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Four Plenary Lectures and Exercises, Respectfully Dedicated to
Professor Mohammed Ayadi.

Four Plenary Lectures and Exercises

Theoretical and Experimental Approach to p-Class Field Towers of Cyclic Cubic Number Fields

by Daniel C. Mayer

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Preface

This *course in algebraic number theory* with title "Theoretical and Experimental Approach to p-Class Field Towers of Cyclic Cubic Number Fields" is intended for an audience consisting of graduate students, doctorands, postdocs, and professional scientists. It is divided into four plenary lectures concerning the theory of *cyclic number fields* $F/\mathbb{Q}$ *of odd prime degree* $\ell := [F : \mathbb{Q}]$ and their unramified p-extension fields E/F for various exemplary cases of p-class groups $\mathrm{Cl}_p F = \mathrm{Syl}_p(\mathrm{Cl}(F))$ with an assigned prime number $p \in \mathbb{P}$. The plenary lectures are separated by three experimental sessions with exercises in the laboratory, where the theory is illuminated by means of the *computational algebra system Magma*. Experiments cover both, algebra and arithmetic as well as group theory.

Particular emphasis lies on the *cubic situation* $\ell = 3$. For a prime number p, denote by $G = \mathrm{Gal}(F_p^{(\infty)}/F)$ the relative pro-p Galois group of the unramified Hilbert p-class field tower $F_p^{(\infty)}$ of a cyclic cubic field F with p-class group $\mathrm{Cl}_p F$. For $p \in \{2, 5\}$ and elementary bicyclic $\mathrm{Cl}_p F \simeq C_p \times C_p$, it is shown (Thm. 19) that the *action of the absolute group* $\mathrm{Gal}(F/\mathbb{Q})$ on G severely restricts the possibilities for the metabelianization $\mathfrak{M} = G/G''$ of G. The length $\ell_p F$ of the p-tower, which coincides with the soluble length $\mathrm{sl}(G)$ of G, is given by $\ell_2 F \in \{1, 2\}$ for $p = 2$ generally, and by $\ell_5 F = 2$ for $p = 5$ in all known examples. Let t denote the number of prime divisors of the conductor c of $F/\mathbb{Q}$, viewed as a subfield of the 3-*ray class field modulo* c of $\mathbb{Q}$. For $p = 3$, the statistical distribution of the two well-known (Thm. 4) possibilities $\ell_3 F \in \{1, 2\}$ is determined in the case $t = 2$ and $\mathrm{Cl}_3 F \simeq C_3 \times C_3$, and a *broad variety of new scenarios* (Thms. 7 – 11) is presented in the case $t = 3$ with elementary bicyclic or tricyclic $\mathrm{Cl}_3 F$, in dependence on the mutual cubic residue conditions between the prime divisors of c. The coronation of this course is the first rigorous proof of cyclic cubic fields F with 3-class field towers of exact length $\ell_3 F = 3$, established by means of Artin patterns and relation ranks (Thms. 12 – 16). For elementary tricyclic $\mathrm{Cl}_3 F \simeq C_3 \times C_3 \times C_3$, finite *three-stage towers* (Thm. 17) were completely unknown up to now, for any kind of algebraic number fields.

Part I.

Foundations

1. Construction of Cyclic Fields

1.1. Multiplicity of Conductors and Discriminants

For a fixed odd prime number $\ell \geq 3$, let F be a *cyclic number field* of degree ℓ, that is, $F/\mathbb{Q}$ is a Galois extension of degree $[F : \mathbb{Q}] = \ell$ with absolute automorphism group $\mathrm{Gal}(F/\mathbb{Q}) = \langle \sigma \mid \sigma^\ell = 1 \rangle$. According to the **Theorem of Kronecker, Weber and Hilbert** on abelian extensions of the rational number field $\mathbb{Q}$, the *conductor* c of F is the smallest positive integer such that $F =: F_c$ is contained in the cyclotomic field $\mathbb{Q}(\zeta_c)$, where ζ_c denotes a primitive c-th root of unity, more precisely, in the *ℓ-ray class field modulo c* of $\mathbb{Q}$, which lies in the maximal real subfield $\mathbb{Q}(\zeta_c + \zeta_c^{-1})$. It is well known that $c = \ell^e \cdot q_1 \cdots q_\tau$, where $e \in \{0, 2\}$ and the q_i are pairwise distinct prime numbers $q_i \equiv +1 \pmod{\ell}$, for $1 \leq i \leq \tau$. The *discriminant* of $F = F_c$ is the perfect $(\ell - 1)$-th power $d_F = c^{\ell-1}$, and the number of rational primes which are (totally) ramified in F is given by

$$t := \begin{cases} \tau & \text{if } e = 0 \text{ (ℓ is unramified in F)}, \\ \tau + 1 & \text{if } e = 2 \text{ (ℓ is ramified in F)}. \end{cases} \tag{1.1.1}$$

In the last case, we formally put $q_{\tau+1} := \ell^2$. The number of non-isomorphic cyclic number fields $F_{c,1}, \ldots, F_{c,m}$ of degree ℓ, sharing the common conductor c, is given by the *multiplicity* formula [25, p. 831]

$$m = m(c) = (\ell - 1)^{t-1}. \tag{1.1.2}$$

1.2. Construction as Ray Class Fields

For the construction of all cyclic number fields $F = F_c$ of degree ℓ with ascending conductors $b \leq c \leq B$ between an assigned lower bound b and upper bound B by means of the computational algebra system Magma [23], the class field theoretic routines by Fieker [16] can be used without the need of preparing a list of suitable generating polynomials of ℓ-th degree. The big advantage of this technique is that the cyclic number fields of degree ℓ are produced as *multiplets* $(F_{c,1}, \ldots, F_{c,m})$ of pairwise non-isomorphic fields sharing the common conductor c with *multiplicities*

$m \in \{1, \ell-1, (\ell-1)^2, (\ell-1)^3, \ldots\}$ in dependence on the number $t \in \{1, 2, 3, 4, \ldots\}$ of primes dividing the conductor c, according to Formula (1.1.2).

Throughout this course, we design algorithms by the *principle of successive refinement*. We begin with a loop which sieves ℓ-admissible conductors, for a given odd prime number $\ell \geq 3$. If $c = q_1^{n_1} \cdots q_t^{n_t}$ is the prime factorization of c, the function `Factorization()` returns a collection of pairs $((q_1, n_1), \ldots, (q_t, n_t))$. In order to obtain **cyclic cubic fields**, we must put $\ell = 3$.

Algorithm 1. (Filtering ℓ-admissible conductors.)
Input: prime `l`, lower bound `b`, upper bound `B`.
Code: uses the subroutine `Process()`.

```
fld := 0; // counter of fields, inclusively multiplicity
for c in [b..B] do
    cPD := Factorization(c);
    adm := true; // admissible
    for i in [1..#cPD] do
        if not ( ((1 eq cPD[i][1] mod l) and (1 eq cPD[i][2])) or
                 ((l eq cPD[i][1]) and (2 eq cPD[i][2])) ) then
            adm := false;
            break; // save CPU time
        end if;
    end for; // i
    if adm then
        Process(l,c,fld); // subroutine
    end if; // adm
end for; // c
```

Output: managed by the subroutine `Process()`.

Now we implement the subroutine `Process()`. Although the conductor $c = 1$ is not admissible in the present context, we suggest to avoid the ray class group modulo 1, and to replace it by the ordinary class group, if necessary. Initially, the wider multiplet of all abelian ℓ-extensions of the rational number field $\mathbb{Q}$, with conductors *dividing* c, is constructed and optimized. In terms of the ℓ-ray class rank modulo c, $\varrho = \varrho_\ell(c)$, it consists of $\frac{\ell^\varrho - 1}{\ell - 1}$ members. This is the number of subgroups of index ℓ in the ray class group modulo c. By testing the correct discriminant $d = c^{\ell-1}$, the narrower multiplet $(F_{c,1}, \ldots, F_{c,m})$ of all cyclic extensions of degree ℓ with *precise* conductor c is extracted, and the *abelian type invariants* of the complete class group $\mathrm{Cl}(F_{c,\mu})$ and of the ℓ-class group $\mathrm{Cl}_\ell F_{c,\mu}$ are determined for $1 \leq \mu \leq m$.

Algorithm 2. (Construction of multiplets.)
Input: prime `l`, `l`-admissible conductor `c`, counter of fields `fld`.
Code:

```
SetClassGroupBounds("GRH");
Q := RationalsAsNumberField();
OQ := MaximalOrder(Q); // this is Z
CQ,mCQ := ClassGroup(OQ);
RQ,mRQ := RayClassGroup(c*OQ); // modulo c*Z
if (1 eq c) then // avoid the ray mod 1
    XQ := CQ;
    mXQ := mCQ;
else
    XQ := RQ;
    mXQ := mRQ;
end if; // c=1
sS1 := Subgroups(XQ: Quot := [l]); // full multiplet of l-ray class fields
sA1 := [AbelianExtension(Inverse(mQQ)*mXQ)
        where QQ,mQQ := quo<XQ|x`subgroup>:  x in sS1];
sN1 := [NumberField(x):  x in sA1]; // relative fields
sR1 := [MaximalOrder(x):  x in sA1];
sF1 := [AbsoluteField(x):  x in sN1]; // absolute fields
sM1 := [MaximalOrder(x):  x in sF1];
sB1 := [OptimizedRepresentation(x):  x in sF1]; // first optimization
sK1 := [NumberField(DefiningPolynomial(x)):  x in sB1];
sO1 := [Simplify(LLL(MaximalOrder(x))):  x in sK1]; // second optimization
// initialization of multiplets
totMult := #sO1; // inclusively wrong partial conductors
locMult := 0; // only with correct conductor
Collection := []; // all l-class groups of the multiplet
for j in [1..totMult] do
    ON := sO1[j];
    N := NumberField(ON);
    G,Z,D:= GaloisGroup(N);
    if ((l eq #G) and IsAbelian(G)) then // cyclic Galois group
        C,mC := ClassGroup(ON);
        CN := AbelianInvariants(C);
        SN := pPrimaryInvariants(C,l); // the Sylow l-subgroup
        Disc := c^2; // root discriminant
        if (Disc^((l-1) div 2) eq AbsoluteDiscriminant(ON)) then
            fld := fld + 1; // one field more
            locMult := locMult + 1; // local counter within multiplet
            printf "Nr %4o",fld; // 4 digits
            printf ":  c=%6o",c; // 6 digits
            printf "=%o",Factorization(c);
            printf ", j/m=%o/%o",j,totMult; // component nr
            printf ", SN=%o, CN=%o\n",SN,CN;
            Append(~Collection,SN); // save l-class group
        end if; // correct discriminant
    end if; // cyclic
end for; // j (end of multiplet)
```

Output: counter `fld`, conductor `c`, factors, class group `CN`, `l`-class group `SN`.

When a *statistic evaluation* of all multiplets is desired, the loop in Algorithm 1 can be surrounded by the following initialization and finalization, which is designed for the special case of cyclic cubic fields, $\ell = 3$. The middle part must be inserted at the end of Algorithm 2.

Algorithm 3. (Statistics of multiplets.)
Input: local counter `locMult` and `Collection`.
Code:

```
// distribution of multiplicities from 1 to 20
Mult := [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0];
// collections of cyclic cubic multiplets
Singlets := []; // m=1
Doublets := []; // m=2
Quartets := []; // m=4
Octets := []; // m=8
Hexadecuplets := []; // m=16

// after (end of multiplet)
// finalization of multiplets
if (20 ge locMult) then
    Mult[locMult] := Mult[locMult] + 1;
end if;
if (1 le locMult) then
    if (1 eq locMult) then
        Append(~Singlets,Collection);
        printf "m=1, no %o\n",Mult[1];
    elif (2 eq locMult) then
        Append(~Doublets,Collection);
        printf "m=2, no %o\n",Mult[2];
    elif (4 eq locMult) then
        Append(~Quartets,Collection);
        printf "m=4, no %o\n",Mult[4];
    elif (8 eq locMult) then
        Append(~Octets,Collection);
        printf "m=8, no %o\n",Mult[8];
    elif (16 eq locMult) then
        Append(~Hexadecuplets,Collection);
        printf "m=16, no %o\n",Mult[16];
    end if;
end if; // locMult

printf "multiplicity:\n";
for i in [1..20] do
    printf "%2o:%o\n",i,Mult[i];
end for; // i
printf "\nSinglets:\n";
for i in [1..#Singlets] do
    printf "%o:  ( ",i;
    for j in [1..#Singlets[i]] do
        printf "%o ",Singlets[i][j];
    end for;
    printf ")\n";
end for;
printf "\nDoublets:\n";
for i in [1..#Doublets] do
    printf "%o:  ( ",i;
    for j in [1..#Doublets[i]] do
        printf "%o ",Doublets[i][j];
    end for;
    printf ")\n";
end for;
// and so on until
printf "\nHexadecuplets:\n";
for i in [1..#Hexadecuplets] do
    printf "%o:  ( ",i;
    for j in [1..#Hexadecuplets[i]] do
```

```
        printf "%o ",Hexadecuplets[i][j];
    end for;
    printf ")\n";
end for;
printf "\n\n\n";
```

Output: singlets, doublets, quartets, octets, hexadecuplets, with `l`-class groups.

Warning 1. It should be pointed out that the totally complex cyclotomic field $\mathbb{Q}(\zeta_c)$ with $c \geq 3$ is the *ray class field modulo* $c{\cdot}w$ of $\mathbb{Q}$, where ζ_c denotes a primitive c-th root of unity, e.g. $\zeta_c = \exp(2\pi\sqrt{-1}/c)$, and w is the unique real archimedean place of $\mathbb{Q}$, which splits into $\frac{\phi(c)}{2}$ pairs of conjugate complex archimedean places in the cyclotomic field. However, the multiplet of cyclic fields $F_{c,\mu}$, $1 \leq \mu \leq m$, of degree ℓ with ℓ-admissible conductor c is contained in the *ℓ-ray class field modulo* c of $\mathbb{Q}$, which is contained in the maximal real subfield $\mathbb{Q}(\zeta_c + \zeta_c^{-1})$ of $\mathbb{Q}(\zeta_c)$.

1.3. Census of Multiplets and Class Groups

With the aid of the computational algebra system Magma [10, 11, 23], we have implemented the Algorithms 1, 2, 3 and constructed the multiplets $(F_{c,1}, \ldots, F_{c,m})$ of all cyclic *quintic*, respectively cyclic *cubic*, number fields F with conductors $0 < c < 100\,000$ as subfields of ray class fields [16] modulo 5-admissible, respectively 3-admissible, conductors c over the rational number field $\mathbb{Q}$.

1.3.1. Statistics of Multiplets

The complete census of multiplets for cyclic *quintic* fields is given in Table 1.1. Obviously, the fields F in *quartets* are dominating, but not the corresponding conductors c. We always emphasize the *minimal conductor* $c_{\min}$ as a prototype.

Table 1.1.: Statistics of Cyclic Quintic Multiplets

Multiplets	m	# Conductors c	# Fields F	$c_{\min}$
$c <$		100000	100000	
Singlets	1	2388	2388	11
Quartets	4	845	3380	275
Hexadecuplets	16	49	784	8525
Total		3282	6552	

For cyclic *cubic* fields, we split the information into four ranges of conductors, with lengths increasing in steps of 25000. The complete census of multiplets is given in Table 1.2. Here, fields F in *doublets* are dominating, but not conductors c.

Table 1.2.: Statistics of Cyclic Cubic Multiplets

Multiplets	m	# Conductors c				# Fields F				$c_{\min}$
$c <$		25000	50000	75000	10^5	25000	50000	75000	10^5	
Singlets	1	1372	2557	3682	4785	1372	2557	3682	4785	7
Doublets	2	993	1953	2921	3863	1986	3906	5842	7726	63
Quartets	4	149	351	556	783	596	1404	2224	3132	819
Octets	8	1	7	17	26	8	56	136	208	15561
Total		2515	4868	7176	9457	3962	7923	11884	15851	

1.3.2. Statistics of Class Group Structures

Table 1.3.: Statistics of Cyclic Quintic 5-Class Group Structures

$\mathrm{Cl}_5 F$	# Fields F	$c_{\min}$
$c <$	100000	
1	2388	11
(5)	3260	275
$(5,5)$	869	2651
$(5,5,5)$	35	13981
Total	6552	

Statistics of the structures of 5-class groups $\mathrm{Cl}_5 F$, respectively 3-class groups $\mathrm{Cl}_3 F$, is shown in Table 1.3 for cyclic *cubic* fields, respectively 1.4 for cyclic *quintic* fields. The 3-class groups reveal a wealth of various structures up to order 3^5.

