1=A+I+R$	A	I	R
α_1	2	3	1	0	2
α_2	2	3	1	1	1
α_3	2	3	1	2	0
β_1	2	3	2	0	1
β_2	2	3	2	1	0
γ	2	3	3	0	0
δ_1	1	2	1	0	1
δ_2	1	2	1	1	0
ε	1	2	2	0	0
ζ_1	1	2	1	0	1
ζ_2	1	2	1	1	0
η	1	2	2	0	0
ϑ	0	1	1	0	0

5.7. **Statistical Distribution** of DPF Types

Table 1. Absolute frequencies of differential principal factorization types

Type	100	200	300	400	500	600	700	800	900	1000	%
α_1	1	2	3	4	5	5	5	9	9	9	
α_2	10	17	23	30	35	42	52	57	63	75	8.3
α_3	0	0	0	1	1	3	5	5	7	8	
β_1	0	2	4	7	8	11	15	18	22	23	
β_2	7	24	40	54	80	94	108	126	146	161	17.9
γ	25	55	88	117	148	187	222	259	290	324	36.0
δ_1	0	0	1	1	3	4	4	4	6	7	
δ_2	8	14	19	23	31	35	38	44	51	53	5.9
ε	26	45	67	95	110	128	150	165	184	208	23.1
ζ_1	0	1	1	1	1	1	1	1	1	1	
ζ_2	0	0	0	0	0	1	1	4	4	5	
η	1	2	4	5	5	6	6	6	6	7	
ϑ	3	6	8	9	11	13	15	17	18	19	
Total	81	168	258	347	438	530	622	715	807	900	100.0

5.8. **Parry/Walter** Class Number Relation

$L_j = L^{\sigma^j}$ ... conjugates of the non-Galois field L, $0 \le j \le 4$,
$U_0 := \langle U_K \cdot \prod_{j=0}^{4} U_{L_j} \rangle$ ... subgroup of subfield units of U_N,
$(U_N : U_0)$ ... index of subfield units of N,
$E := v_5((U_N : U_0))$ ... logarithmic index of U_0 in U_N,
h_F ... class number of the field F,
$V_F := v_5(h_F)$ ... 5-valuation of h_F.

$$h_N = \frac{(U_N : U_0)}{5^5} \cdot h_L^4, \tag{13}$$

$$V_N = E - 5 + 4 \cdot V_L, \quad 0 \le E \le 6. \tag{14}$$

Open Problems:

1. $E = 0$ never occurs for $D < 10^3$.
   Theoretical disproof ? Or verification by some $D > 10^3$?
2. Are there provable relations between E and A, I, R ?

5.9. Polya Fields of Degree 6 or 20

F ... number field with set $\mathbb{P}_F$ of prime ideals,
$p \in \mathbb{P}$ ... prime number,
$f \in \mathbb{N}$ ... positive integer.

Ostrowski ideal of F for the prime power p^f:

$$(15) \qquad \mathfrak{b}_{p^f}(F) := \prod \{ \ \mathfrak{p} \in \mathbb{P}_F \mid N_{F/\mathbb{Q}}(\mathfrak{p}) = p^f \ \}.$$

Polya property of F (according to Zantema, Leriche):

$$(16) \qquad (\forall\, p \in \mathbb{P})\,(\forall\, f \in \mathbb{N})\,(\exists\, \gamma \in F) \quad \mathfrak{b}_{p^f}(F) = \gamma\mathcal{O}_F.$$

Proposition. Polya property of $N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$, expressed as a condition for $L = \mathbb{Q}(\sqrt[5]{D})$: N is a Polya field $\Longleftrightarrow$

$$(17) \qquad (\forall\, p \in \mathbb{P},\ p \mid f_{N/K})\,(\exists\, \alpha \in L) \quad N_{L/\mathbb{Q}}(\alpha) = p.$$

Main Theorem. Let $G := \mathrm{Gal}(N/\mathbb{Q})$. All the following statements are equivalent:

