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Differential Principal Factors (DPF) in Pure Metacyclic Fields

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Towers of p-Class Fields over Algebraic Number Fields

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1. **General Pure Metacyclic Fields**

$p \geq 3$ ... odd prime number,
$D \geq 2$ ... pth power free integer,
$\zeta_p = \exp(2\pi\sqrt{-1}/p)$ ... primitive pth root of unity,
$N = \mathbb{Q}(\zeta_p, \sqrt[p]{D})$ ... **pure metacyclic** field, deg. $(p-1)\cdot p$,
$L = \mathbb{Q}(\sqrt[p]{D})$ ... **pure** subfield of degree p of N,
$K = \mathbb{Q}(\zeta_p)$ ... pth cyclotomic field.

1.1. **General Galois Cohomology**

$G = \mathrm{Gal}(N/K) = \langle\sigma\rangle \simeq C_p$, rel. group of Kummer ext.,
$E_{N/K} = U_N \cap \ker(N_{N/K})$, where U_N = unit group of N.

Herbrand Quotient of the G-module U_N:

$$\frac{\#\mathrm{H}^0(G,U_N)}{\#\mathrm{H}^1(G,U_N)} = \frac{(\ker(\Delta):\mathrm{im}(\mathcal{N}))}{(\ker(\mathcal{N}):\mathrm{im}(\Delta))} = \frac{(U_K : N_{N/K}(U_N))}{(E_{N/K} : U_N^{\sigma-1})},$$

$$\#H^1(G,U_N) = \#H^0(G,U_N)\cdot[N:K] = p^{U+1},\ 0 \leq U \leq \frac{p-1}{2}.$$

Iwasawa Isomorphism: $\quad H^1(G,U_N) \simeq \overbrace{\mathcal{P}_{N/K}/\mathcal{P}_K}^{\text{DPF}},$

$$(\mathcal{P}_{N/K} : \mathcal{P}_K) = (E_{N/K} : U_N^{\sigma-1}) = (U_K : N_{N/K}(U_N)) \cdot p.$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \underbrace{\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}}_{\text{abs. DPF}} \times \underbrace{\left(\mathcal{P}_{N/K}/\mathcal{P}_K \cap \ker(N_{N/L})\right)}_{\text{rel. DPF (norm kernel)}},\ U+1 = A+R.$$

Groups of ambiguous ideals: $\mathcal{P}_{N/K} := \mathcal{P}_N^G$, $\mathcal{P}_{L/\mathbb{Q}} := \mathcal{P}_L \cap \mathcal{P}_N^G$, $\mathcal{I}_{E/E_0} := \{\mathfrak{A} \in \mathcal{I}_E \mid \mathfrak{A}^{[E:E_0]} \in \mathcal{I}_{E_0}\}$.

1.2. **General Norm Kernel**

Theorem. D.C. Mayer (10,13) (Primitive ambiguous ideals)
Let $p \in \mathbb{P}$ be a prime, and $q \in \mathbb{N}$ be an integer coprime to p. Suppose F/F_0 is a number field extension of degree p, E_0/F_0 is an extension of degree q, and $E = F \cdot E_0$ is the compositum of F and E_0. (Most important situation: $q = p - 1$, $p \neq 2$.)
(1) The norm map $N_{E/F} : \mathcal{I}_E \to \mathcal{I}_F$ satisfies

$$N_{E/F}(\mathcal{I}_{E/E_0}) \leq \mathcal{I}_{F/F_0} \text{ and } N_{E/F}(\mathcal{I}_{E_0}) \leq \mathcal{I}_{F_0},$$

and induces an epimorphism

$$N_{E/F} : \mathcal{I}_{E/E_0}/\mathcal{I}_{E_0} \to \mathcal{I}_{F/F_0}/\mathcal{I}_{F_0}.$$

(2) There are isomorphisms of elementary abelian p-groups

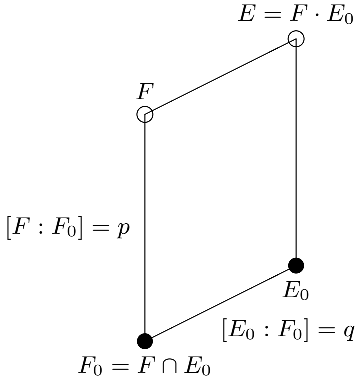
$$\mathcal{I}_{F/F_0}/\mathcal{I}_{F_0} \simeq (\mathcal{I}_{E/E_0}/\mathcal{I}_{E_0})/\ker(N_{E/F})$$

and

$$\mathcal{I}_{E/E_0}/\mathcal{I}_{E_0} \simeq (\mathcal{I}_{F/F_0}/\mathcal{I}_{F_0}) \times \ker(N_{E/F})$$

(Natural decomposition of primitive ambiguous ideal groups).

Figure 1. Compositum of extensions F/F_0 and E_0/F_0 with coprime degrees



Classification into coarse types by U and A, $\bullet$ coarse type splits into subtypes, $\circ$ does not split.

Figure 2. Classification of simply real dihedral [17] and pure cubic [2,3] fields, $p = 3$

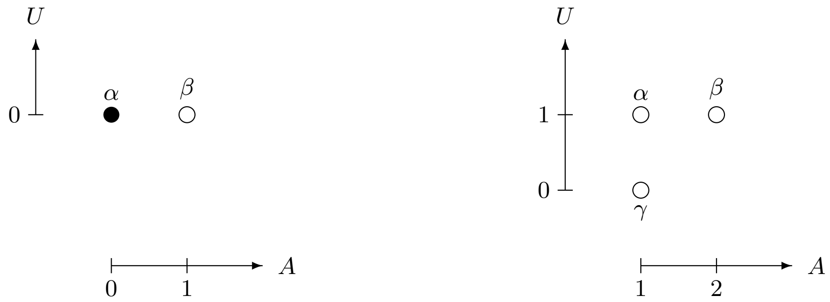


Figure 3. Classification of totally real dihedral [17] and pure quintic [13] fields, $p = 5$

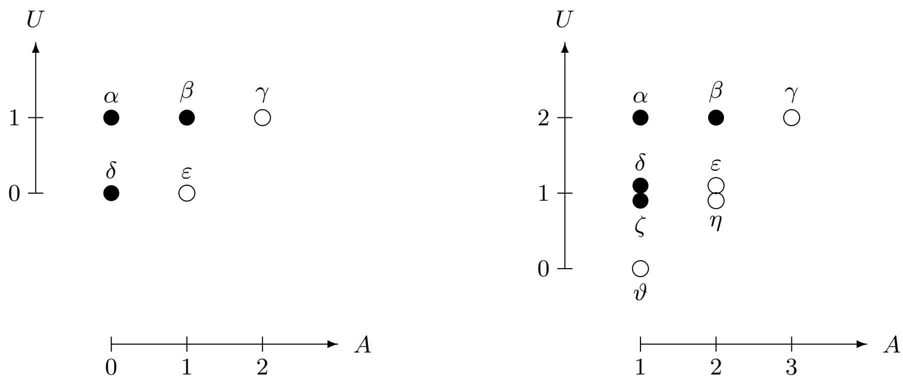
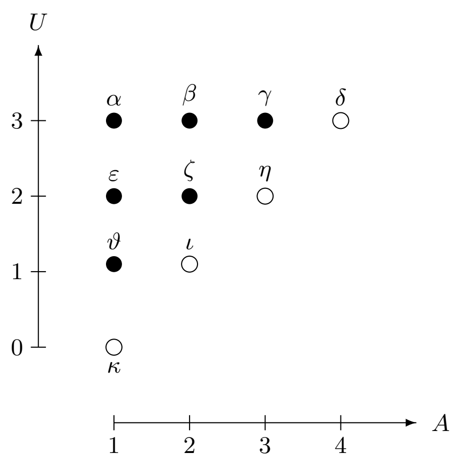