Table 1.4.: Statistics of Cyclic Cubic 3-Class Group Structures

Cl_3F	# Fields F	$c_{\min}$
$c <$	100000	
1	4785	7
(3)	6910	63
$(3,3)$	3498	657
$(3,3,3)$	481	3913
$(3,3,3,3)$	13	25389
$(3,9)$	105	4711
$(3,3,9)$	43	7657
$(3,3,3,9)$	6	15561
$(9,9)$	5	41977
$(3,9,9)$	3	66157
$(9,27)$	2	36667
Total	15851	

1.3.3. Class Group Structures Grouped by Multiplets

The complete census of the structures of 5-class groups Cl_5F, respectively 3-class groups Cl_3F, *grouped by multiplets*, is shown in Table 1.5, respectively 1.6. This point of view admits considerably deeper insight into the connections between the number t of primes dividing the conductor c and the structure of 3-class groups $\mathrm{Cl}_3F_{c,\mu}$ of all components of the multiplet $(F_{c,\mu})_{1\le\mu\le m}$, $m = 2^{t-1}$, of cyclic cubic fields sharing the common conductor c.

Table 1.5.: Statistics of 5-Class Group Structures of Cyclic Quintic Multiplets

$(\mathrm{Cl}_5 F_{c,\mu})_{\mu=1}^m$	m	# Conductors c	# Fields F	$c_{\min}$
$c <$		100000	100000	
1	1	2388	2388	11
$(5)^4$	4	815	3260	275
$(5,5)^4$	4	28	112	2651
$(5,5,5)^4$	4	2	8	23411
$(5,5)^{16}$	16	41	656	8525
$(5,5,5)^3, (5,5)^{13}$	16	5	$15+65$	13981
$(5,5,5)^4, (5,5)^{12}$	16	3	$12+36$	47275
Total		3282	6552	

Experimental Result 1. Table 1.6 will be the leading principle for the entire layout of this course.

- 3-class groups $\mathrm{Cl}_3 F_{c,\mu}$ of rank 4 appear in 14 among the 26 **octets** only. One, two or three components of an octet may have 3-class groups of type $(3,3,3,3)$ or $(3,3,3,9)$. Octets will be reserved for future investigations.
- Since they are trivial, we shall not deal with 4785 *singlets* having the trivial 3-class group 1, and with 3455 *doublets* having cyclic 3-class group (3).
- In this course, we pay attention to 352, respectively 56, **doublets** with 3-class number $h_3 = 9$ and elementary bicyclic 3-class group $(3,3)$, respectively $h_3 \geq 27$ and non-elementary bicyclic 3-class groups $(3,9)$, $(9,9)$, $(9,27)$, in Section § 4, respectively Section § 5, and to all 783 **quartets** in the Sections §§ 6.2.2, 6.2.3, 6.5.1, 6.5.2, 6.5.3, 6.5.4.

Table 1.6.: Statistics of 3-Class Group Structures of Cyclic Cubic Multiplets

$(\mathrm{Cl}_3 F_{c,\mu})_{\mu=1}^{m}$	m	# Conductors c	# Fields F	$c_{\min}$
$c <$		100000	100000	
1	1	4785	4785	7
$(3)^2$	2	3455	6910	63
$(3,3)^2$	2	352	704	657
$(3,9)^2$	2	50	100	4711
$(9,9),(3,9)$	2	4	$4+4$	41977
$(9,27),(3,9)$	2	1	$1+1$	36667
$(9,27),(9,9)$	2	1	$1+1$	42127
$(3,3)^4$	4	579	2316	819
$(3,3,3),(3,3)^3$	4	79	$79+237$	4977
$(3,3,3)^2,(3,3)^2$	4	80	$160+160$	3913
$(3,3,9),(3,3)^3$	4	18	$18+54$	7657
$(3,3,9)^2,(3,3)^2$	4	10	$20+20$	27873
$(3,9,9),(3,3)^3$	4	1	$1+3$	67347
$(3,9,9),(3,3,9),(3,3)^2$	4	2	$2+2+4$	66157
$(3,3,3)^4$	4	12	48	38311
$(3,3,9),(3,3,3)^3$	4	1	$1+3$	91819
$(3,3,9)^2,(3,3,3)^2$	4	1	$2+2$	97747
$(3,3,3)^8$	8	12	96	30303
$(3,3,3,3),(3,3,3)^7$	8	9	$9+63$	25389
$(3,3,3,9),(3,3,3)^7$	8	2	$2+14$	15561
$(3,3,3,9)^2,(3,3,3)^6$	8	1	$2+6$	49959
$(3,3,3,9),(3,3,3,3)^2,(3,3,3)^5$	8	2	$2+4+10$	54873
Total		9457	15851	

2. Arithmetic of Cyclic Cubic Fields

2.1. Rank of 3-Class Groups of Cyclic Cubic Fields

Since the rank $\varrho_3 F$ of the 3-class group $\mathrm{Cl}_3 F$ of a cyclic cubic field F depends on the mutual cubic residue conditions between the prime divisors $q_1, \ldots, q_t$ of the conductor c, Gras [19, pp. 21–22] has introduced directed graphs with t vertices $q_1, \ldots, q_t$ whose directed edges $q_i \to q_j$ describe values of cubic residue symbols. We use a simplified notation of these graphs, fitting in a single line, but occasionally requiring the repetition of a vertex.

Definition 1. Let ζ_3 be a fixed primitive third root of unity. For each pair (q_i, q_j) with $1 \le i \ne j \le t$, the value of the *cubic residue symbol* $\left(\frac{q_i}{q_j}\right)_3 = \zeta_3^{a_{i,j}}$ is determined uniquely by the integer $a_{i,j} \in \{0, 1, 2\}$. Let a *directed edge* $q_i \to q_j$ be defined if and only if $\left(\frac{q_i}{q_j}\right)_3 = 1$, that is, q_i is a cubic residue modulo q_j (and thus $a_{i,j} = 0$). The **combined cubic residue symbol** $[q_1, \ldots, q_t]_3 :=$

$$\left\{ q_i \to q_j \;\middle|\; i \ne j, \left(\frac{q_i}{q_j}\right)_3 = 1 \right\} \bigcup \left\{ q_i \;\middle|\; (\forall j \ne i)\ \left(\frac{q_i}{q_j}\right)_3 \ne 1, \left(\frac{q_j}{q_i}\right)_3 \ne 1 \right\} \quad (2.1.1)$$

where the subscripts i and j run from 1 to t, is defined as the union of the set of all directed edges which occur in the graph associated with $q_1, \ldots, q_t$ in the sense of Gras, and the set of all isolated vertices. For $t = 3$, we additionally need the invariant $\delta := a_{1,2}a_{2,3}a_{3,1} - a_{1,3}a_{3,2}a_{2,1}$ in order to distinguish two subcases of the case with three isolated vertices.

Theorem 1. *(Rank Distribution,* ***G. Gras****, [19].)*
Let F be a cyclic cubic field of conductor $c = q_1 \cdots q_t$ with $1 \le t \le 3$. We indicate ***mutual cubic residues*** *simply by writing $q_1 \leftrightarrow q_2$ instead of $q_1 \to q_2 \to q_1$.*

- *If $t = 1$, then $m = 1$, F forms a singlet, $[q_1]_3 = \{q_1\}$, and $\varrho_3 F = 0$.*
- *If $t = 2$, then $m = 2$, F is member of a doublet (F_1, F_2), and there arise two possibilities.*

1. $(\varrho_3 F_1, \varrho_3 F_2) = (1, 1)$, *if*

$$[q_1, q_2]_3 = \begin{cases} \{q_1, q_2\} & \text{or} \\ \{q_i \to q_j\} & \text{with } i \neq j. \end{cases} \tag{2.1.2}$$

2. $(\varrho_3 F_1, \varrho_3 F_2) = (2, 2)$, *if*

$$[q_1, q_2]_3 = \{q_1 \leftrightarrow q_2\}. \tag{2.1.3}$$

- *If* $t = 3$, *then* $m = 4$, F *is member of a quartet* $(F_1, \ldots, F_4)$, *and there arise five cases.*

1. $(\varrho_3 F_1, \varrho_3 F_2, \varrho_3 F_3, \varrho_3 F_4) = (2, 2, 2, 2)$, *if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta \neq 0\} & \text{or} \\ \{q_i \to q_j; q_k\} & \text{or} \\ \{q_i \to q_j \to q_k\} & \text{or} \\ \{q_i \to q_j \to q_k \to q_i\} & \text{or} \\ \{q_i \leftrightarrow q_j; q_k\} & \text{or} \\ \{q_i \leftrightarrow q_j \to q_k\} & \text{or} \\ \{q_i \leftrightarrow q_j \leftarrow q_k\} & \text{or} \\ \{q_k \to q_i \leftrightarrow q_j \leftarrow q_k\} & \text{or} \\ \{q_k \to q_i \leftrightarrow q_j \to q_k\} & \end{cases} \tag{2.1.4}$$

with i, j, k *pairwise distinct.*

2. $(\varrho_3 F_1, \varrho_3 F_2, \varrho_3 F_3, \varrho_3 F_4) = (3, 2, 2, 2)$, *if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta = 0\} & \text{or} \\ \{q_i \leftarrow q_j \to q_k\} & \text{with } i, j, k \text{ pairwise distinct.} \end{cases} \tag{2.1.5}$$

3. $(\varrho_3 F_1, \varrho_3 F_2, \varrho_3 F_3, \varrho_3 F_4) = (3, 3, 2, 2)$, *if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \to q_j \leftarrow q_k\} & \text{or} \\ \{q_i \to q_j \leftarrow q_k \to q_i\} & \end{cases} \tag{2.1.6}$$

with i, j, k *pairwise distinct.*

4. $(\varrho_3 F_1, \varrho_3 F_2, \varrho_3 F_3, \varrho_3 F_4) = (3,3,3,3)$, *if*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \leftarrow q_j \leftrightarrow q_k \rightarrow q_i\} & \textit{or} \\ \{q_i \leftrightarrow q_j \leftrightarrow q_k\} & \textit{or} \\ \{q_i \leftrightarrow q_j \leftrightarrow q_k \rightarrow q_i\} & \end{cases} \tag{2.1.7}$$

with i, j, k *pairwise distinct.*

5. $(\varrho_3 F_1, \varrho_3 F_2, \varrho_3 F_3, \varrho_3 F_4) = (4,4,4,4)$, *if*

$$[q_1, q_2, q_3]_3 = \{q_1 \leftrightarrow q_2 \leftrightarrow q_3 \leftrightarrow q_1\}. \tag{2.1.8}$$

Proof. See [19, Prp. VI.5, pp. 21–22]. □

Remark 1. Ayadi [3, pp. 45–47] investigated the cases $t = 2$, Formula (2.1.3), and $t = 3$, Formulas (2.1.4), (2.1.5), (2.1.6), in Theorem 1. For $t = 3$, he denoted the nine subcases of Formula (2.1.4) by Graph 1,2,3,4,5,6,7,8,9 of Category III, the two subcases of Formula (2.1.5) by Graph 1 and 2 of Category I, and the two subcases of Formula (2.1.6) by Graph 1 and 2 of Category II. For the Categories I and II, Ayadi did not consider the fields with 3-class rank $\varrho_3 F_\mu = 3$.

2.2. Power Residues, Categories and Graphs

The following variant of the Euler criterion for ℓ-th power residues, with an odd prime ℓ, is the foundation of all further algorithms in this section.

Algorithm 4. (Euler criterion for ℓ-th power residues.)
Input: prime `l`, prime(-power) module `m`, residue `r`.
Code:

```
chr := 0; // residue character
g := PrimitiveRoot(m);
if (0 eq g) then
    printf "no primitive root g modulo %o\n",m;
else
    i := 0;
    bool := false;
    for e in [1..m-1] do
        pot := g^e;
        dif := pot - r;
        if (0 eq dif mod m) then
            bool := true;
            i := e; // storage of exponent e
            break;
        end if;
    end for; // e
    // EULER criterion
    if (true eq bool) and (0 lt Valuation(i,l)) then
        chr := +1; // residue
    elif not (0 eq r mod m) then
        chr := -1; // (coarse) non-residue
    end if;
    printf "(%o/%o)_%o=%o, g=%o, i=%o\n",r,m,l,chr,g,i;
end if;
```

Output: `l`-th residue character `chr` of `r` mod `m`, primitive root `g`, exponent `i`.

The preceding Algorithm 4 can be implemented as a function `PowerResidue()`, which returns the ℓ-th power residue character, rather than printing it as output. Now we design a procedure for $\ell = 3$ which determines the category and graph of any 3-admissible conductor c with $t = 3$ prime divisors. The graph consists of three prime divisors, q_1, q_2, q_3, as vertices, and a directed edge $q_i \to q_k$ whenever $i \neq k$ and $(q_i/q_k)_3 = 1$, for instance $13 \to 7$, since $3^3 - 13 = 14 = 2 \cdot 7$ and thus 13 is cubic residue mod 7, i.e. $(13/7)_3 = 1$. The procedure determines the number of bidirectional edges (*mutual cubic residues*), all directed edges, *universally attractive* vertices, and *universally repulsive* vertices, which is sufficient in order to determine the category and graph, according to the Gras Theorem 1, except for one special configuration without any edges: since we do not classify non-residues by the character values ζ_3 and ζ_3^2, we cannot determine the invariant δ. Thus the decision if Category III, graph 1, has to be replaced by Category I, graph 1, will be supplemented later by means of the rank distribution.

Algorithm 5. (Automatic determination of category and graph.)
Input: critical prime `iCrit`, three prime divisors `iPrm1`, `iPrm2`, `iPrm3` of the conductor `c`.
Code: uses the function `PowerResidue()`.

```
P := iCrit; // critical prime for p-th power residues
Q := iPrm1; // first prime divisor q
R := iPrm2; // second prime divisor r
S := iPrm3; // third prime divisor s
// six tests (in three pairs)
Char := [];
C12 := PowerResidue(P,Q,R); // Q -> R
C21 := PowerResidue(P,R,Q); // R -> Q
C13 := PowerResidue(P,Q,S); // Q -> S
C31 := PowerResidue(P,S,Q); // S -> Q
C23 := PowerResidue(P,R,S); // R -> S
C32 := PowerResidue(P,S,R); // S -> R
Append(~Char,C12);
Append(~Char,C21);
Append(~Char,C13);
Append(~Char,C31);
Append(~Char,C23);
Append(~Char,C32);
// number of bidirectional edges
AnzBid := 0;
if (1 eq C12) and (1 eq C21) then
    AnzBid := AnzBid + 1;
end if;
if (1 eq C13) and (1 eq C31) then
    AnzBid := AnzBid + 1;
end if;
if (1 eq C23) and (1 eq C32) then
    AnzBid := AnzBid + 1;
end if;
// three bidirectional edges
if (3 eq AnzBid) then
    kat := 5;
    grp := 1;
else
    // number of edges
```

```
AnzKnt := 0;
for j in [1..#Char] do
    if (1 eq Char[j]) then
        AnzKnt := AnzKnt + 1;
    end if;
end for; // j
// number of attractive edges
Att := 0;
if (1 eq C21) and (1 eq C31) then
    Att := Att + 1;
end if; if (1 eq C12) and (1 eq C32) then
    Att := Att + 1;
end if; if (1 eq C13) and (1 eq C23) then
    Att := Att + 1;
end if;
// number of repulsive edges
Rep := 0;
if (1 eq C12) and (1 eq C13) then
    Rep := Rep + 1;
end if; if (1 eq C21) and (1 eq C23) then
    Rep := Rep + 1;
end if; if (1 eq C31) and (1 eq C32) then
    Rep := Rep + 1;
end if;
if (2 eq AnzBid) then
    if (4 eq AnzKnt) then
        kat := 4;
        grp := 2;
    else // then AnzKnt=5
        kat := 4;
        grp := 3;
    end if;
elif (1 eq AnzBid) then
    if (2 eq AnzKnt) then
        kat := 3;
        grp := 5;
    elif (3 eq AnzKnt) then
        if (1 eq Rep) then
            kat := 3;
            grp := 6;
        elif (1 eq Att) then
            kat := 3;
            grp := 7;
        end if;
    elif (4 eq AnzKnt) then
        if (1 eq Att) and (1 eq Rep) then
            kat := 3;
            grp := 9;
        elif (1 eq Att) and (2 eq Rep) then
            kat := 4;
            grp := 1;
        elif (2 eq Att) and (1 eq Rep) then
            kat := 3;
            grp := 8;
        end if;
    end if;
elif (0 eq AnzBid) then
    if (0 eq AnzKnt) then
        kat := 3; // or 1 (final decision later)
        grp := 1;
    elif (1 eq AnzKnt) then
        kat := 3;
```

```
            grp := 2;
        elif (2 eq AnzKnt) then
            if (1 eq Rep) then
                kat := 1;
                grp := 2;
            elif (1 eq Att) then
                kat := 2;
                grp := 1;
            else
                kat := 3;
                grp := 3;
            end if;
        elif (3 eq AnzKnt) then
            if (1 eq Att) and (1 eq Rep) then
                kat := 2;
                grp := 2;
            else
                kat := 3;
                grp := 4;
            end if;
        end if;
    end if;
end if;
printf "Category=%o, Graph=%o\n",kat,grp;
printf "%o-symbols of (%o,%o,%o):  ",P,Q,R,S;
for j in [1..#Char] do
    printf "%o,",Char[j];
end for;
printf "\n";
```

Output: category, graph, and power residue symbols.