(1) N is a Polya field.
(2) Subgroup $(\mathcal{I}_N^G \cdot \mathcal{P}_N)/\mathcal{P}_N \le \mathrm{Cl}(N)$ of strongly ambiguous classes of $N/\mathbb{Q}$ is trivial.
(3) $\mathcal{I}_{N/\mathbb{Q}}/\mathcal{P}_\mathbb{Q} = \mathcal{P}_{N/\mathbb{Q}}/\mathcal{P}_\mathbb{Q}$.
(4) $\mathcal{I}_{L/\mathbb{Q}}/\mathcal{P}_\mathbb{Q} = \mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_\mathbb{Q}$, reduced from the metacyclic normal field to the pure field.
(5) $T = A$, in terms of dimensions of spaces of differential factors over $\mathbb{F}_5$ (necessarily yields an upper bound for the number of primes ramified in $L/\mathbb{Q}$, $T \le 2$ in the cubic case, $p = 3$, and $T \le 3$ in the quintic case, $p = 5$).
(6) $(\forall\, p \in \mathbb{P},\ p \mid f)\,(\exists\, \alpha \in L) \quad N_{L/\mathbb{Q}}(\alpha) = p$.

References.

(1) P. Barrucand and H. Cohn, *A rational genus, class number divisibility, and unit theory for pure cubic fields*, J. Number Theory **2** (1970), no. 1, 7–21.
(2) P. Barrucand and H. Cohn, *Remarks on principal factors in a relative cubic field*, J. Number Theory **3** (1971), no. 2, 226–239.
(3) W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.
(4) W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), *Handbook of Magma functions* (Edition 2.24, Sydney, 2018).
(5) F. Halter-Koch, *Eine Bemerkung über kubische Einheiten*, Arch. Math. **27** (1976), 593–595.
(6) T. Honda, *Pure cubic fields whose class numbers are multiples of three*, J. Number Theory **3** (1971), no. 1, 7–12.
(7) K. Iimura, *A criterion for the class number of a pure quintic field to be divisible by* 5, J. Reine Angew. Math. **292** (1977), 201–210.
(8) H. Kobayashi, *Class numbers of pure quintic fields*, J. Number Theory **160** (2016), no. 1, 463–477.
(9) H. Kobayashi, *Class numbers of pure quintic fields*, Ph.D. thesis, Osaka University Knowledge Archive (OUKA), DOI 10.18910/56049.
(10) MAGMA Developer Group, MAGMA *Computational Algebra System*, Version 2.24-3, Sydney, 2018, `(http://magma.maths.usyd.edu.au)`.
(11) D. C. Mayer, *Differential principal factors and units in pure cubic number fields*, Preprint, Dept. of Math., Univ. Graz, 1989.
(12) D. C. Mayer, *Classification of dihedral fields*, Preprint, Dept. of Comp. Science, Univ. of Manitoba, 1991.
(13) D. C. Mayer, *List of discriminants $d_L < 200\,000$ of totally real cubic fields L, arranged according to their multiplicities m and conductors f*, Computer Centre, Department of Computer Science, University of Manitoba, Winnipeg, Canada, 1991, Austrian Science Fund (FWF), Project Nr. J 0497-PHY.
(14) D. C. Mayer, *Discriminants of metacyclic fields*, Canad. Math. Bull. **36** (1) (1993), 103–107.
(15) C. J. Parry, *Class number relations in pure quintic fields*, Symposia Mathematica **15** (1975), 475–485, Academic Press, London. (Convegno di Strutture in Corpi Algebrici, INDAM, Rome, 1973.)
(16) C. J. Parry and C. D. Walter, *The class number of pure fields of prime degree*, Mathematika **23** (1976), no. 2, 220–226; Mathematika **24** (1977), 133.
(17) G. F. Voronoi, *Ob odnom obobshchenii algorifma nepreryvnykh drobei* (On a generalization of the algorithm of continued fractions), Doctoral Dissertation, 1896, Warsaw (Russian).
(18) C. D. Walter, *A class number relation in Frobenius extensions of number fields*, Mathematika **24** (1977), 216–225.
(19) C. D. Walter, *Pure fields of degree* 9 *with class number prime to* 3, Ann. Inst. Fourier Grenoble **30** (1980), no. 2, 1–15.
(20) H. C. Williams, *Determination of principal factors in $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt[3]{D})$*, Math. Comp. **38** (1982), no. 157, 261–274.