Figure 4. Classification of pure septic fields, $p = 7$



1.3. **General DPF Types**

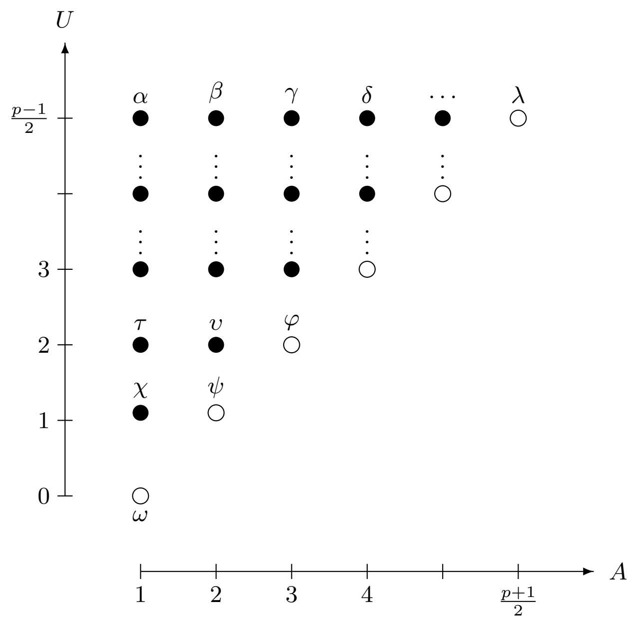
• **Total** $\mathbb{F}_p$-Vector Space of All Differential Principal Factors with Dimension:

$$\dim_{\mathbb{F}_p}(\mathcal{P}_{N/K}/\mathcal{P}_K) = U + 1,$$

• Subspace of **Absolute** Differential Principal Factors with Dimension:

$$A := \dim_{\mathbb{F}_p}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}),\ 1 \le A \le U + 1$$

FIGURE 5. Classification of general pure fields of odd prime degree $p \ge 11$



1.4. General Homogeneous Multiplets

Theorem. [16] Let $(N_1,\ldots,N_m)$ be a multiplet of nonisomorphic pure metacyclic fields $N_i=\mathbb{Q}(\zeta_p,\sqrt[p]{D_i})$, with a primitive pth root of unity ζ_p and distinct normalized pth power free radicands $D_i>1$, which share a common conductor f over the cyclotomic field $K=\mathbb{Q}(\zeta_p)$.

(1) If either $f^{p-1}=p^{p+1}$ or $f=q$ with a prime $q\in\mathbb{P}$ such that $q^{p-1}\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=1$ and the **singulet** N_1 is of species 2 for $f=q$, of species 1a otherwise, and of the unique DPF type ω with $U=R=0$ and $A=1$.

(2) If $f^{p-1}=p^{p+1}\cdot q^{p-1}$ with a prime $q\in\mathbb{P}\setminus\{p\}$ such that $q^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=p-1$ and the **multiplet** $(N_1,\ldots,N_{p-1})$ is of species 1a and of the **homogeneous** DPF type $(\psi,\ldots,\psi)$ with $U=1$, $R=0$ and $A=2$.

(3) If $f^{p-1}=p^2\cdot q^{p-1}$ with a prime $q\in\mathbb{P}\setminus\{p\}$ such that $q^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q does not split in K, then $m=1$ and the **singulet** N_1 is of species 1b and of the unique DPF type ψ with $U=1$, $R=0$ and $A=2$.

(4) If $f=q_1\cdot q_2$ with distinct primes $q_j\in\mathbb{P}\setminus\{p\}$ such that $q_j^{p-1}\not\equiv 1\,(\mathrm{mod}\,p^2)$ and q_j does not split in K for $1\le j\le 2$, then $m=1$ and the **singulet** N_1 is of species 2 and of the unique DPF type ψ with $U=1$, $R=0$ and $A=2$.

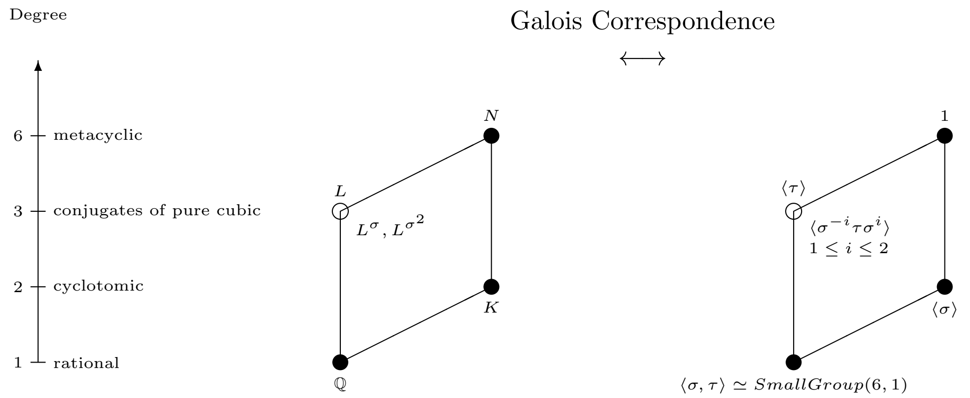
The conditions for the unit norm index can be expressed equivalently by $\quad U=0\Longleftrightarrow N_{N/K}(U_N)=U_K$,
and $\quad U=1\Longleftrightarrow N_{N/K}(U_N)\simeq U_K/\langle\zeta_p\rangle$.

P. Barrucand and H. Cohn [1,2], F. Halter-Koch [3], T. Honda [4], H. C. Williams [22], 1969 – 1982:

3. **Pure Cubic Fields,** $p = 3$

$D \geq 2$ … third power free integer,
$\zeta_3 = \exp(2\pi\sqrt{-1}/3)$ … primitive third root of unity,
$N = \mathbb{Q}(\zeta_3, \sqrt[3]{D})$ … **pure metacyclic** field, degree 6,
$L = \mathbb{Q}(\sqrt[3]{D})$ … **pure cubic** subfield of N,
$K = \mathbb{Q}(\zeta_3)$ … third cyclotomic field, imaginary quadratic.

FIGURE 6. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



Relative different of N with respect to K:

$$\mathfrak{D}_{N/K} = \mathfrak{p}^e \cdot \underbrace{\prod_{i=1}^{n} \mathfrak{q}_i^2}_{\text{absolute}} \cdot \underbrace{\prod_{j=1}^{s} (\mathfrak{L}_j \cdot \mathfrak{L}_j^\tau)^2}_{\text{relative}}$$

where $3\mathcal{O}_L = \mathfrak{p}^3$ or $\mathfrak{p}^2\mathfrak{p}'$, $\quad q_i\mathcal{O}_L = \mathfrak{q}_i^3$, $\quad \ell_j\mathcal{O}_N = (\mathfrak{L}_j \cdot \mathfrak{L}_i^\tau)^3$.

3.1. **Absolute** Differential Principal Factors

T ... number of primes q_i dividing the conductor f of N/K,
$q_i\mathcal{O}_L = \mathfrak{q}_i^3$ ... ramification of q_i in L.