2.3. Detailed Statistics of Categories and Graphs

Statistics of categories and graphs for the cyclic cubic quartets (with $m = 4$) is shown in Table 2.1.

Table 2.1.: Statistics of Categories and Graphs of Cyclic Cubic Quartets

Category	Gph	# Conductors c				# Fields F				$c_{\min}$
$c <$		25000	50000	75000	10^5	25000	50000	75000	10^5	
I	1	7	19	27	38	28	76	108	152	4977
I	2	15	29	44	60	60	116	176	240	7657
Subtotal		22	48	71	98	88	192	284	392	
II	1	10	25	36	47	40	100	144	188	3913
II	2	7	19	34	45	28	76	136	180	6327
Subtotal		17	44	70	92	68	176	280	368	
III	1	11	22	41	52	44	88	164	208	1953
III	2	57	125	181	262	228	500	724	1048	819
III	3	20	50	81	124	80	200	324	496	1197
III	4	5	8	16	17	20	32	64	68	6643
III	5	4	17	27	37	16	68	108	148	14049
III	6	5	16	27	31	20	64	108	124	8541
III	7	4	8	20	34	16	32	80	136	4599
III	8	1	4	6	7	4	16	24	28	20293
III	9	3	7	11	15	12	28	44	60	16471
Subtotal		110	257	410	579	440	1028	1640	2316	
IV	1	0	0	2	7	0	0	8	28	61579
IV	2	0	1	1	2	0	4	4	8	49543
IV	3	0	1	2	5	0	4	8	20	38311
Subtotal		0	2	5	14	0	8	20	56	
Total		149	351	556	783	596	1404	2224	3132	

In the range $1 < c < 10^5$ of 3-admissible conductors, with a total of 9457 hits, there are 783 with exactly $t = 3$ prime divisors, which give rise to quartets, according to the multiplicity formula (1.1.2). With 262 occurrences (33.5%), category III, graph 2, $(q_i \to q_j; q_k)$, is most frequent. On the second place with 124 (15.8%) we have category III, graph 3, $(q_i \to q_j \to q_k)$.

2.4. Ambiguous Principal Ideals

The number of *primitive ambiguous ideals* of a cyclic cubic field F increases with the number t of prime factors of the conductor c. According to Hilbert's Theorem 93, we have:

$$\#\left(\mathcal{I}_F^{\langle\sigma\rangle}/\mathcal{I}_{\mathbb{Q}}\right) = 3^t. \tag{2.4.1}$$

However, the number of *ambiguous principal ideals* of F is a fixed invariant of all cyclic cubic fields, regardless of the number t. It is given by the Theorem on the **Herbrand quotient** of the unit group U_F of F as a Galois module over the group $\mathrm{Gal}(F/\mathbb{Q}) = \langle\sigma\rangle$, which can be expressed by abstract cohomology groups $\#\mathrm{H}^{-1}(\langle\sigma\rangle, U_F)/\#\hat{\mathrm{H}}^0(\langle\sigma\rangle, U_F) = [F:\mathbb{Q}]$ or more ostensively by:

$$\#\left(\mathcal{P}_F^{\langle\sigma\rangle}/\mathcal{P}_{\mathbb{Q}}\right) = \#\left(E_{F/\mathbb{Q}}/U_F^{1-\sigma}\right) = [F:\mathbb{Q}]\cdot\#\left(U_{\mathbb{Q}}/\mathrm{N}_{F/\mathbb{Q}}U_F\right) = 3, \tag{2.4.2}$$

since the unit norm index is given by $\left(U_{\mathbb{Q}} : \mathrm{Norm}_{F/\mathbb{Q}}U_F\right) = 1$. Consequently, if we speak about a *non-trivial primitive ambiguous principal ideal* of F, then we either mean $(\alpha) = \alpha\mathcal{O}_F$ or $(\alpha^2/b) = (\alpha^2/b)\mathcal{O}_F$, where $\mathcal{P}_F^{\langle\sigma\rangle}/\mathcal{P}_{\mathbb{Q}} = \{1, (\alpha), (\alpha^2/b)\}$. The norms of these two elements are divisors of the square $c^2 = q_1^2\cdots q_t^2$ of the conductor c of F, where q_t must be replaced by 3 if $q_t = 9$. When $\mathrm{N}_{F/\mathbb{Q}}\alpha = a\cdot b^2$ with square free coprime integers a, b, then $\mathrm{N}_{F/\mathbb{Q}}(\alpha^2/b) = a^2\cdot b^4/b^3 = a^2\cdot b$. It follows that both norms are cube free integers.

Definition 2. The minimum of the two norms of non-trivial primitive ambiguous principal ideals $(\alpha), (\alpha^2/b)$ of a cyclic cubic field F is called the **principal factor** (of the discriminant $d_F = c^2$ of the field) $B(F) := \min\{a\cdot b^2, a^2\cdot b\}$ of F, that is

$$B(F) = \begin{cases} a\cdot b^2 & \text{if } b < a, \\ a^2\cdot b & \text{if } a < b. \end{cases} \tag{2.4.3}$$

.

Ayadi [3, 4] speaks about the *Parry constant* or *Parry invariant* $B(F)$ of F, and Derhem [15] calls $B(F)$ the *Kummer resolvent* of F.

Algorithm 6. (Principal Factor.)
Input: critical prime `p`, conductor `f`, cyclic number field `N`.
Code:

```
ON := MaximalOrder(N); // algebraic integers of the base field N
// trivial initialization of prime ideals
Pid := Decomposition(ON,p)[1][1];
Qid := Decomposition(ON,p)[1][1];
Rid := Decomposition(ON,p)[1][1];
Sid := Decomposition(ON,p)[1][1];
// number of prime divisors of f totally ramified in N
```

```
nPrimes := 0;
q := 1;
while (f ge q) do
    q := NextPrime(q);
    if (0 eq f mod q) then
        nPrimes := nPrimes + 1;
        // full decomposition of q in N: necessarily a singlet
        sPrimesN := Decomposition(ON,q);
        if (1 eq nPrimes) then
            Pid := sPrimesN[1][1];
        end if;
        if (2 eq nPrimes) then
            Qid := sPrimesN[1][1];
        end if;
        if (3 eq nPrimes) then
            Rid := sPrimesN[1][1];
        end if;
        // and so on for bigger number nPrimes
    end if; // q divides f
end while; // q
// possible absolute principal factorizations in N
nPrincipalN := 0;
if (1 eq nPrimes) then
    for e in [0..p-1] do
        I := (Pid^e);
        if IsPrincipal(I) then
            nPrincipalN := nPrincipalN + 1;
            printf " %o:  [%o]\n",nPrincipalN,e;
        end if;
    end for;
    printf " %o power products of primes among %o\n",nPrincipalN,p;
end if; // 1
if (2 eq nPrimes) then
    for e in [0..p-1] do
        for f in [0..p-1] do
            I := (Pid^e)*(Qid^f);
            if IsPrincipal(I) then
                nPrincipalN := nPrincipalN + 1;
                printf " %o:  [%o,%o]\n",nPrincipalN,e,f;
            end if;
        end for;
    end for;
    printf " %o power products of primes among %o\n",nPrincipalN,p^2;
end if; // 2
if (3 eq nPrimes) then
    for e in [0..p-1] do
        for f in [0..p-1] do
            for g in [0..p-1] do
                I := (Pid^e)*(Qid^f)*(Rid^g);
                if IsPrincipal(I) then
                    nPrincipalN := nPrincipalN + 1;
                    printf " %o:  [%o,%o,%o]\n",nPrincipalN,e,f,g;
                end if;
            end for;
        end for;
    end for;
    printf " %o power products of primes among %o\n",nPrincipalN,p^3;
end if; // 3
// and so on for bigger number nPrimes
```

Output: prime factorization of norms of ambiguous principal ideals.

3. Unramified Extensions

3.1. The Absolute 3-Genus Field

The *absolute* 3*-genus field* $F^* = (F/\mathbb{Q})^*$ of a cyclic cubic field F is the maximal unramified 3-extension F^*/F with abelian absolute Galois group $\mathrm{Gal}(F^*/\mathbb{Q})$. If the conductor $c = q_1 \cdots q_t$ of $F = F_c$ has t prime divisors, then F^* is the compositum of the multiplet $(F_{c,1}, \ldots, F_{c,m})$ of all cyclic cubic fields sharing the common conductor c, where $m = m(c) = 2^{t-1}$, according to the multiplicity formula (1.1.2). The absolute Galois group $\mathrm{Gal}(F^*/\mathbb{Q})$ is the elementary abelian 3-group C_3^t. In particular, if $t = 1$ then $F^* = F$, and if $t = 2$, $c = q_1q_2$, then $F^* = F_{c,1} \cdot F_{c,2}$ is a bicyclic bicubic field with conductor c and discriminant

$$d(F^*) = d(F_{q_1}) \cdot d(F_{q_2}) \cdot d(F_{c,1}) \cdot d(F_{c,2}) = q_1^2 \cdot q_2^2 \cdot (q_1q_2)^2 \cdot (q_1q_2)^2 = c^6. \quad (3.1.1)$$

Parry [42] has investigated the arithmetic of a general *bicyclic bicubic field* K with four cyclic cubic subfields $K_1, \ldots, K_4$. In particular, he determined the *class number relation*

$$h = \frac{I}{3^5} \cdot \prod_{i=1}^{4} h_i, \quad (3.1.2)$$

where $I := (U : V)$ with $V := \prod_{i=1}^{4} U_i$ denotes the *index of subfield units* of K [42, Prp. 7, p. 496]. Generally, I is a divisor of $27 = 3^3$ [42, Lem. 11, p. 500].

For a cyclic cubic field F with $t = 2$, the 3-class numbers of the 3-genus field F^* and its subfields can be summarized as follows.

Theorem 2. *Let $F^* = F_{q_1} \cdot F_{q_2} \cdot F_{c,1} \cdot F_{c,2}$ be the genus field of the two cyclic cubic fields $F_{c,1}$ and $F_{c,2}$ with conductor $c = q_1q_2$. Denote the* 3*-valuations of the class numbers h, h_1, h_2, h_3, h_4 of F^*, F_{q_1}, F_{q_2}, $F_{c,1}$, $F_{c,2}$, respectively, by v, v_1, v_2, v_3, v_4. Then $v_1 = v_2 = 0$, and*

$$v \begin{cases} = 0,\ v_3 = v_4 = 1,\ I = 27, & \textit{if not } q_1 \leftrightarrow q_2, \\ = 1, & \textit{if } q_1 \leftrightarrow q_2,\ v_3 = v_4 = 2,\ I = 9, \\ = 2, & \textit{if } q_1 \leftrightarrow q_2,\ v_3 = v_4 = 2,\ I = 27, \\ \geq 3, & \textit{if } q_1 \leftrightarrow q_2,\ v_3 \geq 3,\ v_4 \geq 3,\ I \geq 9. \end{cases} \quad (3.1.3)$$

Proof. According to Theorem 1, we generally have $v_1 = v_2 = 0$, $v_3 \ge 1$, $v_4 \ge 1$ if not $q_1 \leftrightarrow q_2$, and $v_3 \ge 2$, $v_4 \ge 2$ if $q_1 \leftrightarrow q_2$. Now, the claim is an immediate consequence of Formula (3.1.2), which yields

$$v = v_3 h = v_3 I - 5 + \sum_{i=1}^{4} v_3 h_i = v_3 I - 5 + v_1 + v_2 + v_3 + v_4 = v_3 I - 5 + v_3 + v_4.$$

The combination of [42, Thm. 9, p. 497] and [42, Cor. 1, p. 498] shows that $v = 0$ if and only if not $q_1 \leftrightarrow q_2$, and $v = 0$ implies $v_3 I = 3$, whence necessarily $v_3 = v_4 = 1$. However, if $q_1 \leftrightarrow q_2$, then $v_3 = 2$ is equivalent with $v_4 = 2$, according to [4, Thm. 4.1, p. 472]. □

Remark 2. For $v_3 = v_4 = 2$, we have $\mathrm{Cl}_3 F_{q_1 q_2, \mu} \simeq (3,3)$. The smallest occurrences of $v_3 = v_4 = 3$ are the conductors $7 \cdot 673 = 4\,711$ (Eau de Cologne) and $7 \cdot 769 = 5\,383$ with $\mathrm{Cl}_3 F_{q_1 q_2, \mu} \simeq (9,3)$, $\mu = 1, 2$. They will be considered in Section § 5.

For a cyclic cubic field F with $t = 3$ and conductor $c = q_1 q_2 q_3$, the 3-genus field F^* contains 13 bicyclic bicubic subfields. Three of them are the *sub genus fields* F_i^*, $1 \le i \le 3$, of the cyclic cubic fields with conductors $q_1 q_2$, $q_2 q_3$, $q_3 q_1$, respectively. In the sequel, we always start with the leading three sub genus fields F_i^*, $1 \le i \le 3$, separated by a semicolon from the trailing ten remaining bicyclic bicubic subfields, when we give a family of invariants for these 13 subfields $S_1, \ldots, S_{13}$,

$$\text{in particular, } [\mathrm{Cl}_3 S_i]_{1 \le i \le 13} := [\mathrm{Cl}_3 F_1^*, \ldots, \mathrm{Cl}_3 F_3^*; \mathrm{Cl}_3 S_4, \ldots, \mathrm{Cl}_3 S_{13}]. \qquad (3.1.4)$$

Algorithm 7. (Class groups of 13 bicyclic bicubic fields.)
Input: critical prime `p`, prime power factorization `cPD` of the conductor `c`.
Code:

```
numPF := #cPD;
if (1 le numPF) then
    prime1 := cPD[1][1];
    if (p eq prime1) then
        prime1 := p^2;
    end if;
    if (2 le numPF) then
        prime2 := cPD[2][1];
        if (3 le numPF) then
            prime3 := cPD[3][1];
            if (4 le numPF) then
                prime4 := cPD[4][1];
            end if; // 4
        end if; // 3
    end if; // 2
end if; // 1
f1 := prime1*prime2;
Disc1 := f1^2;
f2 := prime1*prime3;
Disc2 := f2^2;
// and so on until
```

```
Disc := c^2; // root discriminant
if (Disc^((p-1) div 2) eq AbsoluteDiscriminant(ON)) then
    fld := fld + 1;
    locMult := locMult + 1;
    if (1 eq locMult) then
        Component1 := N;
    elif (2 eq locMult) then
        Component2 := N;
    elif (3 eq locMult) then
        Component3 := N;
    else
        Component4 := N;
    end if;
elif (Disc1^((p-1) div 2) eq AbsoluteDiscriminant(ON)) then
    locMult1 := locMult1 + 1;
    if (1 eq locMult1) then
        Sub11 := N;
    elif (2 eq locMult1) then
        Sub12 := N;
    end if;
elif (Disc2^((p-1) div 2) eq AbsoluteDiscriminant(ON)) then
    locMult2 := locMult2 + 1;
    if (1 eq locMult2) then
        Sub21 := N;
    elif (2 eq locMult2) then
        Sub22 := N;
    end if;
end if; // and so on until
elif (2 eq locMult) then
    Append(~Doublets,Collection);
    printf "\nm=2, no.  %o\n",Mult[2];
    GenusField := Compositum(Component1,Component2);
    OGK := MaximalOrder(GenusField);
    CGK,mCGK := ClassGroup(OGK);
    AGK := AbelianInvariants(CGK);
    PGK := pPrimaryInvariants(CGK,p);
    VGK := Valuation(#CGK,p);
    printf "Genus Field 2:  SGK=%o, CGK=%o\n",PGK,AGK;
elif (4 eq locMult) then
    Append(~Quartets,Collection);
    printf "\nm=4, no.  %o\n",Mult[4];
    // ten bicyclic bicubic fields
    B1 := Compositum(Sub11,Sub21);
    B2 := Compositum(Sub11,Sub22);
    B3 := Compositum(Sub12,Sub22);
    B4 := Compositum(Sub12,Sub21);
    B5 := Compositum(Component1,Component3);
    B6 := Compositum(Component1,Component4);
    B7 := Compositum(Component1,Component2);
    B8 := Compositum(Component2,Component4);
    B9 := Compositum(Component2,Component3);
    B10 := Compositum(Component3,Component4);
    // their class groups
    OB := MaximalOrder(B1);
    CB,mCB := ClassGroup(OB);
    AB := AbelianInvariants(CB);
    PB := pPrimaryInvariants(CB,p);
    VB := Valuation(#CB,p);
    printf "Bicubic 1:  SB=%o, CB=%o\n",PB,AB;
    // and so on until
    OB := MaximalOrder(B10);
    CB,mCB := ClassGroup(OB);
```

```
    AB := AbelianInvariants(CB);
    PB := pPrimaryInvariants(CB,p);
    VB := Valuation(#CB,p);
    printf "Bicubic 10:  SB=%o, CB=%o\n",PB,AB;
end if;
```

Output: class groups of ten bicyclic bicubic subfields of the genus field.

3.2. Capitulation Kernels of Cyclic Cubic Fields

Finally, we recall the connection between the size of the capitulation kernel $\ker(T_{E/F})$ and the unit norm index $(U_F : \mathrm{Norm}_{E/F}U_E)$ of an unramified cyclic cubic extension E/F of a cyclic cubic field F. Here, $T_{E/F} : \mathrm{Cl}_3 F \to \mathrm{Cl}_3 E$, $\mathfrak{a}\mathcal{P}_F \mapsto (\mathfrak{a}\mathcal{O}_E)\mathcal{P}_E$, denotes the transfer of 3-classes from F to E.

Theorem 3. *The order of the* 3*-capitulation kernel of* E/F *is given by*

$$\#\ker(T_{E/F}) = \begin{cases} 3, \\ 9, \\ 27, \end{cases} \quad \textit{if and only if} \quad (U_F : \mathrm{Norm}_{E/F}U_E) = \begin{cases} 1, \\ 3, \\ 9. \end{cases} \tag{3.2.1}$$

Proof. According to the Herbrand Theorem on the cohomology of the unit group U_E as a Galois module with respect to $G = \mathrm{Gal}(E/F)$, we have the relation $\#\ker(T_{E/F}) = [E : F] \cdot (U_F : \mathrm{Norm}_{E/F}U_E)$, since $\ker(T_{E/F}) \simeq H^1(G, U_E)$ when E/F is unramified of odd prime degree $[E : F] = 3$ and $U_F/\mathrm{Norm}_{E/F}U_E \simeq \hat{H}^0(G, U_E)$. The cyclic cubic base field F has signature $(r_1, r_2) = (3, 0)$ and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 3 + 0 - 1 = 2$. Thus, there are three possibilities for the unit norm index $(U_F : \mathrm{Norm}_{E/F}U_E) \in \{1, 3, 9\}$. □

Remark 3. When F is a cyclic cubic field with 3-class group $O := \mathrm{Cl}_3 F$ of elementary tricyclic type $(3, 3, 3)$, viewed as a vector space of dimension 3 over the finite field $\mathbb{F}_3$, then $\#\ker(T_{E/F}) = 3$ if and only if $(\exists\, 1 \le i \le 13)\ \ker(T_{E/F}) = L_i$ is a line in Table 6.9, $\#\ker(T_{E/F}) = 9$ if and only if $(\exists\, 1 \le i \le 13)\ \ker(T_{E/F}) = P_i$ is a plane in Table 6.10, and $\#\ker(T_{E/F}) = 27$ if and only if $\ker(T_{E/F}) = O$ is the entire vector space. See Section § 6.3.

Algorithm 8. (Artin pattern for type $(3, 3)$.)
Input: critical prime `p`, algebraic number field `F` of type (p, p).
Code:

```
O := MaximalOrder(F); // algebraic integers of the base field F
C,mC := ClassGroup(O);
epsilon := 0; // counter of (3,3,3)
polarization1 := 3;
polarization2 := 3; // co-polarization
fixedpoints := 0;
```

```
capitulations := 0; // occupation number of 0
occupation := 0; // cardinality of occupation support (including 0)
repetitions := 0; // maximal occupation number (except for 0)
intersection := 0; // meet of repetitions and fixed points
sS := Subgroups(C: Quot := [p]); // subgroups of index p
sI := []; // fixed ordering of subgroups
for j in [1..p+1] do
    Append(~sI,0);
end for; // j
n := Ngens(C);
H := (Order(C.(n-1)) div p)*C.(n-1); // 1st p-generator
K := (Order(C.n) div p)*C.n; // 2nd p-generator
ct := 0; // local counter
for x in sS do
    ct := ct+1;
    if H in x`subgroup then
        sI[1] := ct;
    end if; // n-1
    if K in x`subgroup then
        sI[2] := ct;
    end if; // n
    for e in [1..p-1] do
        if (H+(e*K)) in x`subgroup then
            sI[e+2] := ct;
        end if; // product
    end for; // e
end for; // x
// p+1 unramified extensions of degree p
sA := [AbelianExtension(Inverse(mQ)*mC)
        where Q,mQ := quo<C|x`subgroup>:  x in sS];
sN := [NumberField(x):  x in sA]; // relative extensions
sR := [MaximalOrder(x):  x in sA];
sF := [AbsoluteField(x):  x in sN]; // absolute extensions
sM := [MaximalOrder(x):  x in sF];
sM := [OptimizedRepresentation(x):  x in sF]; // 1st optimization
sA := [NumberField(DefiningPolynomial(x)):  x in sM];
sO := [Simplify(LLL(MaximalOrder(x))):  x in sA]; // 2nd optimization
// Artin pattern with two components TTT and TKT
TTT := []; // transfer target type
for j in [1..#sO] do
    CO := ClassGroup(sO[j]);
    Append(~TTT,pPrimaryInvariants(CO,p));
    if (3 eq #pPrimaryInvariants(CO,p)) then
        epsilon := epsilon + 1;
    end if;
    val := Valuation(Order(CO),p);
    if (2 eq val) then
        polarization2 := val;
    elif (4 le val) then
        if (3 eq polarization1) then
            polarization1 := val;
        else
            polarization2 := val;
        end if;
    end if; // val
end for; // j
// using the fixed ordering of subgroups
TKT := []; // transfer kernel type
for j in [1..#sR] do // use sR rather than sO
    Collector := [];
    I := sR[j]!!mC(H);
    if IsPrincipal(I) then
```

```
        Append(~Collector,sI[1]);
    end if; // I
    I := sR[j]!!mC(K);
    if IsPrincipal(I) then
        Append(~Collector,sI[2]);
    end if; // I
    for e in [1..p-1] do
        I := sR[j]!!mC(H+(e*K));
        if IsPrincipal(I) then
            Append(~Collector,sI[e+2]);
        end if; // I
    end for; // e
    if (2 le #Collector) then // total capitulation
        Append(~TKT,0);
    else // partial capitulation
        Append(~TKT,Collector[1]);
    end if; // #Collector
end for; // j
TAB := []; // Taussky conditions A and B
image := []; // local image collection
for j in [1..#TKT] do
    if (j eq TKT[j]) then // fixed point
        Append(~TAB,"A");
        fixedpoints := fixedpoints + 1;
    elif (0 eq TKT[j]) then // total
        Append(~TAB,"A");
        capitulations := capitulations + 1;
    else // non-fixed point partial
        Append(~TAB,"B");
    end if; // fixed point or total or partial
    if not (TKT[j] in image) then
        Append(~image,TKT[j]);
    end if; // image
end for; // j
occupation := #image;
doublet := 0; // local memory
for digit in [1..p+1] do
    counter := 0; // local counter
    for j in [1..#TKT] do
        if (digit eq TKT[j]) then
            counter := counter + 1;
        end if;
    end for; // j
    if (counter ge 2) then
        doublet := digit; // last assignment persists
    end if;
    if (counter gt repetitions) then
        repetitions := counter;
    end if;
end for; // digit
if (doublet ge 1) then
    if (doublet eq TKT[doublet]) then
        intersection := 1;
    end if;
end if;
printf "TKT %o; TAB %o; INV %o; %o; %o; TTT ( ",TKT,TAB,epsilon,polarization1,polarization2;
for j in [1..#TTT] do
    printf "%o ",TTT[j];
end for; // j
printf ")\n";
```

Output: transfer kernel type TKT, transfer target type TTT, other invariants.

Algorithm 9. (Artin pattern for type $(3,3,3)$.)
Input: critical prime p, algebraic number field N of type (p,p,p).
Code:

```
ON := MaximalOrder(N); // algebraic integers of the base field N
CN,mCN := ClassGroup(ON);
PN := pPrimaryInvariants(CN,p);
if ([p,p,p] eq PN) then
    sS2 := Subgroups(C: Quot := [p]);
    sA2 := [AbelianExtension(Inverse(mQ1)*mC)
            where Q1,mQ1 := quo<C|x`subgroup>:  x in sS2];
    sN2 := [NumberField(x):  x in sA2]; // relative extensions
    sO := [MaximalOrder(x):  x in sA2];
    sF2 := [AbsoluteField(x):  x in sN2]; // absolute extensions
    sM2 := [MaximalOrder(x):  x in sF2];
    sB2 := [OptimizedRepresentation(x):  x in sF2]; // 1st optimization
    sK2 := [NumberField(DefiningPolynomial(x)):  x in sB2];
    sO2 := [Simplify(LLL(MaximalOrder(x))):  x in sK2]; // 2nd optimization
    n := Ngens(C);
    u := (Order(C.n) div p)*C.n; // 1st p-generator
    v := (Order(C.(n-1)) div p)*C.(n-1); // 2nd p-generator
    w := (Order(C.(n-2)) div p)*C.(n-2); // 3rd p-generator
    ArtinMap := []; // fixed ordering of subgroups
    for x in sS2 do
        Collector := [];
        if u in x`subgroup then
            Append(~Collector,1);
        end if; // n
        if v in x`subgroup then
            Append(~Collector,2);
        end if; // n-1
        if w in x`subgroup then
            Append(~Collector,3);
        end if; // n-2
        for e in [1..p-1] do
            if u+e*v in x`subgroup then
                Append(~Collector,3+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            if v+e*w in x`subgroup then
                Append(~Collector,2+p+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            if w+e*u in x`subgroup then
                Append(~Collector,1+2*p+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            for f in [1..p-1] do
                if u+e*v+f*w in x`subgroup then
                    Append(~Collector,3*p+(e-1)*(p-1)+f);
                end if;
            end for; // f
        end for; // e
        Append(~ArtinMap,Collector);
    end for; // x
    printf "Artin Map:  ";
    for i in [1..p^2+p+1] do
        printf "("; for k in [1..#ArtinMap[i]] do
            printf "%o,",ArtinMap[i][k];
        end for; printf ")";
    end for;
    printf "\n";
    ATI := []; // transfer target type
    for j in [1..#sO2] do
        C2,mC2 := ClassGroup(sO2[j]);
        Append(~ATI,pPrimaryInvariants(C2,p));
    end for; // j printf "TTT = (";
    for i in [1..p^2+p+1] do
        printf "%o,",ATI[i];
    end for;
    printf ")\n";
    // using the fixed ordering of subgroups
    TKT := []; // transfer kernel type
    for j in [1..#sO] do
        Collector := [];
        I := sO[j]!!mC(u);
        if IsPrincipal(I) then
            Append(~Collector,1);
        end if;
```

```
        I := sO[j]!!mC(v);
        if IsPrincipal(I) then
            Append(~Collector,2);
        end if;
        I := sO[j]!!mC(w);
        if IsPrincipal(I) then
            Append(~Collector,3);
        end if;
        for e in [1..p-1] do
            I := sO[j]!!mC(u+e*v);
            if IsPrincipal(I) then
                Append(~Collector,3+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            I := sO[j]!!mC(v+e*w);
            if IsPrincipal(I) then
                Append(~Collector,2+p+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            I := sO[j]!!mC(w+e*u);
            if IsPrincipal(I) then
                Append(~Collector,1+2*p+e);
            end if;
        end for; // e
        for e in [1..p-1] do
            for f in [1..p-1] do
                I := sO[j]!!mC(u+e*v+f*w);
                if IsPrincipal(I) then
                    Append(~Collector,3*p+(e-1)*(p-1)+f);
                end if;
            end for; // f
        end for; // e
        if (p^2+p+1 eq #Collector) then // total capitulation
            Append(~TKT,[0]);
        else
            Append(~TKT,Collector);
        end if;
    end for; // j
    printf "TKT = (";
    for i in [1..p^2+p+1] do
        printf "%o,",TKT[i];
    end for;
    printf ")\n";
end if; // (p,p,p)
```

Output: `ArtinMap`, transfer target type `TTT`, transfer kernel type `TKT`.

4. Classical Results

4.1. Cyclic Cubic Doublets of Type (3,3)

4.1.1. Conductors with Two Prime Divisors

First we focus on conductors c with *two* prime divisors, $t = 2$. According to the multiplicity formula $m(c) = (3-1)^{t-1}$, there are 2 cyclic cubic fields $F_{c,1}, F_{c,2}$ sharing the common conductor c. If one of them has 3-class group $(3,3)$, then the same is true for the other [4, Thm. 4.1, p. 472]. Necessarily, the prime divisors of the conductor c are mutual cubic residues with respect to each other, $\left(\frac{q_1}{q_2}\right)_3 = \left(\frac{q_2}{q_1}\right)_3 = +1$, with graph $q_1 \leftrightarrow q_2$, by Theorem 1. Further, it turns out that both members $F = F_{c,\mu}$, $\mu \in \{1,2\}$, of the *doublet* have the same 3-capitulation type $\varkappa(F)$ [3, Prp. 3.3, p. 27] and the same 3-class field tower group $G := \mathrm{G}_3^{(\infty)}F := \mathrm{Gal}(F_3^{(\infty)}/F)$, which can be determined with the aid of the following Theorem 4. Thus, we have only one graph, $q_1 \leftrightarrow q_2$, and do not need categories in the case $t = 2$. In the sequel, we always use *identifiers* $\langle o, i\rangle$ of the SmallGroups Library [7, 8] in order to characterize a group G by its order $o = \#G$ and a counting number i, enclosed in angle brackets.

Theorem 4. *(Principal factor criterion,* ***Ayadi, 1995****, [3, 4].)*
Let c be a conductor divisible by two primes, $t = 2$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for both cyclic cubic fields $F_{c,\mu}$, $1 \le \mu \le 2$, with conductor c. Denote by n the number of prime divisors of the norm $B(F) = \mathrm{N}_{F/\mathbb{Q}}(\alpha)$ of a non-trivial primitive ambiguous principal ideal (α), i.e. a ***principal factor****, of any of the two fields $F_{c,\mu}$.*
Then $n \in \{1,2\}$, and the second 3-class group $\mathfrak{M} := \mathrm{G}_3^{(2)}F := \mathrm{Gal}(F_3^{(2)}/F)$ of both fields $F = F_{c,\mu}$ is given by

$$\mathrm{G}_3^{(2)}F \simeq \begin{cases} \langle 9,2\rangle \text{ with capitulation type a.1, } \varkappa F = (0000), & \text{if } n = 2, \\ \langle 27,4\rangle \text{ with capitulation type A.1, } \varkappa F = (1111), & \text{if } n = 1. \end{cases} \tag{4.1.1}$$

The length of the Hilbert 3-class field tower is $\ell_3 F = 1$ with $F_3^{(\infty)} = F_3^{(1)}$ if $n = 2$, and $\ell_3 F = 2$ with $F_3^{(\infty)} = F_3^{(2)}$ if $n = 1$. In both cases, $G := \mathrm{G}_3^{(\infty)}F = \mathrm{G}_3^{(2)}F$.

Proof. See [3, Prp. 3.6, p. 32, Thm. 3.1, p. 34, Thm. 3.3, p. 37]. □

Concerning the 3-capitulation types a.1 and A.1, viewed as transfer kernel types (TKT), and the related concept of transfer target types (TTT), see [27].

The criteria in Theorem 4 can also be expressed with the aid of the 3-class number $h_3F^* = \#\mathrm{Cl}_3F^*$ of the 3-**genus field** F^* instead of principal factors.

Corollary 1. *(Genus field criterion,* ***Ayadi, 2001****, [3, 4].)*
Under the assumptions of Theorem 4, the second 3-class group of both fields $F = F_{c,\mu}$, $1 \le \mu \le 2$, is characterized by the 3-valuation v of the class number of F^*.

$$\mathrm{G}_3^{(2)}F \simeq \begin{cases} \langle 9,2\rangle \text{ with transfer target type } \tau F = [1,1,1,1], & \text{if } v = 1, \\ \langle 27,4\rangle \text{ with transfer target type } \tau F = [11,2,2,2], & \text{if } v = 2. \end{cases} \tag{4.1.2}$$

Proof. See [4, Thm. 4.2 and Thm. 4.3, p. 473]. □

Remark 4. Note that the criteria in Theorem 4 are not rational (in terms of the prime divisors of the conductor). A non-trivial primitive ambiguous principal ideal (α) must be determined either directly or by means of a fundamental system of units of F, for instance with Magma [23].