• Space of Absolute Differential Factors in $L/\mathbb{Q}$
Dimension:

$$\dim_{\mathbb{F}_3}(\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}) = T(= n + s), \tag{1}$$

Basis:

$$\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}} \simeq \bigoplus_{i=1}^{T} \mathbb{F}_3\, \mathfrak{q}_i\,. \tag{2}$$

• Subspace of Absolute Differential **Principal** Factors
Dimension:

$$A := \dim_{\mathbb{F}_3}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}), \tag{3}$$

Bounds for the dimension (cohomology: ≤ 2, subspace: $\le T$):

$$1 \le A \le \min(2, T). \tag{4}$$

3.2. **Relative** Differential Principal Factors

s ... number of primes $\ell_i \equiv +1 \pmod 3$ dividing f,
$\ell_i \mathcal{O}_N = (\mathfrak{L}_i \cdot \mathfrak{L}_i^\tau)^3$ ... 2-splitting of ℓ_i in N.

• Space of Relative Differential Factors in N/K
Dimension:

$$(5) \qquad \dim_{\mathbb{F}_3}\left((\mathcal{I}_{N/K}/\mathcal{I}_K) \bigcap \ker(N_{N/L})\right) = s$$

Basis with elements $\mathfrak{K}_{(\ell_i)} := \mathfrak{L}_i \cdot (\mathfrak{L}_i^\tau)^2$:

$$(6) \qquad (\mathcal{I}_{N/K}/\mathcal{I}_K) \bigcap \ker(N_{N/L}) \simeq \bigoplus_{i=1}^{s} \mathbb{F}_3 \, \mathfrak{K}_{(\ell_i)} \, .$$

• Subspace of Relative Differential **Principal** Factors
Dimension:

$$(7) \qquad R := \dim_{\mathbb{F}_3}\left((\mathcal{P}_{N/K}/\mathcal{P}_K) \bigcap \ker(N_{N/L})\right),$$

Bounds (cohomology: ≤ 1, subspace: $\leq s$):

$$(8) \qquad 0 \leq R \leq \min(1, s).$$

3.3. **Semi-Local Contributions**

Non-split primes only contribute to absolute DPF.
2-split primes contribute to both kinds of DPF.

TABLE 1. Complete Census of $\mathbb{F}_3$-Dimensions

Entire Space of DPF $\mathcal{I}_{N/K}/\mathcal{I}_K$ $n+2s$	Absolute $\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_\mathbb{Q}$ $n+s$	Relative $\ker(N_{N/L})$ s
$\forall\, 1 \le i \le n$	$\mathbb{F}_3\mathfrak{q}_i$ n	— —
$\forall\, 1 \le i \le s$	$\mathbb{F}_3\mathfrak{L}_i^{1+\tau}$ s	$\mathbb{F}_3\mathfrak{L}_i^{1+2\tau}$ s

3.4. **Smallest Examples of DPF Type** α

Primes $\ell \equiv 1 \pmod 3$ start with $7, 13, 19, 31, 37, 43, 61, 67 \ldots$
The fields $\mathbb{Q}(\sqrt[3]{\ell})$ have conductor $f = 3 \cdot \ell$, $\quad n = 1$, $\quad s = 1$.
Barrucand and Cohn [(2)] investigated principal factors in 1971:

Trivially, all the fields share principal $\mathfrak{L}^{1+\tau} = \sqrt[3]{\ell}\mathcal{O}_L$,
but $7, 13, 19, 31, 37, 43, 73, 79, 97$ show the usual $A = R = 1$ with principal $\mathfrak{L}^{1+2\tau} = \Lambda\mathcal{O}_N$,
whereas $61, 67$ reveal the exceptional $A = 2$ with principal $\mathfrak{q} = \kappa\mathcal{O}_L$, $N(\kappa) = 3$.
Consequently, $\mathbb{Q}(\sqrt[3]{\ell})$ is of **type** β for $\ell = 61, 67$ with $\left(\frac{3}{\ell}\right)_3 = 1$, and of **type** α otherwise.

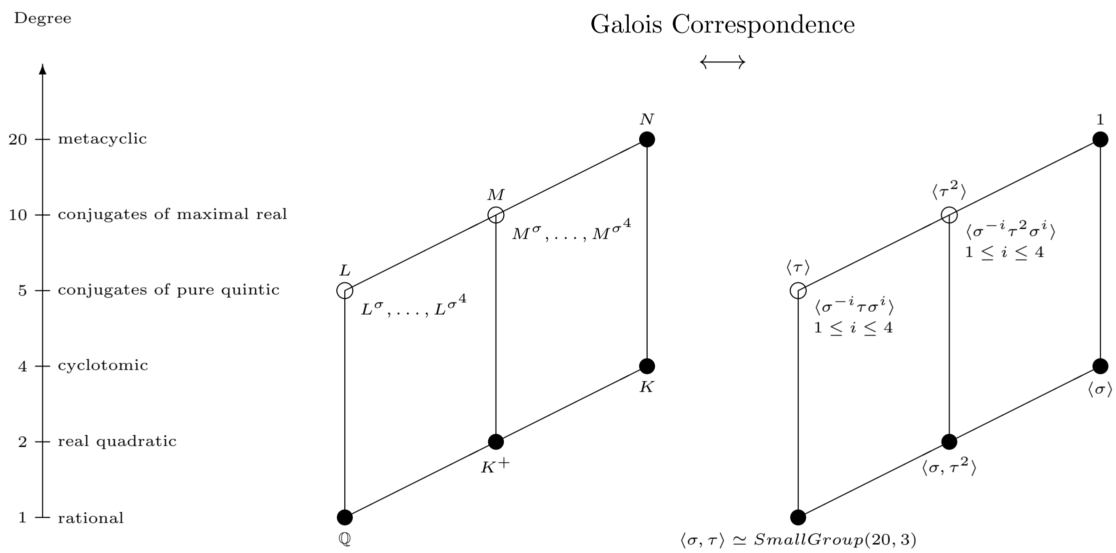
(Sparsely, there also occurs another type with $A = 1$ and $R = 0$ (D.C. Mayer, 2002):
type γ with $\zeta_3 = N_{N/K}(Z)$ for $\ell = 541, 919$, i.e. bigger prime radicands $\ell \equiv 1 \pmod 3$.)

D. C. Mayer, 1991 (10,12) and 2014 – 2018 (13,14,15,16):

5. **Pure Quintic Fields,** $p = 5$

$D \ge 2$ ... fifth power free integer,
$\zeta_5 = \exp(2\pi\sqrt{-1}/5)$ ... primitive fifth root of unity,
$N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$ ... **pure metacyclic** field, degree 20,
$M = \mathbb{Q}(\sqrt{5}, \sqrt[5]{D})$ ... maximal real subfield of N, deg. 10,
$L = \mathbb{Q}(\sqrt[5]{D})$ ... **pure quintic** subfield of N,
$K = \mathbb{Q}(\zeta_5)$ ... fifth cyclotomic field, cyclic quartic,
$K^+ = \mathbb{Q}(\sqrt{5})$ ... maximal real subfield of K, real quadratic.

FIGURE 7. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



Relative different of N with respect to K:

$$\mathfrak{D}_{N/K} = \underbrace{\mathfrak{p}^e \cdot \prod_{i=1}^{n} \mathfrak{q}_i^4}_{\text{absolute}} \cdot \underbrace{\prod_{i=1}^{s_2} (\mathcal{L}_i \cdot \mathcal{L}_i^\tau)^4}_{\text{intermediate}} \cdot \underbrace{\prod_{i=s_2+1}^{s_4} (\mathfrak{L}_i \cdot \mathfrak{L}_i^{\tau^2} \cdot \mathfrak{L}_i^{\tau} \cdot \mathfrak{L}_i^{\tau^3})^4}_{\text{relative}}$$

5.1. **Absolute** Differential Principal Factors

T ... number of primes q_i dividing the conductor f of N/K,
$q_i\mathcal{O}_L = \mathfrak{q}_i^5$ ... ramification of q_i in L.

• Space of Absolute Differential Factors in $L/\mathbb{Q}$
Dimension:

$$\dim_{\mathbb{F}_5}(\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}) = T(= n + s_2 + s_4), \tag{9}$$

Basis:

$$\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}} \simeq \bigoplus_{i=1}^{T} \mathbb{F}_5\,\mathfrak{q}_i\,. \tag{10}$$

• Subspace of Absolute Differential **Principal** Factors
Dimension:

$$A := \dim_{\mathbb{F}_5}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}), \tag{11}$$

Bounds for the dimension (cohomology: ≤ 3, subspace: $\le T$):

$$1 \le A \le \min(3, T). \tag{12}$$

5.2. **Intermediate** Differential Principal Factors

s_2 ... number of primes $\ell_i \equiv -1 \pmod 5$ dividing f,
s_4 ... number of primes $\ell_{s_2+i} \equiv +1 \pmod 5$ dividing f,
$\ell_i\mathcal{O}_M = (\mathcal{L}_i \cdot \mathcal{L}_i^\tau)^5$ ... 2-splitting of ℓ_i in M.

• Space of Intermediate Differential Factors in M/K^+

Dimension:

$$(13) \qquad \dim_{\mathbb{F}_5}\Big((\mathcal{I}_{M/K^+}/\mathcal{I}_{K^+})\bigcap \ker(N_{M/L})\Big) = s_2 + s_4$$

Basis with elements $\mathcal{K}_{(\ell_i)} := \mathcal{L}_i \cdot (\mathcal{L}_i^\tau)^4$ (see § 5.4):

$$(14) \qquad (\mathcal{I}_{M/K^+}/\mathcal{I}_{K^+})\bigcap \ker(N_{M/L}) \simeq \bigoplus_{i=1}^{s_2+s_4} \mathbb{F}_5\,\mathcal{K}_{(\ell_i)}\,.$$

• Subspace of Intermediate Differential **Principal** Factors
Dimension:

$$(15) \qquad I := \dim_{\mathbb{F}_5}\Big((\mathcal{P}_{M/K^+}/\mathcal{P}_{K^+})\bigcap \ker(N_{M/L})\Big)\,,$$

Bounds (cohomology: ≤ 2, subspace: $\leq s_2 + s_4$):

$$(16) \qquad 0 \leq I \leq \min(2, s_2 + s_4).$$

5.3. **Relative** Differential Principal Factors

s_4 ... number of primes $\ell_{s_2+i} \equiv +1 \pmod 5$ dividing f,
$\ell_i \mathcal{O}_N = (\mathfrak{L}_i \cdot \mathfrak{L}_i^{\tau} \cdot \mathfrak{L}_i^{\tau^2} \cdot \mathfrak{L}_i^{\tau^3})^5$ ... 4-splitting of ℓ_i in N.

• Space of Relative Differential Factors in N/K
Dimension:

$$\dim_{\mathbb{F}_5}\left((\mathcal{I}_{N/K}/\mathcal{I}_K)\bigcap \ker(N_{N/M})\right) = 2s_4 \tag{17}$$

Basis with $\mathfrak{K}_{(\ell_i),1} := \mathfrak{L}_i \cdot (\mathfrak{L}_i^{\tau})^2 \cdot (\mathfrak{L}_i^{\tau^2})^4 \cdot (\mathfrak{L}_i^{\tau^3})^3$ and
$\mathfrak{K}_{(\ell_i),2} := \mathfrak{L}_i \cdot (\mathfrak{L}_i^{\tau})^3 \cdot (\mathfrak{L}_i^{\tau^2})^4 \cdot (\mathfrak{L}_i^{\tau^3})^2$ (see § 5.4):

(18)

$$(\mathcal{I}_{N/K}/\mathcal{I}_K)\bigcap \ker(N_{N/M}) \simeq \bigoplus_{i=s_2+1}^{s_2+s_4} \left(\mathbb{F}_5\,\mathfrak{K}_{(\ell_i),1} \oplus \mathbb{F}_5\,\mathfrak{K}_{(\ell_i),2}\right)\ .$$

• Subspace of Relative Differential **Principal** Factors
Dimension:

$$R := \dim_{\mathbb{F}_5}\left((\mathcal{P}_{N/K}/\mathcal{P}_K)\bigcap \ker(N_{N/M})\right), \tag{19}$$

Bounds (cohomology: ≤ 2, subspace: $\le 2s_4$):

$$0 \le R \le \min(2, 2s_4). \tag{20}$$

5.4. **Central Orthogonal Idempotents (COI)**

Four COI, formally in terms of characters, $\chi_j \in \langle\tau\rangle^*$, $\chi_0 = 1$:

$$\psi_j := \frac{1}{4}\sum_{k=0}^{3} \chi_j(\tau^{-k})\tau^k \quad \text{for } 0 \le j \le 3.$$

Four COI, in terms of invertible residue classes mod 5,
$\langle\zeta_4\rangle \simeq C_4 \simeq U(\mathbb{Z}/5\mathbb{Z})$, $\zeta_4 \mapsto 2$, $\zeta_4^2 \mapsto 4$, $\zeta_4^3 \mapsto 3$:

$$\psi_0 = -(1+\tau+\tau^2+\tau^3), \quad \psi_1 = -(1+2\tau+4\tau^2+3\tau^3),$$

$$\psi_2 = -(1+4\tau+\tau^2+4\tau^3), \quad \psi_3 = -(1+3\tau+4\tau^2+2\tau^3).$$

COI corresponding to **Basis** elements of **Norm Kernels**:

$\ker(N_{N/M})$:

$$\psi_1 \hat{=} \mathfrak{K}_1 = \mathfrak{L}^{(1+4\tau^2)+2\tau(1+4\tau^2)} = \mathfrak{L}^{1+2\tau+4\tau^2+3\tau^3} \hat{=} (1243),$$
$$\psi_3 \hat{=} \mathfrak{K}_2 = \mathfrak{L}^{(1+4\tau^2)+3\tau(1+4\tau^2)} = \mathfrak{L}^{1+3\tau+4\tau^2+2\tau^3} \hat{=} (1342).$$

$\ker(N_{M/L})$:

$$\psi_2 \hat{=} \mathcal{K} = \mathcal{L}^{1+4\tau} = \mathfrak{L}^{(1+\tau^2)+4\tau(1+\tau^2)} = \mathfrak{L}^{1+4\tau+\tau^2+4\tau^3} \hat{=} (1414).$$

5.5. **Semi-Local Contributions**

Non-split primes only contribute to absolute DPF.
2-split primes contribute to several kinds of DPF.
4-split primes contribute to all kinds of DPF.