Example 1. The first occurrences for both situations, $9 \mid c$ and $\gcd(c,3) = 1$, are:
$B(F) = \mathrm{N}_{F/\mathbb{Q}}(\alpha) = 3^2 \cdot 73$, $n = 2$, and thus $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 9,2\rangle$, for $c = 657 = 3^2 \cdot 73$,
$B(F) = \mathrm{N}_{F/\mathbb{Q}}(\alpha) = 7\cdot 181$, $n = 2$, and thus $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 9,2\rangle$, for $c = 1\,267 = 7\cdot 181$,
$B(F) = \mathrm{N}_{F/\mathbb{Q}}(\alpha) = 271$, $n = 1$, and thus $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 27,4\rangle$, for $c = 2\,439 = 3^2\cdot 271$,
$B(F) = \mathrm{N}_{F/\mathbb{Q}}(\alpha) = 853$, $n = 1$, and thus $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 27,4\rangle$, for $c = 5\,971 = 7\cdot 853$,
according to Ayadi, Azizi, and Ismaili [4, p. 474], and confirmed by our own computations.

Remark 5. The statistical distribution of the three possible graphs for $t = 2$, according to Theorem 1, in the range $c < 10^5$ of conductors is the following. From Table 1.2 we know there are 3863 doublets with conductors $c = q_1q_2$ in this range. They can be partitioned as follows. Graph 1 with symbol $[q_1,q_2]_3 = \{q_1,q_2\}$ occurs 1740 times, graph 2 with symbol $[q_1,q_2]_3 = \{q_1 \to q_2\}$ or $\{q_2 \to q_1\}$ occurs 1715 times, and graph 3 with symbol $[q_1,q_2]_3 = \{q_1 \leftrightarrow q_2\}$ occurs 408 times. As required, the sum is $1740 + 1715 + 408 = 3863$.

This statistical result is in good accordance with the *probability tree* in Table 4.1, because the proportion $4:4:1$ for the three graphs is nearly satisfied by the census $1740 : 1715 : 408$, since $4 \cdot 408 = 1632 \approx 1715 \approx 1740$.

Another partition according to the 3-class number $h = h_3F^* = \#\mathrm{Cl}_3F^*$ of the 3-*genus field* $F^* = F_{c,1} \cdot F_{c,2}$ is also very illuminating. We have $h = 1$ for $3455 = 1740 + 1715$ hits, which correspond to the union of graphs 1 and 2. On the other hand, the 408 occurrences of graph 3 split into 210 cases with $h = 3$, corresponding to the group $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 9,2\rangle$, 142 cases with $h = 9$, corresponding to the group $\mathrm{G}_3^{(2)}F_{c,\mu} \simeq \langle 27, 4\rangle$, and 56 cases with $h \geq 27$, corresponding to elevated 3-class groups $\mathrm{Cl}_3 F_{c,\mu} \simeq (9, 3)$ and bigger. The sum is $210 + 142 + 56 = 408$, as required. So the **overall proportion** of the *abelian group* $\langle 9, 2\rangle$ is $\frac{210}{352} \approx 59.7\%$, as opposed to the *extra special group* $\langle 27, 4\rangle$ with $\frac{142}{352} \approx 40.3\%$. See, however, section 1.2 in the Conclusion for an *inverted population* in the special case $9 \mid c$.

Table 4.1.: Probability tree for the symbol $[q_1, q_2]_3$

P_1	$\left(\frac{q_1}{q_2}\right)_3$	P_2	$\left(\frac{q_2}{q_1}\right)_3$	$P_1 \cdot P_2$	for $[q_1, q_2]_3$
		$2/3$	ζ, ζ^2	$4/9$	for $\{q_1, q_2\}$
		$\nearrow$			
$2/3$	ζ, ζ^2	$\searrow$			
$\nearrow$		$1/3$	1	$2/9$	for $\{q_1 \leftarrow q_2\}$
$\searrow$		$2/3$	ζ, ζ^2	$2/9$	for $\{q_1 \rightarrow q_2\}$
$1/3$	1	$\nearrow$			
		$\searrow$			
		$1/3$	1	$1/9$	for $\{q_1 \leftrightarrow q_2\}$

Part II.

Current Research

5. Singular and Super-Singular Doublets

In his thesis [3], Ayadi investigated doublets of cyclic cubic fields $(F_{c,\mu})_{1\le\mu\le2}$ with conductor $c = q_1q_2$ divisible by two primes, which are mutual cubic residues with respect to each other $q_1 \leftrightarrow q_2$, and 3-class group $\mathrm{Cl}_3F_{c,\mu} = (3,3)$. However, since the constitution of the 3-genus fields F^* of cyclic cubic fields F is recursive, we shall see that the sub genus fields F_c^* with partial conductors $c \in \{q_1q_2, q_2q_3, q_3q_1\}$ exert a considerable impact on the quartets of cyclic cubic fields $(F_{f,\mu})_{1\le\mu\le4}$ with conductor $f = q_1q_2q_3$ divisible by three primes and 3-genus field F_f^*. Consequently, the behavior of quartets cannot be analyzed satisfactorily, if one restricts to doublets with elementary bicyclic 3-class group. For doublets $(F_{c,\mu})_{1\le\mu\le2}$ with conductor $c = q_1q_2$ and non-elementary bicyclic 3-class group, a further distinction arises from the 3-valuation v of the class number $\#\mathrm{Cl}F_c^*$ of the 3-genus field F_c^*:

Definition 3. A doublet $(F_{c,\mu})_{1\le\mu\le2}$ of cyclic cubic fields is called

$$\begin{cases} \textbf{regular} & \text{if } v \in \{0,1,2\}, \\ \textbf{singular} & \text{if } v = 3, \\ \textbf{super-singular} & \text{if } v \in \{4,5,6\}. \end{cases} \tag{5.0.1}$$

Table 5.1 shows all *singular* doublets $(F_{c,\mu})_{1\le\mu\le2}$ of cyclic cubic fields with conductors $c < 10^5$. Both components have 3-class group $\mathrm{Cl}_3F_{c,\mu} = (3,9)$, the 3-genus field F^* has 3-class group $\mathrm{Cl}_3F^* = (3,3,3)$, and thus valuation $v = 3$. The second 3-class group is always $G_3^{(2)}F_{c,\mu} \simeq \langle 81, 3\rangle$, for both fields $1 \le \mu \le 2$. This group has punctured Artin pattern $\mathrm{AP} = (\varkappa, \tau)$ with transfer kernel type $\varkappa = (000;0)$, a.1, and transfer target type $\tau = [(21)^3; 1^3]$. See [39].

Within the frame $c < 10^5$ of our present computations, there are 13 singular doublets but no super-singular doublets with conductor c divisible by 9.

Table 5.1.: Thirty-Seven Singular Doublets with $v = 3$

No.	c	Factors	No.	c	Factors
1	4 711	$7 \leftrightarrow 673$	20	58 329	$9 \leftrightarrow 6481$
2	11 167	$13 \leftrightarrow 859$	21	62 257	$13 \leftrightarrow 4789$
3	12 439	$7 \leftrightarrow 1777$	22	64 971	$9 \leftrightarrow 7219$
4	16 177	$7 \leftrightarrow 2311$	23	65 383	$151 \leftrightarrow 433$
5	17 593	$73 \leftrightarrow 241$	24	66 829	$7 \leftrightarrow 9547$
6	20 421	$9 \leftrightarrow 2269$	25	69 183	$9 \leftrightarrow 7687$
7	25 963	$7 \leftrightarrow 3709$	26	71 611	$19 \leftrightarrow 3769$
8	27 571	$79 \leftrightarrow 349$	27	72 099	$9 \leftrightarrow 8011$
9	32 689	$97 \leftrightarrow 337$	28	73 873	$31 \leftrightarrow 2383$
10	35 163	$9 \leftrightarrow 3907$	29	77 281	$109 \leftrightarrow 709$
11	37 933	$7 \leftrightarrow 5419$	30	78 093	$9 \leftrightarrow 8677$
12	40 573	$13 \leftrightarrow 3121$	31	81 009	$9 \leftrightarrow 9001$
13	40 873	$7 \leftrightarrow 5839$	32	89 109	$9 \leftrightarrow 9901$
14	43 081	$67 \leftrightarrow 643$	33	89 863	$73 \leftrightarrow 1231$
15	44 397	$9 \leftrightarrow 4933$	34	94 357	$157 \leftrightarrow 601$
16	49 743	$9 \leftrightarrow 5527$	35	95 913	$9 \leftrightarrow 10657$
17	51 847	$139 \leftrightarrow 373$	36	96 709	$97 \leftrightarrow 997$
18	55 951	$7 \leftrightarrow 7993$	37	96 817	$7 \leftrightarrow 13831$
19	56 223	$9 \leftrightarrow 6247$			

Table 5.2 shows all *super-singular* doublets $(F_{c,\mu})_{1\le\mu\le 2}$ of cyclic cubic fields with conductors $c < 10^5$. On 23 December 2001, Aissa Derhem [15] asked me, if there exist such fields with 3-class numbers $h_3F_{c,\mu} \in \{81, 243\}$. On Christmas Day 2001, I answered that in the 1982 tables of Ennola and Turunen, there only appears $c = 5\,383$ with $h_3F_{c,\mu} = 27$, and I announced that I am going to launch an extensive computation of regulators and class numbers of cyclic cubic fields by means of Voronoi's algorithm [45] and the Euler product method for the analytical class number formula. I needed three months until I arrived at $h_3F_{c,\mu} \in \{81, 243\}$ for $c \in \{36\,667, 41\,977, 42\,127, 42\,991\}$ on 2 April 2002. See the details in

(1) `http://www.algebra.at/AissaDan17.htm` and

(2) `http://www.algebra.at/KarimAissaDan23.htm`.

Later I saw that $c = 36\,667$ was known to Georges Gras in 1973 already [19, Exm. VI.7, pp. 36–38].

Table 5.2.: Nineteen Super-Singular Doublets with $v \in \{4, 5, 6\}$

No.	c	Factors	$\mathrm{Cl}_3 F^*$	v	$\mathrm{Cl}_3 F_{c,\mu}$	$G_3^{(2)} F_{c,\mu}$
1	5 383	$7 \leftrightarrow 769$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
2	12 403	$79 \leftrightarrow 157$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
3	21 763	$7 \leftrightarrow 3109$	$(3,3,3,3)$	4	$(3,9),(3,9)$	$\langle 243,13\rangle,\langle 729,12\rangle$
4	28 177	$19 \leftrightarrow 1483$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
5	32 311	$79 \leftrightarrow 409$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
6	36 667	$37 \leftrightarrow 991$	$(3,9,27)$	6	$(3,9),(9,27)$	$\langle 243,14\rangle,**$
7	38 503	$139 \leftrightarrow 277$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle,\langle 729,17\rangle$
8	41 977	$13 \leftrightarrow 3229$	$(3,3,9)$	4	$(3,9),(9,9)$	$\langle 243,15\rangle,*$
9	42 127	$103 \leftrightarrow 409$	$(3,3,9,9)$	6	$(9,9),(9,27)$	$*,**,$
10	42 991	$13 \leftrightarrow 3307$	$(3,3,9)$	4	$(3,9),(9,9)$	$\langle 243,15\rangle,*$
11	49 849	$79 \leftrightarrow 631$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle,\langle 729,17\rangle$
12	55 657	$7 \leftrightarrow 7951$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle,\langle 729,17\rangle$
13	57 811	$13 \leftrightarrow 4447$	$(3,3,3,3)$	4	$(3,9),(3,9)$	$\langle 243,13\rangle^2$
14	59 803	$79 \leftrightarrow 757$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
15	59 911	$181 \leftrightarrow 331$	$(3,3,3,3)$	4	$(3,9),(3,9)$	$\langle 243,13\rangle^2$
16	68 857	$37 \leftrightarrow 1861$	$(3,9,9)$	5	$(3,9),(9,9)$	$\langle 243,14\rangle,*$
17	75 859	$7 \leftrightarrow 10837$	$(3,3,3,3)$	4	$(3,9),(3,9)$	$\langle 729,12\rangle^2$
18	84 103	$31 \leftrightarrow 2713$	$(3,3,9)$	4	$(3,9),(3,9)$	$\langle 243,14\rangle^2$
19	97 249	$79 \leftrightarrow 1231$	$(3,9,9)$	5	$(3,9),(9,9)$	$\langle 243,14\rangle,*$

6. Recent Results

6.1. Finite 3-Groups of Type (3,3)

In the following tables, we list those invariants of finite 3-groups with elementary bicyclic commutator quotient $(3,3)$ which qualify metabelian groups $\mathfrak{M}$ as second 3-class groups $\mathrm{Gal}(F_3^{(2)}/F)$ and non-metabelian groups G as 3-class tower groups $\mathrm{Gal}(F_3^{(\infty)}/F)$ of cyclic cubic number fields F. The process of searching for suitable groups in descendant trees by means of the strategy of pattern recognition is governed by the *Artin transfer pattern* $\mathrm{AP} = (\tau, \varkappa)$, where $\tau = \tau_1$, resp. $\varkappa = \varkappa_1$, denotes the first layer of the transfer target type (TTT), resp. transfer kernel type (TKT). Additionally, we give the top layer of the TTT, which consists of the abelian quotient invariants of the commutator subgroup G', corresponding to the 3-class group of the first Hilbert 3-class field $F_3^{(1)}$. The *nuclear rank* ν is responsible for the search complexity. The p-multiplicator rank μ of a group G is precisely its *relation rank* $d_2(G) = \dim_{\mathbb{F}_3} \mathrm{H}^2(G, \mathbb{F}_3)$, which decides whether G is admissible as $\mathrm{Gal}(F_3^{(\infty)}/F)$, according to the Shafarevich Theorem [43]. In the case of cyclic cubic fields F, it is limited by the *Shafarevich bound* $\mu \le \varrho + r + \theta$, where $\varrho = d_1(G) = \dim_{\mathbb{F}_3} \mathrm{H}^1(G, \mathbb{F}_3)$ denotes the *generator rank* of G, which coincides with the 3-class rank ϱ of F, $r = r_1 + r_2 - 1 = 2$ is the torsion free Dirichlet unit rank of the field F with signature $(r_1, r_2) = (3, 0)$, and $\theta = 0$ indicates the absence of a (complex) primitive third root of unity in the totally real field F. Finally, $\pi(\mathfrak{M}) = \mathfrak{M}/\gamma_c\mathfrak{M}$ denotes the parent of $\mathfrak{M}$.

In Table 6.1, we begin with metabelian groups $\mathfrak{M}$ of generator rank $d_1(\mathfrak{M}) = 2$. The Shafarevich bound [32] is given by $\mu \le \varrho + r + \theta = 2 + 2 + 0 = 4$.

For the metabelian groups $\mathfrak{M}$ with non-trivial cover $\mathrm{cov}(\mathfrak{M})$, we need non-metabelian groups G in the cover, which are given in Table 6.2, where we begin with groups G of generator rank $d_1(G) = 2$. For $d_1(G) = 3$, see § 6.4.