TABLE 2. Complete Census of $\mathbb{F}_5$-Dimensions

Entire Space of DPF	Absolute	Intermediate	Relative	
$\mathcal{I}_{N/K}/\mathcal{I}_K$	$\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_\mathbb{Q}$	$\ker(N_{M/L})$	$\ker(N_{N/M})$	
$n+2s_2+4s_4$	$n+s_2+s_4$	s_2+s_4	$2s_4$	
$\forall\, 1\le i\le n$	$\mathbb{F}_5\mathfrak{q}_i$	—	—	
	n	—	—	
$\forall\, 1\le i\le s_2$	$\mathbb{F}_5\mathcal{L}_i^{1+\tau}$	$\mathbb{F}_5\mathcal{L}_i^{1+4\tau}$	—	
	s_2	s_2	—	
$\forall\, 1\le i\le s_4$	$\mathbb{F}_5\mathfrak{L}_i^{1+\tau+\tau^2+\tau^3}$	$\mathbb{F}_5\mathfrak{L}_i^{1+4\tau+\tau^2+4\tau^3}$	$\mathbb{F}_5\mathfrak{L}_i^{1+2\tau+4\tau^2+3\tau^3}$	$\leftarrow \mathfrak{K}_1$
		$\mathcal{K}\uparrow$	$\oplus\mathbb{F}_5\mathfrak{L}_i^{1+3\tau+4\tau^2+2\tau^3}$	$\leftarrow \mathfrak{K}_2$
	s_4	s_4	$2s_4$	

5.6. **Smallest Examples of DPF Type** α

The primes $\ell \equiv +1 \pmod 5$ start with $11, 31, 41, 61, 71, \ldots$
The fields $\mathbb{Q}(\sqrt[5]{\ell})$ have $f^4 = 5^2\cdot\ell^4$, $\quad n=1$, $\quad s_4=1$.
It was puzzling to investigate their principal factorizations:

Trivially, they all share principal $\mathfrak{L}^{1+\tau+\tau^2+\tau^3} = \sqrt[5]{\ell}\mathcal{O}_L$,
but $11, 41, 61, 71$ show the usual $A=I=R=1$ with principal $\mathcal{K}=\lambda\mathcal{O}_M$, $\mathfrak{K}=\Lambda\mathcal{O}_N$,
whereas 31 reveals the exceptional $A=1$, $R=2$ with principal $\mathfrak{K}_1=\Lambda_1\mathcal{O}_N$, $\mathfrak{K}_2=\Lambda_2\mathcal{O}_N$.
Consequently, $\mathbb{Q}(\sqrt[5]{\ell})$ is of **type** $\boldsymbol{\alpha_1}$ for $\ell=31$, and of **type** $\boldsymbol{\alpha_2}$ otherwise (see § 5.8).

(Sparsely, there also occur some other types with common $R=1$ and $\mathfrak{K}=\Lambda\mathcal{O}_N$:
type ζ_1 with $\zeta_5 = N_{N/K}(Z)$ for $\ell=101$,
type β_1 with $\mathfrak{q}=\kappa\mathcal{O}_L$, $N(\kappa)=5$ for $\ell=191$ with $\left(\frac{5}{\ell}\right)_5=1$,
and type δ_1 with $\eta=N_{N/K}(H)$ for $\ell=211$, i.e. bigger prime radicands $\ell\equiv+1 \pmod 5$.)

5.7. **Special Galois Cohomology**

Herbrand Quotient:

$$\frac{\#\mathrm{H}^0(G,U_N)}{\#\mathrm{H}^{-1}(G,U_N)} = \frac{(\ker(\Delta):\mathrm{im}(\mathcal{N}))}{(\ker(\mathcal{N}):\mathrm{im}(\Delta))} = \frac{(U_K:N_{N/K}(U_N))}{(E_{N/K}:U_N^{\sigma-1})},$$

$$\#H^1(G,U_N) = \#H^0(G,U_N)\cdot[N:K] = 5^{U+1},\ 0\le U\le 2.$$

Iwasawa Isomorphism: $H^1(G,U_N) \simeq \mathcal{P}_{N/K}/\mathcal{P}_K$.

$$(\mathcal{P}_{N/K}:\mathcal{P}_K) = (E_{N/K}:U_N^{\sigma-1}) = (U_K:N_{N/K}(U_N))\cdot 5 \in \{5,25,125\}.$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}\times\left(\mathcal{P}_{N/K}/\mathcal{P}_K\cap\ker(N_{N/L})\right),\ U{+}1 = A{+}R_0,$$

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq$$
$$\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}\times\left(\mathcal{P}_{M/K^+}/\mathcal{P}_{K^+}\cap\ker(N_{M/L})\right)\times\left(\mathcal{P}_{N/K}/\mathcal{P}_K\cap\ker(N_{N/M})\right),$$
respectively $U+1 = A+I+R$ ($R_0 = I+R$ splits further).

5.8. **Quintic DPF Types**

TABLE 3. Classification with respect to U, A, I, R

Type	U	$U+1=A+I+R$	A	I	R
α_1	2	3	1	0	2
α_2	2	3	1	1	1
α_3	2	3	1	2	0
β_1	2	3	2	0	1
β_2	2	3	2	1	0
γ	2	3	3	0	0
δ_1	1	2	1	0	1
δ_2	1	2	1	1	0
ε	1	2	2	0	0
ζ_1	1	2	1	0	1
ζ_2	1	2	1	1	0
η	1	2	2	0	0
ϑ	0	1	1	0	0

Types $\delta_1, \delta_2, \varepsilon$ and ζ_1, ζ_2, η share common invariants A, I, R, respectively.
If $\eta_0 = \frac{1}{2}(1+\sqrt{5})$ denotes the fundamental unit of K^+,
types $\delta_1, \delta_2, \varepsilon$ have $\eta_0 = N_{N/K}(H)$ for some $H \in U_N$, and
types ζ_1, ζ_2, η have $\zeta_5 = N_{N/K}(Z)$ for some $Z \in U_N$.

5.9. **Statistical Distribution** of DPF Types [13,15]

By means of MAGMA, we computed a table of all pure quintic fields $L = \mathbb{Q}(\sqrt[5]{D})$ with normalized radicands $2 \le D < 10^3$.

Table 4. Absolute frequencies of differential principal factorization types

Type	100	200	300	400	500	600	700	800	900	1000	%
α_1	1	2	3	4	5	5	5	9	9	9	
α_2	10	17	23	30	35	42	52	57	63	75	8.3
α_3	0	0	0	1	1	3	5	5	7	8	
β_1	0	2	4	7	8	11	15	18	22	23	
β_2	7	24	40	54	80	94	108	126	146	161	17.9
γ	25	55	88	117	148	187	222	259	290	324	36.0
δ_1	0	0	1	1	3	4	4	4	6	7	
δ_2	8	14	19	23	31	35	38	44	51	53	5.9
ε	26	45	67	95	110	128	150	165	184	208	23.1
ζ_1	0	1	1	1	1	1	1	1	1	1	
ζ_2	0	0	0	0	0	1	1	4	4	5	
η	1	2	4	5	5	6	6	6	6	7	
ϑ	3	6	8	9	11	13	15	17	18	19	
Total	81	168	258	347	438	530	622	715	807	900	100.0

Absolute differential principal factorizations are dominating. The high champion is DPF type γ followed by type ε.

5.10. **Parry/Walter** (18,19,20,21) Class Number Relation

$L_j = L^{\sigma^j}$ ... conjugates of the non-Galois field L, $0 \le j \le 4$,
$U_0 := \langle U_K \cdot \prod_{j=0}^{4} U_{L_j} \rangle$ ... subgroup of subfield units of U_N,
$(U_N : U_0)$ ... index of subfield units of N,
$E := v_5((U_N : U_0))$ ... logarithmic index of U_0 in U_N,
h_F ... class number of the field F,
$V_F := v_5(h_F)$ ... 5-valuation of h_F.

$$h_N = \frac{(U_N : U_0)}{5^5} \cdot h_L^4, \tag{21}$$

$$V_N = E - 5 + 4 \cdot V_L, \quad 0 \le E \le 6. \tag{22}$$

Open Problems:

1. $E = 0$ never occurs for $D < 10^3$.
   Theoretical disproof ? Or verification by some $D > 10^3$?
2. Are there provable relations between E and A, I, R ?