Table 6.1.: Invariants of Metabelian 3-Groups $\mathfrak{M}$ with $\mathfrak{M}/\mathfrak{M}' \simeq (3,3)$

$\mathfrak{M}$	cc	Type	$\varkappa$	τ	τ_2	ν	μ	$\pi(\mathfrak{M})$
$\langle 9,2\rangle$	1	a.1	(0000)	$(1)^4$	0	3	3	
$\langle 27,4\rangle$	1	A.1	(1111)	$1^2,(2)^3$	1	0	2	$\langle 9,2\rangle$
$\langle 81,7\rangle$	1	a.3	(2000)	$1^3,(1^2)^3$	1^2	0	3	$\langle 27,3\rangle$
$\langle 81,8\rangle$	1	a.3	(2000)	$21,(1^2)^3$	1^2	0	3	$\langle 27,3\rangle$
$\langle 81,10\rangle$	1	a.2	(1000)	$21,(1^2)^3$	1^2	0	3	$\langle 27,3\rangle$
$\langle 81,9\rangle$	1	a.1	(0000)	$21,(1^2)^3$	1^2	1	4	$\langle 27,3\rangle$
$\langle 243,28\ldots 30\rangle$	1	a.1	(0000)	$21,(1^2)^3$	21	0	3	$\langle 81,9\rangle$
$\langle 243,25\rangle$	1	a.3	(2000)	$2^2,(1^2)^3$	21	0	3	$\langle 81,9\rangle$
$\langle 243,27\rangle$	1	a.2	(1000)	$2^2,(1^2)^3$	21	0	3	$\langle 81,9\rangle$
$\langle 243,8\rangle$	2	c.21	(0231)	$(21)^4$	1^3	1	3	$\langle 27,3\rangle$
$\langle 729,54\rangle=U$	2	c.21	(0231)	$22,(21)^3$	21^2	2	4	$\langle 243,8\rangle$
$\langle 243,3\rangle$	2	b.10	(0043)	$(21)^2,(1^3)^2$	1^3	2	4	$\langle 27,3\rangle$
$\langle 729,40\rangle=B$	2	b.10	(0043)	$2^2,21,(1^3)^2$	21^2	2	5	$\langle 243,3\rangle$
$\langle 729,34\rangle=H$	2	b.10	(0043)	$(21)^2,(1^3)^2$	1^4	2	5	$\langle 243,3\rangle$
$\langle 729,35\rangle=I$	2	b.10	(0043)	$(21)^2,(1^3)^2$	1^4	1	4	$\langle 243,3\rangle$
$\langle 729,37\rangle=A$	2	b.10	(0043)	$(21)^2,(1^3)^2$	21^2	2	5	$\langle 243,3\rangle$
$\langle 729,38\rangle=C$	2	b.10	(0043)	$(21)^2,(1^3)^2$	21^2	1	4	$\langle 243,3\rangle$
$\langle 729,41\rangle=D$	2	d.19	(4043)	$32,21,(1^3)^2$	21^2	1	4	$\langle 243,3\rangle$
$\langle 2187,301\vert 305\rangle$	2	G.16	(4231)	$32,(21)^3$	2^21	1	4	$\langle 729,54\rangle$

Table 6.2.: Invariants of Non-Metabelian 3-Groups G with $G/G' \simeq (3,3)$

G	cc	Type	$\varkappa$	τ	τ_2	ν	μ	G/G''
$\langle 2187,263\ldots 265\rangle$	2	d.19	(4043)	$32,21,(1^3)^2$	21^2	0	3	$\langle 729,41\rangle$
$\langle 2187,307\vert 308\rangle$	2	c.21	(0231)	$32,(21)^3$	21^2	0	3	$\langle 729,54\rangle$
$\langle 6561,619\vert 623\rangle$	3	G.16	(4231)	$32,(21)^3$	2^21	1	3	$\langle 2187,301\vert 305\rangle$

Theorem 5. *Let F be a cyclic cubic number field with elementary bicyclic 3-class group $\mathrm{Cl}_3F \simeq (3,3)$. Denote by $\mathfrak{M} = \mathrm{Gal}(F_3^{(2)}/F)$ the second 3-class group of F, and by $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ the 3-class field tower group of F. Then, the Artin pattern $(\tau,\varkappa)$ of F identifies the groups $\mathfrak{M}$ and G, and determines the length ℓ_3F of the 3-class field tower of F, according to the following* ***deterministic laws****.*

1. *If $\tau = [(1)^4]$, $\varkappa = (0000)$ (type a.1), then $G \simeq \langle 9,2\rangle$ and $\ell_3F = 1$.*
2. *If $\tau \sim [1^2,(2)^3]$, $\varkappa \sim (1111)$ (type A.1), then $G \simeq \langle 27,4\rangle$.*
3. *If $\tau \sim [1^3,(1^2)^3]$, $\varkappa \sim (2000)$ (type a.3), then $G \simeq \langle 81,7\rangle$.*
4. *If $\tau \sim [21,(1^2)^3]$, $\varkappa \sim (2000)$ (type a.3), then $G \simeq \langle 81,8\rangle$.*
5. *If $\tau \sim [21,(1^2)^3]$, $\varkappa \sim (1000)$ (type a.2), then $G \simeq \langle 81,10\rangle$.*

6. *If* $\tau \sim [2^2,(1^2)^3]$, $\varkappa \sim (2000)$ *(type* a.3*), then* $G \simeq \langle 243,25\rangle$.

7. *If* $\tau \sim [2^2,(1^2)^3]$, $\varkappa \sim (1000)$ *(type* a.2*), then* $G \simeq \langle 243,27\rangle$.

Except for the abelian tower in item 1, the tower is metabelian with $\ell_3 F = 2$.

Proof. Generally, a cyclic cubic field F has signature $(r_1,r_2)=(3,0)$, torsion free unit rank $r=r_1+r_2-1=2$, does not contain primitive third roots of unity, and thus possesses the maximal admissible relation rank $d_2 \le d_1+r=4$ for the group G, when its 3-class rank, i.e. the generator rank of G, is $d_1=\varrho=2$. Consequently, $\ell_3 F \ge 3$ in the case of $d_2\mathfrak{M} \ge 5$.

For item 1, we have $\mathfrak{M}=\mathrm{Gal}(F_3^{(2)}/F) \simeq \langle 9,2\rangle \simeq (3,3) \simeq \mathrm{Cl}_3 F \simeq \mathrm{Gal}(F_3^{(1)}/F)$, whence $\ell_3 F = 1$.

For item 2 to item 7, the group $\mathfrak{M}$ is of maximal class (coclass $\mathrm{cc}(\mathfrak{M})=1$), and thus coincides with G, whence $\ell_3 F = 2$.

In each case, the Artin pattern $(\tau,\varkappa)$ identifies $\mathfrak{M}=G$ uniquely, and the relation ranks are $d_2\langle 9,2\rangle=3$, $d_2\langle 27,4\rangle=2$, $d_2\langle 81,7\rangle=3$, $d_2\langle 81,8\rangle=3$, $d_2\langle 81,10\rangle=3$, $d_2\langle 243,25\rangle=3$, $d_2\langle 243,27\rangle=3$, each of them less than 4. □

Corollary 2. *Under the assumptions of Theorem 5, the abelian type invariants* τ_2 *of the* 3*-class group* $\mathrm{Cl}_3 F_3^{(1)}$ *of the first Hilbert* 3*-class field of* F *are required for the unambiguous identification of the groups* G *respectively* $\mathfrak{M}$.

If $\tau \sim [21,(1^2)^3]$, $\varkappa \sim (0000)$, a.1, *then* $G \simeq \begin{cases} \langle 81,9\rangle & \text{for } \tau_2=[1^2], \\ \langle 243,28\ldots30\rangle & \text{for } \tau_2 \sim [21]. \end{cases}$

If $\tau \sim [(21)^2,(1^3)^2]$, $\varkappa \sim (0043)$, b.10, *then* $\mathfrak{M} \simeq \begin{cases} \langle 729,34\ldots36\rangle & \text{for } \tau_2=[1^4], \\ \langle 729,37\ldots39\rangle & \text{for } \tau_2 \sim [21^2]. \end{cases}$

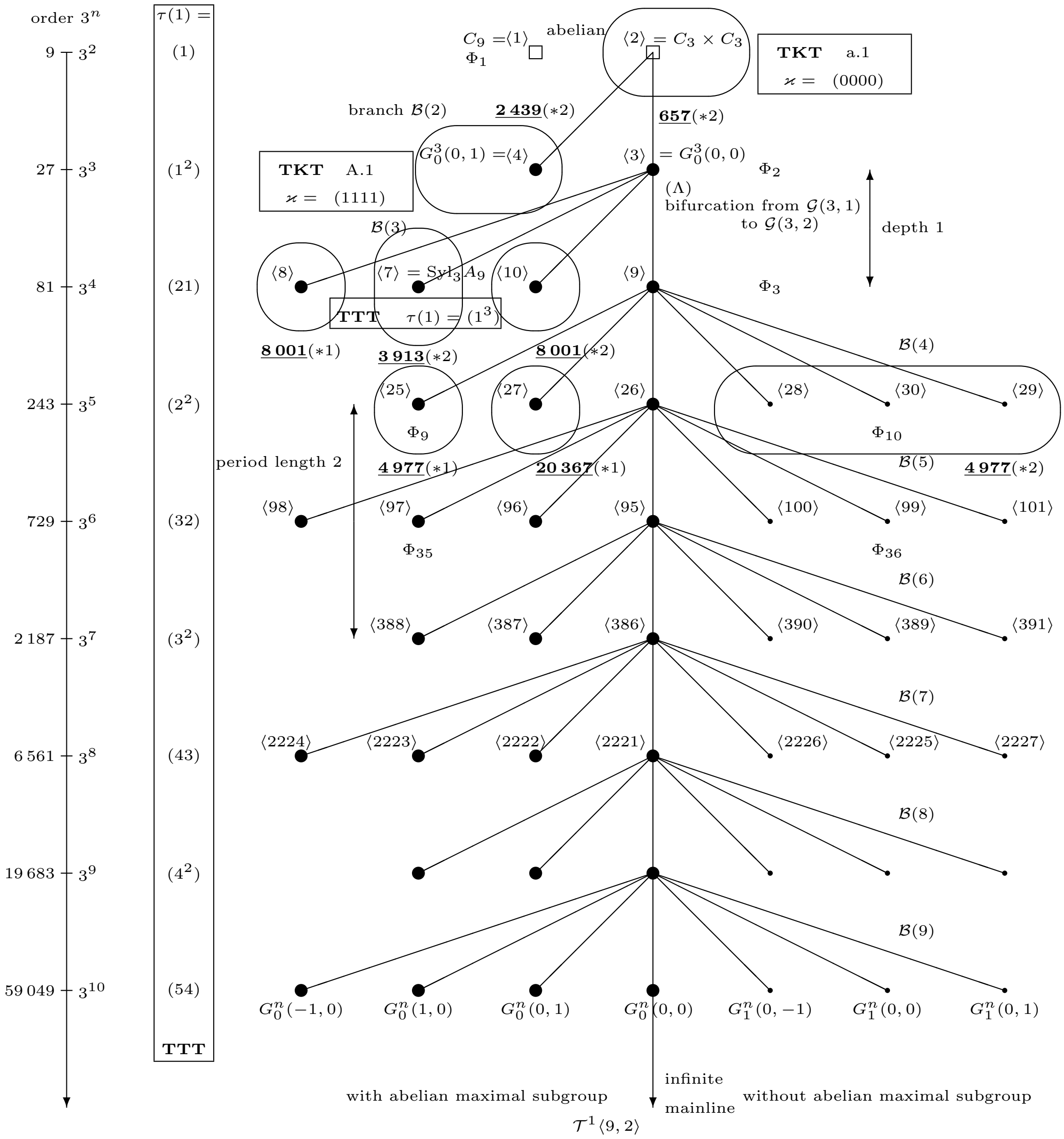
Proof. The Artin pattern $(\tau,\varkappa)$ of F alone is not able to identify the groups $\mathfrak{M}$ and G unambiguously. Ascione uses the notation $\langle 729,34\rangle=H$, $\langle 729,35\rangle=I$, $\langle 729,37\rangle=A$, $\langle 729,38\rangle=C$. □

Since many new groups will arise for $t=3$, we insert a section on p-group theory.

6.1.1. Descendant Trees of Finite 3-Groups

Basic definitions, facts, and notation concerning *descendant trees* of finite p-groups are summarized briefly in [28, § 2, pp. 410–411]. They are discussed thoroughly in the broadest detail in the initial sections of [29, 37]. Trees are crucial for recent progress in the theory of p-class field towers [33, 34, 35], in particular in order to describe the mutual location of $\mathrm{G}_3^{(2)}K$ and $\mathrm{G}_3^{(\infty)}K$ for number fields K.

Figure 6.1.: Distribution of Conductors for $\mathrm{G}_3^{(2)}F$ on the Coclass Tree $\mathcal{T}^1\langle 9,2\rangle$



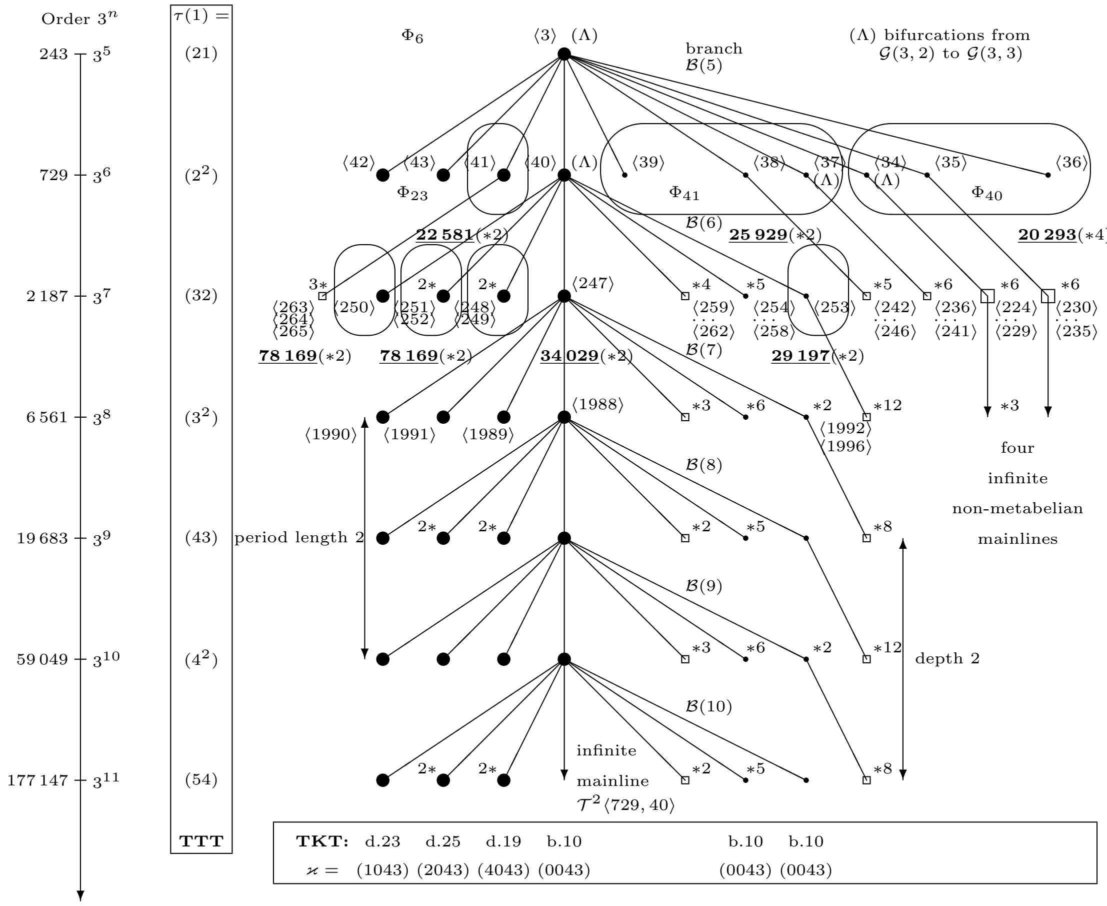
TKT	a.3	a.3	a.2	a.1	a.1	a.1	a.1
$\varkappa =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

Generally, the *vertices* of *coclass trees* in the Figures 6.1, 6.2 and 6.4 represent isomorphism classes of finite 3-groups. Two vertices are connected by a *directed edge* $G \to H$ if H is isomorphic to the last lower central quotient $G/\gamma_c(G)$, where $c = \mathrm{cl}(G)$ denotes the nilpotency class of G, and either $|G| = 3|H|$, that is, the last lower central $\gamma_c(G)$ is cyclic of order 3, or $|G| = 9|H|$, that is, $\gamma_c(G)$ is bicyclic of type $(3,3)$. See also [28, § 2.2, pp. 410–411] and [29, § 4, pp. 163–164].

Vertices of the tree diagrams in Figure 6.1 are classified with various symbols:

1. big full discs ● represent metabelian groups $\mathfrak{M}$ with *defect* $k(\mathfrak{M}) = 0$ [27],
2. small full discs • represent metabelian groups $\mathfrak{M}$ with defect $k(\mathfrak{M}) = 1$.

Figure 6.2.: Distribution of Conductors for $\mathrm{G}_3^{(2)}F$ on the Coclass Tree $\mathcal{T}^2\langle 729, 40\rangle$



In the Figures 6.2 and 6.4,

1. big full discs ● represent *metabelian* groups $\mathfrak{M}$ with bicyclic centre of type $(3,3)$ and defect $k(\mathfrak{M}) = 0$ [28, § 3.3.2, p. 429],
2. small full discs • represent metabelian groups $\mathfrak{M}$ with cyclic centre of order 3 and defect $k(\mathfrak{M}) = 1$,
3. small contour squares □ represent *non-metabelian* groups $\mathfrak{G}$.

A symbol $n*$ adjacent to a vertex denotes the multiplicity of a *batch* of n siblings, that is, immediate descendants sharing a common parent. The groups of particular importance are labelled by a number in angles, which is the *identifier* in the SmallGroups Library [7, 8] of GAP [18] and Magma [23]. We omit the orders, which are given on the left hand scale. The transfer kernel type (TKT) $\varkappa$ [27, Thm. 2.5, Tbl. 6–7], in the bottom rectangle concerns all vertices located vertically above. The first component $\tau(1)$ of the transfer target type (TTT) [30, Dfn. 3.3, p. 288] in the left rectangle concerns vertices G on the same horizontal level with defect $k(G) = 0$. The periodicity with length 2 of branches, $\mathcal{B}(j) \simeq \mathcal{B}(j+2)$ for $j \geq 4$, respectively $j \geq 7$, sets in with branch $\mathcal{B}(4)$, respectively $\mathcal{B}(7)$, having a root of order 3^4, respectively 3^7, in Figure 6.1, resp. Figures 6.2, 6.4.