5.11. Polya Fields of Degree 6 or 20 (13)

F ... number field with set $\mathbb{P}_F$ of prime ideals,
$p \in \mathbb{P}$ ... prime number,
$f \in \mathbb{N}$ ... positive integer.

Ostrowski ideal of F for the prime power p^f:

$$\mathfrak{b}_{p^f}(F) := \prod \{ \, \mathfrak{p} \in \mathbb{P}_F \mid N_{F/\mathbb{Q}}(\mathfrak{p}) = p^f \, \}. \tag{23}$$

Polya property of F (according to Zantema, Leriche):

$$(\forall \, p \in \mathbb{P}) \, (\forall \, f \in \mathbb{N}) \, (\exists \, \gamma \in F) \quad \mathfrak{b}_{p^f}(F) = \gamma \mathcal{O}_F. \tag{24}$$

Proposition. Polya property of $N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$, expressed as a condition for $L = \mathbb{Q}(\sqrt[5]{D})$: N is a Polya field $\Longleftrightarrow$

$$(\forall \, p \in \mathbb{P}, \; p \mid f_{N/K}) \, (\exists \, \alpha \in L) \quad N_{L/\mathbb{Q}}(\alpha) = p. \tag{25}$$

Main Theorem. Let $G := \mathrm{Gal}(N/\mathbb{Q})$. All the following statements are equivalent:

1. N is a Polya field.
2. Subgroup $(\mathcal{I}_N^G \cdot \mathcal{P}_N)/\mathcal{P}_N \le \mathrm{Cl}(N)$ of strongly ambiguous classes of $N/\mathbb{Q}$ is trivial.
3. $\mathcal{I}_{N/\mathbb{Q}}/\mathcal{P}_\mathbb{Q} = \mathcal{P}_{N/\mathbb{Q}}/\mathcal{P}_\mathbb{Q}$, all primitive ambiguous ideals are principal.
4. $\mathcal{I}_{L/\mathbb{Q}}/\mathcal{P}_\mathbb{Q} = \mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_\mathbb{Q}$, reduced from the metacyclic normal field to the pure field.
5. $T = A$, in terms of dimensions of spaces of differential factors over $\mathbb{F}_5$ (necessarily yields an upper bound for the number of primes ramified in $L/\mathbb{Q}$, $T \le 2$ in the cubic case, $p = 3$, and $T \le 3$ in the quintic case, $p = 5$).
6. $(\forall \, p \in \mathbb{P}, \; p \mid f_{N/K}) \, (\exists \, \alpha \in L) \quad N_{L/\mathbb{Q}}(\alpha) = p$.

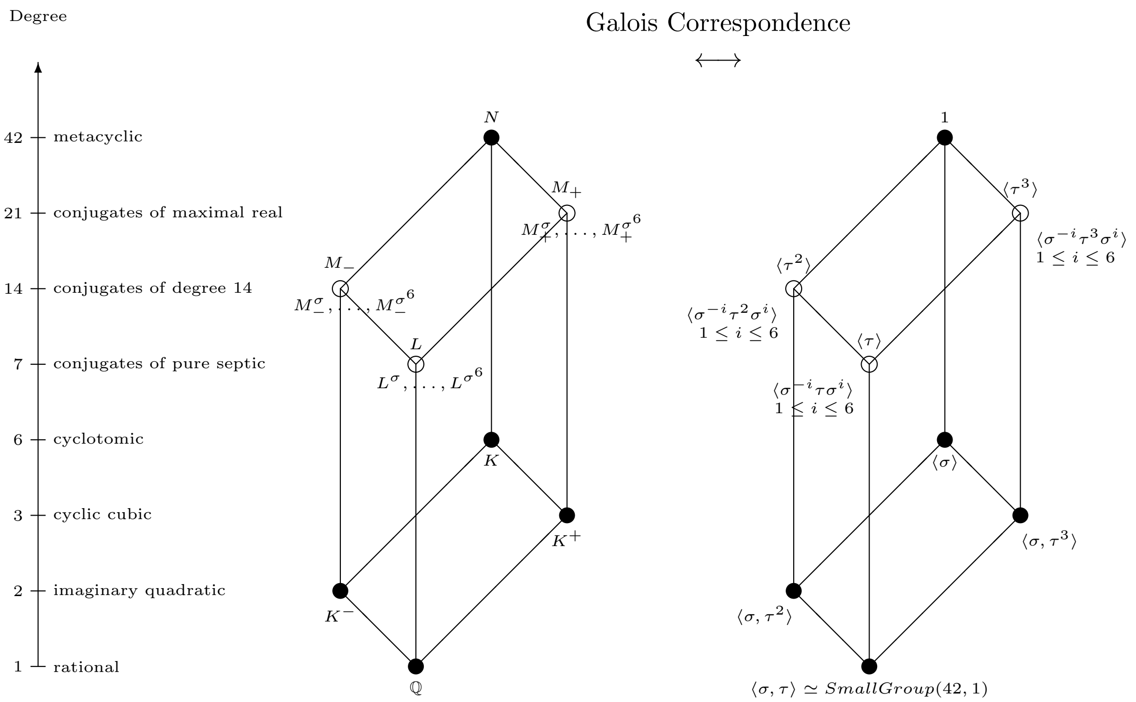
D. C. Mayer, 2018 – 2019:

7. **Pure Septic Fields,** $p = 7$

$D \geq 2 \ldots$ seventh power free integer,
$\zeta_7 = \exp(2\pi\sqrt{-1}/7) \ldots$ primitive seventh root of unity,
$N = \mathbb{Q}(\zeta_7, \sqrt[7]{D}) \ldots$ **pure metacyclic** field, degree 42,
$M_+ = \mathbb{Q}(\varrho, \sqrt[7]{D}) \ldots$ maximal real subfield of N, deg. 21,
$M_- = \mathbb{Q}(\sqrt{-7}, \sqrt[7]{D}) \ldots$ complex degree 14 subfield of N,
$L = \mathbb{Q}(\sqrt[7]{D}) \ldots$ **pure septic** subfield of N,
$K = \mathbb{Q}(\zeta_7) \ldots$ seventh cyclotomic field, cyclic sextic,
$K^+ = \mathbb{Q}(\varrho) \ldots$ maximal real subfield of K, cyclic cubic,
$K^- = \mathbb{Q}(\sqrt{-7}) \ldots$ imaginary quadratic subfield of K.

Remark. The cubic irrationality which generates K^+ is a solution of $\varrho^3 - \varrho^2 - 2\varrho + 1 = 0$.

FIGURE 8. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



7.1. **Absolute** Differential Principal Factors

T ... number of primes q_i dividing the conductor f of N/K,
$q_i\mathcal{O}_L = \mathfrak{q}_i^7$ ... ramification of q_i in L.

• Space of Absolute Differential Factors in $L/\mathbb{Q}$
Dimension:

$$\dim_{\mathbb{F}_7}(\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}) = T(= n + s_2 + s_3 + s_6), \tag{26}$$

Basis:

$$\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}} \simeq \bigoplus_{i=1}^{T} \mathbb{F}_7\,\mathfrak{q}_i\,. \tag{27}$$

• Subspace of Absolute Differential **Principal** Factors
Dimension:

$$A := \dim_{\mathbb{F}_7}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}), \tag{28}$$

Bounds for the dimension (cohomology: ≤ 4, subspace: $\le T$):

$$1 \le A \le \min(4, T). \tag{29}$$

7.2. **Central Orthogonal Idempotents (COI)**

Six COI, formally in terms of characters, $\chi_j \in \langle\tau\rangle^*$, $\chi_0 = 1$:

$$\psi_j := \frac{1}{6}\sum_{k=0}^{3} \chi_j(\tau^{-k})\tau^k \quad \text{for } 0 \le j \le 5.$$

Six COI, in terms of invertible residue classes mod 7,

$\langle\zeta_6\rangle \simeq C_6 \simeq U(\mathbb{Z}/7\mathbb{Z})$, $\zeta_6 \mapsto 3$, $\zeta_6^2 \mapsto 2$, $\zeta_6^3 \mapsto 6$, $\zeta_6^4 \mapsto 4$, $\zeta_6^5 \mapsto 5$:

$$\psi_0 = -(1+\tau+\tau^2+\tau^3+\tau^4+\tau^5), \quad \psi_1 = -(1+5\tau+4\tau^2+6\tau^3+2\tau^4+3\tau^5),$$

$$\psi_2 = -(1+4\tau+2\tau^2+\tau^3+4\tau^4+2\tau^5), \quad \psi_3 = -(1+6\tau+\tau^2+6\tau^3+\tau^4+6\tau^5),$$

$$\psi_4 = -(1+2\tau+4\tau^2+\tau^3+2\tau^4+4\tau^5), \quad \psi_5 = -(1+3\tau+2\tau^2+6\tau^3+4\tau^4+5\tau^5).$$

COI corresponding to **Basis** elements of **Norm Kernels**:

$\ker(N_{N/M_-}) \cap \ker(N_{N/M_+})$:

$$\psi_1 \hat{=} \mathfrak{K}_1 = \mathfrak{L}^{(1+6\tau^3)+5\tau(1+6\tau^3)+4\tau^2(1+6\tau^3)} = \mathfrak{L}^{1+5\tau+4\tau^2+6\tau^3+2\tau^4+3\tau^5} \hat{=} (154623),$$
$$\psi_5 \hat{=} \mathfrak{K}_2 = \mathfrak{L}^{(1+6\tau^3)+3\tau(1+6\tau^3)+2\tau^2(1+6\tau^3)} = \mathfrak{L}^{1+3\tau+2\tau^2+6\tau^3+4\tau^4+5\tau^5} \hat{=} (132645).$$

$\ker(N_{M_+/L})$:

$$\psi_2 \hat{=} \mathfrak{K}_1 = \mathfrak{L}^{(1+\tau^3)+4\tau(1+\tau^3)+2\tau^2(1+\tau^3)} = \mathfrak{L}^{1+4\tau+2\tau^2+\tau^3+4\tau^4+2\tau^5} \hat{=} (142142),$$
$$\psi_4 \hat{=} \mathfrak{K}_2 = \mathfrak{L}^{(1+\tau^3)+2\tau(1+\tau^3)+4\tau^2(1+\tau^3)} = \mathfrak{L}^{1+2\tau+4\tau^2+\tau^3+2\tau^4+4\tau^5} \hat{=} (124124).$$

$\ker(N_{M_-/L})$:

$$\psi_3 \hat{=} \mathcal{K} = \mathcal{L}^{1+6\tau} = \mathfrak{L}^{(1+\tau^2+\tau^4)+6\tau(1+\tau^2+\tau^4)} = \mathfrak{L}^{1+6\tau+\tau^2+6\tau^3+\tau^4+6\tau^5} \hat{=} (161616).$$

7.3. **Semi-Local Contributions**

Non-split primes only contribute to absolute DPF.
2- and 3-split primes contribute to several kinds of DPF.
6-split primes contribute to all kinds of DPF.

TABLE 5. Complete Census of $\mathbb{F}_7$-Dimensions

Entire Space of DPF $\mathcal{I}_{N/K}/\mathcal{I}_K$ $n+2s_2+3s_3+6s_4$	Absolute $\mathcal{I}_{L/\mathbb{Q}}/\mathcal{I}_{\mathbb{Q}}$ $n+s_2+s_3+s_6$	M_--Intermediate $\ker(N_{M_-/L})$ s_2+s_6	M_+-Intermediate $\ker(N_{M_+/L})$ $2(s_3+s_6)$	Relative $\ker(N_{N/M_-})$ $\cap\ker(N_{N/M_+})$ $2s_6$
$\forall\,1\le i\le n$	$\mathbb{F}_7\mathfrak{q}_i$ n	— —	— —	— —
$\forall\,1\le i\le s_2$	$\mathbb{F}_7\mathcal{L}_i^{1+\tau}$ s_2	$\mathbb{F}_7\mathcal{L}_i^{1+4\tau}$ s_2	— —	— —
$\forall\,1\le i\le s_3$	$\mathbb{F}_7\mathcal{L}_i^{1+\tau+\tau^2}$ s_3	— —	$\mathbb{F}_7\mathcal{L}_i^{1+4\tau+2\tau^2}$ $\oplus\mathbb{F}_7\mathcal{L}_i^{1+2\tau+4\tau^2}$ $2s_3$	— —
$\forall\,1\le i\le s_6$	$\mathbb{F}_7\mathfrak{L}_i^{1+\tau+\tau^2+\tau^3+\tau^4+\tau^5}$ s_6	$\mathbb{F}_7\mathfrak{L}_i^{1+6\tau+\tau^2+6\tau^3+\tau^4+6\tau^5}$ s_6	$\mathbb{F}_7\mathfrak{L}_i^{1+4\tau+2\tau^2+\tau^3+4\tau^4+2\tau^5}$ $\oplus\mathbb{F}_7\mathfrak{L}_i^{1+2\tau+4\tau^2+\tau^3+2\tau^4+4\tau^5}$ $2s_6$	$\mathbb{F}_7\mathfrak{L}_i^{1+5\tau+4\tau^2+6\tau^3+2\tau^4+3\tau^5}$ $\oplus\mathbb{F}_7\mathfrak{L}_i^{1+3\tau+2\tau^2+6\tau^3+4\tau^4+5\tau^5}$ $2s_6$

7.4. **Smallest Examples of Coarse DPF Type** α

The primes $\ell\equiv +1\,(\mathrm{mod}\,7)$ start with $29,43,71,\ldots$
The fields $\mathbb{Q}(\sqrt[7]{\ell})$ possess $f^6=7^2\cdot\ell^6,\quad n=1,\quad s_6=1.$
It is hard to compute their principal factorizations:

Trivially, they all share principal $\mathfrak{L}^{1+\tau+\tau^2+\tau^3+\tau^4+\tau^5}=\sqrt[7]{\ell}\mathcal{O}_L$,
but additionally we found the following behavior: $A=1$ and $R_0=3$ with
principal $\mathcal{L}^{1+4\tau+2\tau^2}=\lambda_1\mathcal{O}_{M_+}$ **and** $\mathcal{L}^{1+2\tau+4\tau^2}=\lambda_2\mathcal{O}_{M_+}$ (whence $\mathcal{L},\mathcal{L}^\tau,\mathcal{L}^{\tau^2}$ are all principal),
and **either** principal $\mathfrak{L}^{1+5\tau+4\tau^2+6\tau^3+2\tau^4+3\tau^5}=\Lambda\mathcal{O}_N$ **or** principal $\mathfrak{L}^{1+3\tau+2\tau^2+6\tau^3+4\tau^4+5\tau^5}=\Lambda\mathcal{O}_N$.

Our theory of *fine* DPF types, as developed for $p = 5$, showed the crucial impact of *splitting* prime divisors of the conductor f of N/K on the possibility of DPF types with non-maximal extent of absolute principal factorizations $A < U + 1$, which will appear in aggravated form for $p \geq 7$.