6.2. Cyclic Cubic Quartets of Type (3,3)

6.2.1. Conductors with Three Prime Divisors

Now we come to conductors c with *three* prime divisors, $t = 3$. According to the multiplicity formula $m(c) = (3-1)^{t-1}$, there are 4 cyclic cubic fields $F_{c,1}, \ldots, F_{c,4}$ sharing the common conductor c. The members of a *quartet* $(F_{c,\mu})_{1\le\mu\le 4}$ do not necessarily have equal 3-class ranks $\varrho_3 F_{c,\mu}$, according to Theorem 1.

Definition 4. This forces us to introduce *three categories* of quartets $(F_{c,\mu})_{1\le\mu\le 4}$.
Category I:
one member $F_{c,1}$ has 3-class rank 3, the remaining three $F_{c,2}, \ldots, F_{c,4}$ have rank 2.
Category II:
two members $F_{c,1}, F_{c,2}$ have 3-class rank 3, the other two $F_{c,3}, F_{c,4}$ have rank 2.
Category III: all four members $F_{c,1}, \ldots, F_{c,4}$ have 3-class rank 2.
Additionally to these categories, defined in Ayadi's Thesis [3], we introduce *two further categories.*
Category IV: all four members $F_{c,1}, \ldots, F_{c,4}$ have 3-class rank 3.
Category V: all four members $F_{c,1}, \ldots, F_{c,4}$ have 3-class rank 4.

However, there arises an additional complication: In general, the members of rank 2 neither have the same 3-capitulation type $\varkappa(F_{c,\mu})$ nor the same 3-class tower group $\mathrm{G}_3^{(\infty)} F_{c,\mu}$. Therefore we occasionally use formal powers with exponents denoting iteration. There is still one classical result, Theorem 6, for $t = 3$.

6.2.2. Graphs 1,2,3,4 of Category III

Rational criteria for $\mathrm{G}_3^{(\infty)} F_{c,\mu}$ to be abelian are given in the following theorem which refers to the first four subcases of Formula (2.1.4).

Theorem 6. *(Cubic residue criterion,* ***Ayadi, 1995****, [3, 4].)*
Let c be a conductor divisible by exactly three primes, $t = 3$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for all four cyclic cubic fields $F_{c,\mu}$, $1 \le \mu \le 4$, with conductor c. If there are no mutual cubic residues among the prime divisors of c, that is, if $c = q_1 q_2 q_3$ belongs to the Graphs 1, 2, 3, 4 *of Category* III*, i.e.,*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta \ne 0\} & \text{or} \\ \{q_i \to q_j; q_k\} & \text{or} \\ \{q_i \to q_j \to q_k\} & \text{or} \\ \{q_i \to q_j \to q_k \to q_i\} & \end{cases} \tag{6.2.1}$$

with i, j, k pairwise distinct, then the second 3-class group $\mathrm{G}_3^{(2)} F_{c,\mu}$ of all four fields $F_{c,\mu}$ is isomorphic to the abelian group $\langle 9, 2\rangle \simeq (3,3)$ with capitulation type a.1,

$\varkappa(F_{c,\mu}) = (0000)$, and transfer target type $\tau(F_{c,\mu}) = [1,1,1,1]$, the 3-class tower has length $\ell_3 F_{c,\mu} = 1$, and the 3-class groups of the 13 bicyclic bicubic subfields S_i of the 3-genus field F^* are given by the logarithmic abelian type invariants (ATI)

$$[\mathrm{Cl}_3 S_i]_{1\le i\le 13} = [(0)^3; (1)^{10}]. \tag{6.2.2}$$

Conjecture 1. We conjecture that the converse of Theorem 6 is also true, i.e., that $\mathrm{G}_3^{(2)}F$ is never abelian for conductors c in the Categories I and II, and in the remaining Graphs $5,6,7,8,9$ of Category III.

Example 2. The decisive advantage of the criteria in Theorem 6 is that they are *rational*, i.e., expressible in terms of the prime divisors of the conductor. In Table 6.3, the first occurrence of each of the Graphs $1,2,3,4$ of Category III is presented. Again, we give the smallest examples for both, $9 \mid c$ and $\gcd(c,3) = 1$, separately, indicated by the 3-valuation $v_3 c$ of $c = q_1 q_2 q_3$.

Table 6.3.: The First Examples for the Graphs $1,2,3,4$ of Category III

Graph	$v_3 c$	c	$[q_1,q_2,q_3]_3$	$\mathrm{G}_3^{(2)}F_{c,\mu}$
1	2	1 953	$\{9,7,31;\ \delta\ne 0\}$	$\langle 9,2\rangle^4$
1	0	14 539	$\{7,31,67;\ \delta\ne 0\}$	$\langle 9,2\rangle^4$
2	2	819	$\{13\to 7; 9\}$	$\langle 9,2\rangle^4$
2	0	3 367	$\{13\to 7; 37\}$	$\langle 9,2\rangle^4$
3	2	1 197	$\{7\to 19\to 9\}$	$\langle 9,2\rangle^4$
3	0	1 729	$\{13\to 7\to 19\}$	$\langle 9,2\rangle^4$
4	0	6 643	$\{13\to 7\to 73\to 13\}$	$\langle 9,2\rangle^4$
4	2	17 613	$\{19\to 9\to 103\to 19\}$	$\langle 9,2\rangle^4$

6.2.3. Graphs 5,6,7,8,9 of Category III

Now we begin with interesting **new quartets** of cyclic cubic fields F, having $t = 3$ and $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for $1 \le \mu \le 4$, where the second 3-class groups $\mathrm{G}_3^{(2)}F_{c,\mu}$ were completely **unknown** up to now. According to Theorems 1 and 6, there must occur mutual cubic residues among the prime divisors q_1,q_2,q_3 of c, say q_1,q_2, and thus it is also interesting to consider the coinciding group $\mathrm{G}_3^{(2)}F_{f,\nu}$ of the two members $F_{f,\nu}$, $1 \le \nu \le 2$, of the doublet with *partial conductor* $f = q_1 \cdot q_2$, $f \mid c$, as given by Theorem 4, and listed in the tables of
$(*)$ `http://www.algebra.at/ResearchFrontier2013ThreeByThree.htm`.

Within the frame of our computations, conductors c belonging to the Graphs 6, 7 and 9 of Category III reveal a completely *uniform behaviour* of the associated second 3-class groups $\mathrm{G}_3^{(2)}F_{c,\mu}$ of the four members $F_{c,\mu}$, $1 \le \mu \le 4$, of the quartet with conductor $c = q_1 \cdot q_2 \cdot q_3$, provided that $\mathrm{G}_3^{(2)}F_{f,\nu} \simeq \langle 9,2\rangle$ is *abelian*, black color in (∗), for the two members $F_{f,\nu}$, $1 \le \nu \le 2$, of the doublet with conductor $f = q_1 \cdot q_2$. Therefore, we can summarize their properties in Theorem 7. However, exceptions arise if either $\mathrm{G}_3^{(2)}F_{f,\nu} \simeq \langle 27,4\rangle$ is *extra-special*, red color in (∗), or $\mathrm{Cl}_3F_{f,\nu} \simeq (9,3)$ is *non-elementary*, listed in the tables of
(∗∗) `http://www.algebra.at/ResearchFrontier2013NineByThree.htm`.

Exceptions are indicated by **boldface** font in the Tables 6.4, 6.5, 6.6.

Table 6.4.: Thirty-One Examples for Graph 6 of Category III

No.	c	$[q_1,q_2,q_3]_3$	$\mathrm{G}_3^{(2)}F_{c,\mu}$	f	$[q_1,q_2]_3$	$\mathrm{G}_3^{(2)}F_{f,\nu}$	$[\mathrm{Cl}_3S_i]_{1\le i\le 13}$
1	8 541	$\{9 \leftrightarrow 73 \to 13\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
2	9 373	$\{103 \leftrightarrow 13 \to 7\}$	$\langle 81,7\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
3	11 403	$\{7 \leftrightarrow 181 \to 9\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
4	19 341	$\{9 \leftrightarrow 307 \to 7\}$	$\langle 81,7\rangle^4$	2 763	$\{9 \leftrightarrow 307\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
5	20 839	$\{229 \leftrightarrow 13 \to 7\}$	$\langle 81,7\rangle^4$	2 977	$\{13 \leftrightarrow 229\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
6	25 441	$\{13 \leftrightarrow 103 \to 19\}$	$\langle 81,7\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
7	29 659	$\{223 \leftrightarrow 7 \to 19\}$	$\langle 81,7\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
8	35 919	$\{9 \leftrightarrow 307 \to 13\}$	$\langle 81,7\rangle^4$	2 763	$\{9 \leftrightarrow 307\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
9	37 297	$\{19 \leftrightarrow 151 \to 13\}$	$\langle 81,7\rangle^4$	2 869	$\{19 \leftrightarrow 151\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
10	40 077	$\{73 \leftrightarrow 9 \to 61\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
11	44 019	$\{73 \leftrightarrow 9 \to 67\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
12	44 821	$\{337 \leftrightarrow 7 \to 19\}$	$\langle 81,7\rangle^4$	2 359	$\{7 \leftrightarrow 337\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
13	45 409	$\{499 \leftrightarrow 13 \to 7\}$	$\langle 81,7\rangle^4$	6 487	$\{13 \leftrightarrow 499\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
14	45 477	$\{31 \leftrightarrow 163 \to 9\}$	$\langle 81,7\rangle^4$	5 053	$\{31 \leftrightarrow 169\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
15	47 367	$\{277 \leftrightarrow 19 \to 9\}$	$\langle 81,7\rangle^4$	5 263	$\{19 \leftrightarrow 277\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
16	47 691	$\{9 \leftrightarrow 757 \to 7\}$	$\langle 81,7\rangle^4$	6 813	$\{9 \leftrightarrow 757\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
17	51 093	$\{7 \leftrightarrow 811 \to 9\}$	$\langle 81,7\rangle^4$	5 677	$\{7 \leftrightarrow 811\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
18	53 641	$\{79 \leftrightarrow 97 \to 7\}$	$\langle 81,7\rangle^4$	7 663	$\{79 \leftrightarrow 97\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
19	55 629	$\{7 \leftrightarrow 883 \to 9\}$	$\langle 81,7\rangle^4$	6 181	$\{7 \leftrightarrow 883\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
20	56 329	$\{619 \leftrightarrow 13 \to 7\}$	$\langle 81,7\rangle^4$	8 047	$\{13 \leftrightarrow 619\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
21	56 563	$\{13 \leftrightarrow 229 \to 19\}$	$\langle 81,7\rangle^4$	2 977	$\{13 \leftrightarrow 229\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
22	57 757	$\{7 \leftrightarrow 223 \to 37\}$	$\langle 81,7\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
23	58 093	$\{193 \leftrightarrow 43 \to 7\}$	$\langle 81,7\rangle^4$	8 299	$\{43 \leftrightarrow 193\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
24	63 567	$\{9 \leftrightarrow 1009 \to 7\}$	$\langle 81,7\rangle^4$	9 081	$\{9 \leftrightarrow 1009\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
25	63 783	$\{373 \leftrightarrow 19 \to 9\}$	$\langle 81,7\rangle^4$	7 087	$\{19 \leftrightarrow 373\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
26	67 509	$\{9 \leftrightarrow 577 \to 13\}$	$\langle 81,7\rangle^4$	5 193	$\{9 \leftrightarrow 577\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
27	73 129	$\{7 \leftrightarrow 337 \to 31\}$	$\langle 81,7\rangle^4$	2 359	$\{7 \leftrightarrow 337\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
28	75 733	$\{31 \leftrightarrow 349 \to 7\}$	$\langle 81,7\rangle^4$	10 819	$\{31 \leftrightarrow 349\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
29	**78 169**	$\{859 \leftrightarrow 13 \to 7\}$	$\langle 3^7,250\rangle^2,\langle 3^7,251\rangle^2$	**11 167**	$\{13 \leftrightarrow 859\}$	$\langle 81,3\rangle^2$	$[(0)^2,1^3;(21)^3,(32)^3,(1^3)^4]$
30	92 491	$\{881 \leftrightarrow 7 \to 73\}$	$\langle 81,7\rangle^4$	6 181	$\{7 \leftrightarrow 883\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$
31	99 619	$\{97 \leftrightarrow 79 \to 13\}$	$\langle 81,7\rangle^4$	7 663	$\{79 \leftrightarrow 97\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^8,(1^3)^2]$

Table 6.4 shows all 31 hits of quartets belonging to Graph 6 in Category III, as mentioned among the computational results for $c < 10^5$ in Table 2.1. There is only one irregular case with conductor $c = 78\,169$ and *singular* partial conductor $f = 11\,167$, where $\mathrm{Cl}_3F_{f,\nu} \simeq (9,3)$ and $\mathrm{Cl}_3F_0^* \simeq (3,3,3)$, i.e. $v = 3$. Three cases with $\mathrm{G}_3^{(2)}F_{f,\nu} \simeq \langle 27,4\rangle$ fit in unceremoniously without causing irregular behavior, except that $\mathrm{Cl}_3F_0^* \simeq (3,3)$, i.e. $v = 2$.

Table 6.5 shows all 34 occurrences of quartets belonging to Graph 7 in Category III, as mentioned in Table 2.1. There are two irregular cases with conductors $c = 69\,979$, $86\,821$ and *super-singular* partial conductors $f = 5\,383$, $12\,403$, where $\mathrm{Cl}_3 F_{f,\nu} \simeq (9,3)$ and $\mathrm{Cl}_3 F_0^* \simeq (9,3,3)$, i.e. $v = 4$. There is one irregular case with conductor $c = 61\,243$ and *singular* partial conductor $f = 4\,711$, where $\mathrm{Cl}_3 F_{f,\nu} \simeq (9,3)$ and $\mathrm{Cl}_3 F_0^* \simeq (3,3,3)$, i.e. $v = 3$. Seven cases with $\mathrm{G}_3^{(2)} F_{f,\nu} \simeq \langle 27,4\rangle$ fit in unspectacularly without causing irregular behavior, except that $\mathrm{Cl}_3 F_0^* \simeq (3,3)$, i.e. $v = 2$. However, the behavior is not uniform, since four conductors $c = 76\,741$, $89\,433$, $90\,243$, $99\,801$ with $\mathrm{G}_3^{(2)} F_{f,\nu} \simeq \langle 27,4\rangle$ cause exceptions.