On the other hand, splitting prime divisors of f have been proved to enforce non-trivial p-class numbers of L and N: according to Ishida [(6)], a prime divisor $\ell \equiv +1 \pmod{p}$ of f implies $p \mid h_L$ and $p \mid h_N$, for any $p \geq 3$. Such a prime divisor ℓ splits completely in the cyclotomic field K, that is, into $p - 1$ prime ideals. More recently, Kobayashi [(7,8)] has proved that a prime divisor $\ell \equiv -1 \pmod{5}$ of f implies $5 \mid h_L$ and $5 \mid h_N$ and he conjectures the truth of this behavior for $p \geq 7$. Such a prime divisor ℓ splits into $\frac{p-1}{2}$ prime ideals of K. Therefore, we were surprised that other splitting prime divisors ℓ of f, whose occurrence starts with $p = 7$, do not exert such severe constraints on class numbers, and we conclude with the following interesting proven phenomenon.

Theorem. Let $L = \mathbb{Q}(\sqrt[7]{D})$ be a pure septic field with seventh power free radicand $D > 1$ and with Galois closure $N = \mathbb{Q}(\zeta_7, \sqrt[7]{D})$. If $D \equiv 2, 4 \pmod{7}$ is a prime radicand, then it **splits** into $\frac{7-1}{3} = 2$ prime ideals of $K = \mathbb{Q}(\zeta_7)$. If the radicand belongs to the range $2 \leq D < 200$, then D causes **relative principal factorizations** in the norm kernel $(\mathcal{P}_{N/K}/\mathcal{P}_K) \cap \ker(N_{M_-/L})$, where $M_- = L(\sqrt{-7})$, but nevertheless h_L and h_N are **not divisible by** 7.

Proof. By direct investigation with the aid of the computer algebra system Magma. Explicitly, the radicands are $D \in \{2, 11, 23, 37, 53, 67, 79, 107, 109, 137, 149, 151, 163, 179, 191, 193\}$.

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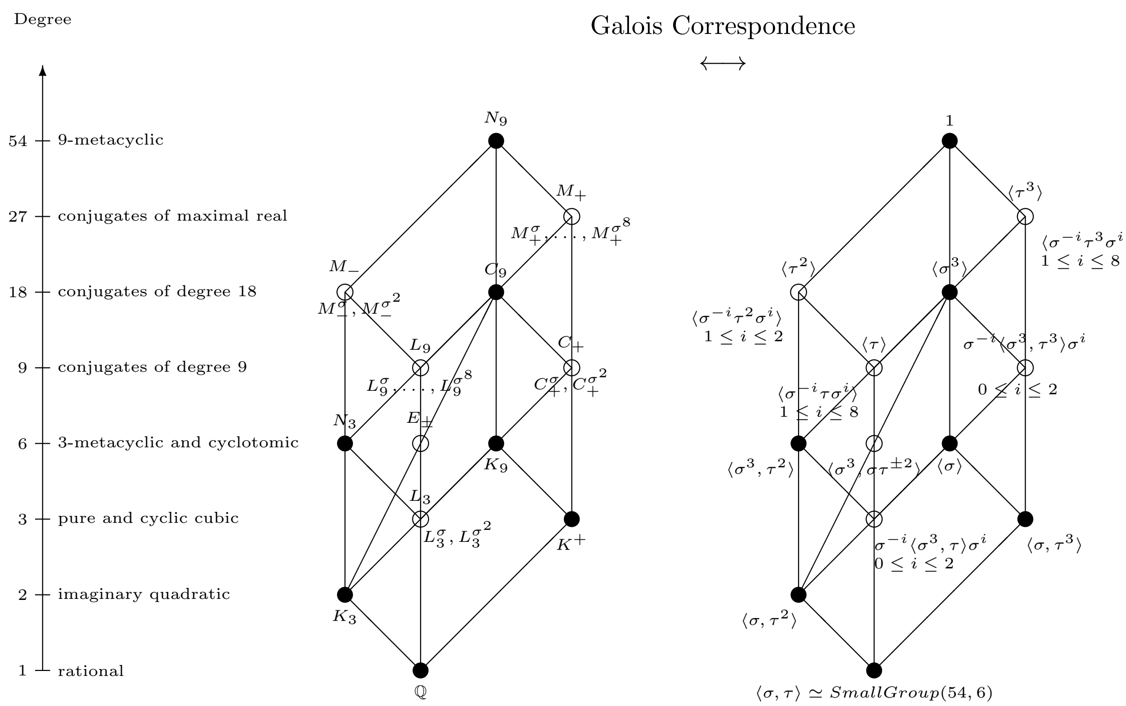
9. **Pure Nonic Fields**, $p = 9 = 3^2$ (composite !)

$D \geq 2$ … **third** power free integer,
$\zeta_9 = \exp(2\pi\sqrt{-1}/9)$ … primitive ninth root of unity,
$N_9 = \mathbb{Q}(\zeta_9, \sqrt[9]{D})$ … **pure metacyclic** field, degree 54,
$M_+ = \mathbb{Q}(\varrho, \sqrt[9]{D})$ … maximal real subfield of N, deg. 27,
$M_- = \mathbb{Q}(\sqrt{-3}, \sqrt[9]{D})$ … complex degree 18 subfield of N,
$L_9 = \mathbb{Q}(\sqrt[9]{D})$ … **pure nonic** subfield of N,
$K_9 = \mathbb{Q}(\zeta_9)$ … ninth cyclotomic field, cyclic sextic,
$K^+ = \mathbb{Q}(\varrho)$ … maximal real subfield of K, cyclic cubic,
$K_3 = \mathbb{Q}(\sqrt{-3})$ … imaginary quadratic subfield of K.

Remark. The cubic irrationality which generates K^+ is a solution of $\varrho^3 - 3\varrho + 1 = 0$.

FIGURE 9. Lattices of subfields of N and of subgroups of $\mathrm{Gal}(N/\mathbb{Q}) = \langle\sigma, \tau\rangle$



We have to point out that the subfield lattice of N_9 determined by the technique of forming composita is not complete, since there exist two additional hidden fields which are not composita.

Theorem. (Two hidden subfields in the lattice)
There exist two conjugate non-Galois intermediate fields

$$E_- = \mathbb{Q}(\zeta_3, \varrho\zeta_9\delta^3) \text{ and } E_+ = \mathbb{Q}(\zeta_3, \varrho\zeta_9\delta^6)$$

between K_3 and C_9, of degree 6 without cubic subfields. They correspond to the conjugate intermediate groups $\langle\sigma^3, \sigma\tau^4\rangle$ and $\langle\sigma^3, \sigma\tau^2\rangle$ between $\langle\sigma^3\rangle$ and $\langle\sigma, \tau^2\rangle$.

Pure nonic fields $L_9 = \mathbb{Q}(\sqrt[9]{D})$ admit a refined classification of their underlying pure cubic fields $L_3 = \mathbb{Q}(\sqrt[3]{D})$ with the aid of 3-class groups of certain subfields within the Galois closure $N_9 = L_9(\zeta_9)$ of L_9. If the cube free radicand $D > 1$ is congruent to ± 1 modulo 9, and thus belongs to Dedekind's species 2, there arises a special behavior if $D \equiv \pm 1 \pmod{27}$. The coarse similarity classes of pure cubic fields, with respect to their differential principal factorization type split into finer similarity classes of pure nonic superfields, with respect to the structure of the 3-class groups mentioned above.

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