Table 6.5.: Thirty-Four Examples for Graph 7 of Category III

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	f	$[q_1, q_2]_3$	$\mathrm{G}_3^{(2)} F_{f,\nu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
1	4 599	$\{9 \leftrightarrow 73 \leftarrow 7\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
2	12 051	$\{13 \leftrightarrow 103 \leftarrow 9\}$	$\langle 81,7\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
3	12 483	$\{73 \leftrightarrow 9 \leftarrow 19\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
4	20 083	$\{151 \leftrightarrow 19 \leftarrow 7\}$	$\langle 81,7\rangle^4$	2 869	$\{19 \leftrightarrow 151\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
5	28 251	$\{9 \leftrightarrow 73 \leftarrow 43\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
6	31 707	$\{9 \leftrightarrow 271 \leftarrow 13\}$	$\langle 81,7\rangle^4$	2 439	$\{9 \leftrightarrow 271\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
7	36 841	$\{277 \leftrightarrow 19 \leftarrow 7\}$	$\langle 81,7\rangle^4$	5 263	$\{19 \leftrightarrow 277\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
8	39 277	$\{7 \leftrightarrow 181 \leftarrow 31\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
9	52 633	$\{103 \leftrightarrow 73 \leftarrow 7\}$	$\langle 81,7\rangle^4$	7 519	$\{73 \leftrightarrow 103\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
10	53 739	$\{7 \leftrightarrow 853 \leftarrow 9\}$	$\langle 81,7\rangle^4$	5 971	$\{7 \leftrightarrow 853\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
11	54 481	$\{181 \leftrightarrow 7 \leftarrow 43\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
12	58 383	$\{13 \leftrightarrow 499 \leftarrow 9\}$	$\langle 81,7\rangle^4$	6 487	$\{13 \leftrightarrow 499\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
13	**61 243**	$\{673 \leftrightarrow 7 \leftarrow 13\}$	$\langle 3^7,253\rangle^4$	**4 711**	$\{7 \leftrightarrow 673\}$	$\langle 81,3\rangle^2$	$[(0)^2, 1^3; (21)^3, (2^2)^3, (1^3)^4]$
14	63 729	$\{9 \leftrightarrow 73 \leftarrow 97\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
15	64 323	$\{7 \leftrightarrow 1021 \leftarrow 9\}$	$\langle 81,7\rangle^4$	7 147	$\{7 \leftrightarrow 1021\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
16	68 419	$\{19 \leftrightarrow 277 \leftarrow 13\}$	$\langle 81,7\rangle^4$	5 263	$\{19 \leftrightarrow 277\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
17	**69 979**	$\{769 \leftrightarrow 7 \leftarrow 13\}$	$\langle 3^6,41\rangle^4$	**5 383**	$\{7 \leftrightarrow 769\}$	$\langle 243,14\rangle^2$	$[(0)^2, 21^2; (2^2)^3, (32)^3, (1^3)^4]$
18	71 613	$\{73 \leftrightarrow 9 \leftarrow 109\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
19	72 423	$\{13 \leftrightarrow 619 \leftarrow 9\}$	$\langle 81,7\rangle^4$	8 047	$\{13 \leftrightarrow 619\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
20	74 691	$\{43 \leftrightarrow 193 \leftarrow 9\}$	$\langle 81,7\rangle^4$	8 299	$\{43 \leftrightarrow 193\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
21	**76 741**	$\{577 \leftrightarrow 19 \leftarrow 7\}$	$\langle 3^7,65\vert 67\rangle^4$	10 963	$\{19 \leftrightarrow 577\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (2^2)^6, (1^3)^4]$
22	80 941	$\{31 \leftrightarrow 373 \leftarrow 7\}$	$\langle 81,7\rangle^4$	11 563	$\{31 \leftrightarrow 373\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
23	81 679	$\{13 \leftrightarrow 103 \leftarrow 61\}$	$\langle 81,7\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
24	84 889	$\{7 \leftrightarrow 181 \leftarrow 67\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
25	**86 821**	$\{79 \leftrightarrow 157 \leftarrow 7\}$	$\langle 3^6,37\rangle^4$	**12 403**	$\{79 \leftrightarrow 157\}$	$\langle 243,14\rangle^2$	$[(0)^2, 21^2; (21)^6, (1^3)^4]$
26	**89 433**	$\{523 \leftrightarrow 9 \leftarrow 19\}$	$\langle 3^7,65\vert 67\rangle^4$	4 707	$\{9 \leftrightarrow 523\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (2^2)^6, (1^3)^4]$
27	**90 243**	$\{271 \leftrightarrow 9 \leftarrow 37\}$	$\langle 3^6,41\rangle^4$	2 439	$\{9 \leftrightarrow 271\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (21)^3, (2^2)^3, (1^3)^4]$
28	91 323	$\{9 \leftrightarrow 73 \leftarrow 139\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
29	91 903	$\{691 \leftrightarrow 19 \leftarrow 7\}$	$\langle 81,7\rangle^4$	13 129	$\{19 \leftrightarrow 691\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
30	92 131	$\{19 \leftrightarrow 373 \leftarrow 13\}$	$\langle 81,7\rangle^4$	7 087	$\{19 \leftrightarrow 373\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
31	92 287	$\{229 \leftrightarrow 13 \leftarrow 31\}$	$\langle 81,7\rangle^4$	2 977	$\{13 \leftrightarrow 229\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
32	96 453	$\{9 \leftrightarrow 1531 \leftarrow 7\}$	$\langle 81,7\rangle^4$	13 779	$\{9 \leftrightarrow 1531\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (1^2)^8, (1^3)^2]$
33	97 489	$\{733 \leftrightarrow 19 \leftarrow 7\}$	$\langle 81,7\rangle^4$	13 927	$\{19 \leftrightarrow 733\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
34	**99 801**	$\{13 \leftrightarrow 853 \leftarrow 9\}$	$\langle 3^6,41\rangle^4$	11 089	$\{13 \leftrightarrow 853\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (21)^3, (2^2)^3, (1^3)^4]$

Table 6.6.: Fifteen Examples for Graph 9 of Category III

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	f	$[q_1, q_2]_3$	$\mathrm{G}_3^{(2)} F_{f,\nu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
1	16 471	$\{13 \to 7 \leftrightarrow 181 \to 13\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
2	24 073	$\{19 \to 181 \leftrightarrow 7 \to 19\}$	$\langle 81,7\rangle^4$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
3	24 309	$\{37 \to 9 \leftrightarrow 73 \to 37\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
4	25 821	$\{9 \to 151 \leftrightarrow 19 \to 9\}$	$\langle 81,7\rangle^4$	2 869	$\{19 \leftrightarrow 151\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
5	30 667	$\{13 \to 7 \leftrightarrow 337 \to 13\}$	$\langle 81,7\rangle^4$	2 359	$\{7 \leftrightarrow 337\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
6	34 299	$\{9 \to 103 \leftrightarrow 37 \to 9\}$	$\langle 81,7\rangle^4$	3 811	$\{37 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
7	42 133	$\{13 \to 7 \leftrightarrow 463 \to 13\}$	$\langle 81,7\rangle^4$	3 241	$\{7 \leftrightarrow 463\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
8	55 993	$\{19 \to 421 \leftrightarrow 7 \to 19\}$	$\langle 81,7\rangle^4$	2 947	$\{7 \leftrightarrow 421\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
9	65 689	$\{13 \to 163 \leftrightarrow 31 \to 13\}$	$\langle 81,7\rangle^4$	5 053	$\{31 \leftrightarrow 163\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
10	67 123	$\{43 \to 7 \leftrightarrow 223 \to 43\}$	$\langle 81,7\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
11	73 801	$\{13 \to 7 \leftrightarrow 811 \to 13\}$	$\langle 81,7\rangle^4$	5 677	$\{7 \leftrightarrow 811\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
12	80 353	$\{13 \to 7 \leftrightarrow 883 \to 13\}$	$\langle 81,7\rangle^4$	6 181	$\{7 \leftrightarrow 883\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
13	83 439	$\{127 \to 9 \leftrightarrow 73 \to 127\}$	$\langle 81,7\rangle^4$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
14	88 939	$\{31 \to 19 \leftrightarrow 151 \to 31\}$	$\langle 81,7\rangle^4$	2 869	$\{19 \leftrightarrow 151\}$	$\langle 9,2\rangle^2$	$[(0)^2, 1; (1^2)^8, (1^3)^2]$
15	**89 487**	$\{9 \to 61 \leftrightarrow 163 \to 9\}$	$\langle 3^6,41\rangle^4$	9 943	$\{61 \leftrightarrow 163\}$	$\langle 27,4\rangle^2$	$[(0)^2, 1^2; (21)^3, (2^2)^3, (1^3)^4]$

Theorem 7. *Suppose that $u := 10^5$ is an assigned upper bound. Let $c < u$ be a conductor divisible by exactly three primes, $t = 3$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for all four cyclic cubic fields $F_{c,\mu}$, $1 \le \mu \le 4$, with conductor c. If $c = q_1 q_2 q_3$ belongs to the Graphs $6, 7, 9$ of Category III, i.e.,*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \leftrightarrow q_j \to q_k\} & \text{or} \\ \{q_i \leftrightarrow q_j \leftarrow q_k\} & \text{or} \\ \{q_k \to q_i \leftrightarrow q_j \to q_k\} \end{cases} \tag{6.2.3}$$

with i, j, k pairwise distinct, and the 3-valuation v of the class number h of the 3-genus field F_0^* of the cyclic cubic fields $F_{f,\nu}$, $1 \le \nu \le 2$, with conductor $f = q_i q_j$ is $v = 1$, that is $\mathrm{G}_3^{(2)} F_{f,\nu} \simeq \langle 9, 2\rangle$, then the second 3-class group $\mathrm{G}_3^{(2)} F_{c,\mu}$ of all four fields $F_{c,\mu}$ is isomorphic to the Sylow 3-subgroup of the symmetric group S_9 of degree 9, i.e. the wreath product of two cyclic groups C_3, $\langle 81, 7\rangle \simeq \mathrm{Syl}_3 S_9 \simeq C_3 \wr C_3$, with 3-capitulation type a.3, $\varkappa(F_{c,\mu}) = (2000)$, and transfer target type $\tau(F_{c,\mu}) = [111, (11)^3]$, the 3-class tower has length $\ell_3 F_{c,\mu} = 2$, and the 3-class groups of the 13 bicyclic bicubic subfields S_i of the 3-genus field F^* are given by

$$[\mathrm{Cl}_3 S_i]_{1\le i\le 13} = [(0)^2, 1; (11)^8, (111)^2]. \tag{6.2.4}$$

If $[q_1, q_2, q_3]_3 = \{q_i \leftrightarrow q_j \to q_k\}$, respectively $\{q_k \to q_i \leftrightarrow q_j \to q_k\}$, belongs to Graph 6, respectively 9, of Category III, then the principal factors of all four fields are equal to the prime divisor $B(F_{c,\mu}) = q_j$, $1 \le \mu \le 4$, which is cubic residue for both other prime divisors q_i, q_k and the result remains valid also for $v = 2$, that is

$\mathrm{G}_3^{(2)} F_{f,\nu} \simeq \langle 27,4\rangle$. *If* $[q_1,q_2,q_3]_3 = \{q_i \leftrightarrow q_j \leftarrow q_k\}$ *belongs to Graph* 7 *of Category* III, *then the principal factors of two fields are* $B(F_{c,\mu}) = q_i q_k$, $1 \le \mu \le 2$, *and* $B(F_{c,\mu}) = q_i^2 q_k$, $3 \le \mu \le 4$, *for the other two fields (up to the order).*

Proof. See the Tables 6.4, 6.5, 6.6, which have been computed with the aid of Magma [10, 11, 16, 23]. □

Conjecture 2. Theorem 7 remains true for any upper bound $u > 10^5$.

Remark 6. The upper bound for the conductors in our 2022 computation of cyclic cubic fields F with $t \in \{2,3\}$ was $c < u = 10^5$. In Theorem 7, the 3-tower group $G = \langle 81,7\rangle$ turned out to be the first new possibility beyond the results of Ayadi [3, 4], which were restricted to the groups $\langle 9,2\rangle$ and $\langle 27,4\rangle$. It is still a 3-group of coclass $\mathrm{cc}(G) = 1$, associated with a metabelian 3-class tower of length $\ell_3 F_{c,\mu} = 2$.

Table 6.7.: Thirty-Seven Examples for Graph 5 of Category III

No.	c	$[q_1,q_2,q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	f	$[q_1,q_2]_3$	$\mathrm{G}_3^{(2)} F_{f,\nu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
1	14 049	$\{7 \leftrightarrow 223; 9\}$	$\langle 3^5,28\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
2	17 073	$\{9 \leftrightarrow 271; 7\}$	$\langle 81,7\rangle^4$	2 439	$\{9 \leftrightarrow 271\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
3	20 367	$\{9 \leftrightarrow 73; 31\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
4	21 231	$\{7 \leftrightarrow 337; 9\}$	$\langle 3^5,25\rangle^2,\langle 3^5,27\rangle^2$	2 359	$\{7 \leftrightarrow 337\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(2^2)^3]$
5	26 523	$\{7 \leftrightarrow 421; 9\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	2 947	$\{7 \leftrightarrow 421\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
6	26 677	$\{37 \leftrightarrow 103; 7\}$	$\langle 3^5,28\rangle^4$	3 811	$\{37 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
7	26 793	$\{13 \leftrightarrow 229; 9\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	2 977	$\{13 \leftrightarrow 229\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
8	29 169	$\{7 \leftrightarrow 463; 9\}$	$\langle 3^5,28\rangle^4$	3 241	$\{7 \leftrightarrow 463\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
9	32 949	$\{9 \leftrightarrow 523; 7\}$	$\langle 81,7\rangle^4$	4 707	$\{9 \leftrightarrow 523\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
10	35 371	$\{31 \leftrightarrow 163; 7\}$	$\langle 3^5,28\rangle^4$	5 053	$\{31 \leftrightarrow 163\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
11	36 351	$\{9 \leftrightarrow 577; 7\}$	$\langle 3^5,25\rangle^2,\langle 3^5,27\rangle^2$	5 193	$\{9 \leftrightarrow 577\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(2^2)^3]$
12	38 619	$\{9 \leftrightarrow 613; 7\}$	$\langle 3^5,28\rangle^4$	5 517	$\{9 \leftrightarrow 613\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
13	**42 399**	$\{7 \leftrightarrow 673; 9\}$	$\langle 3^6,37\rangle^4$	**4 711**	$\{7 \leftrightarrow 673\}$	$\langle 81,3\rangle^2$	$[(0)^2,1^3;(21)^6,(1^3)^4]$
14	46 879	$\{7 \leftrightarrow 181; 37\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
15	48 391	$\{7 \leftrightarrow 223; 31\}$	$\langle 3^5,28\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
16	**48 447**	$\{7 \leftrightarrow 769; 9\}$	$\langle 3^6,37\rangle^4$	**5 383**	$\{7 \leftrightarrow 769\}$	$\langle 243,14\rangle^2$	$[(0)^2,21^2;(21)^6,(1^3)^4]$
17	49 257	$\{13 \leftrightarrow 421; 9\}$	$\langle 3^5,28\rangle^4$	5 473	$\{13 \leftrightarrow 421\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
18	51 903	$\{9 \leftrightarrow 73; 79\}$	$\langle 3^5,25\rangle^2,\langle 3^5,28\rangle^2$	657	$\{9 \leftrightarrow 73\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
19	57 577	$\{13 \leftrightarrow 103; 43\}$	$\langle 3^5,25\rangle^2,\langle 3^5,28\rangle^2$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
20	57 897	$\{9 \leftrightarrow 919; 7\}$	$\langle 81,7\rangle^4$	8 271	$\{9 \leftrightarrow 919\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
21	58 807	$\{31 \leftrightarrow 271; 7\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	8 401	$\{31 \leftrightarrow 271\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
22	61 191	$\{9 \leftrightarrow 523; 13\}$	$\langle 81,7\rangle^4$	4 707	$\{9 \leftrightarrow 523\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
23	62 433	$\{9 \leftrightarrow 991; 7\}$	$\langle 81,7\rangle^4$	8 919	$\{9 \leftrightarrow 991\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
24	68 967	$\{79 \leftrightarrow 97; 9\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	7 663	$\{79 \leftrightarrow 97\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
25	69 601	$\{61 \leftrightarrow 163; 7\}$	$\langle 81,7\rangle^4$	9 943	$\{61 \leftrightarrow 163\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
26	70 371	$\{9 \leftrightarrow 1117; 7\}$	$\langle 81,7\rangle^4$	10 053	$\{9 \leftrightarrow 1117\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$
27	71 721	$\{9 \leftrightarrow 613; 13\}$	$\langle 3^5,28\rangle^4$	5 517	$\{9 \leftrightarrow 613\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
28	77 287	$\{7 \leftrightarrow 181; 61\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	1 267	$\{7 \leftrightarrow 181\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
29	85 653	$\{9 \leftrightarrow 307; 31\}$	$\langle 3^5,28\rangle^4$	2 763	$\{9 \leftrightarrow 307\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
30	87 283	$\{7 \leftrightarrow 337; 37\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	2 359	$\{7 \leftrightarrow 337\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
31	88 569	$\{9 \leftrightarrow 757; 13\}$	$\langle 3^5,28\rangle^4$	6 813	$\{9 \leftrightarrow 757\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
32	89 713	$\{13 \leftrightarrow 103; 67\}$	$\langle 3^5,28\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
33	90 517	$\{67 \leftrightarrow 193; 7\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	12 931	$\{67 \leftrightarrow 193\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
34	91 357	$\{7 \leftrightarrow 421; 31\}$	$\langle 3^5,25\rangle^2,\langle 3^5,27\rangle^2$	2 947	$\{7 \leftrightarrow 421\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(2^2)^3]$
35	95 221	$\{7 \leftrightarrow 223; 61\}$	$\langle 3^5,28\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^3]$
36	97 371	$\{31 \leftrightarrow 349; 9\}$	$\langle 3^5,27\rangle,\langle 3^5,28\rangle^3$	10 819	$\{31 \leftrightarrow 349\}$	$\langle 9,2\rangle^2$	$[(0)^2,1;(1^2)^7,(21)^2,2^2]$
37	97 587	$\{9 \leftrightarrow 1549; 7\}$	$\langle 81,7\rangle^4$	13 941	$\{9 \leftrightarrow 1549\}$	$\langle 27,4\rangle^2$	$[(0)^2,1^2;(1^2)^8,(1^3)^2]$

Unfortunately, the behaviour of Graph 5 and the rare Graph 8 in Category III is not uniform and totally different from the Graphs 6, 7, 9, as the Tables 6.7 and 6.8 show. At least for Table 6.7, the 3-tower groups G for $v \le 2$ are still of coclass $\mathrm{cc}(G) = 1$, associated with metabelian 3-class towers of length $\ell_3 F_{c,\mu} = 2$. The possible groups $\mathfrak{M}$ for **boldface** conductors in the Tables 6.4, 6.5, 6.6, 6.7, 6.8 are the first occurrences of coclass $\mathrm{cc}(\mathfrak{M}) \ge 2$, usually with unknown length $\ell_3 F_{c,\mu}$ of the 3-class field tower.

Table 6.8.: Seven Examples for Graph 8 of Category III

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	f	$[q_1, q_2]_3$	$\mathrm{G}_3^{(2)} F_{f,\nu}$	$[\mathrm{Cl}_3 S_i]_{1 \le i \le 13}$
1	**20 293**	$\{13 \to 7 \leftrightarrow 223 \leftarrow 13\}$	$\langle 3^6, 34\rangle^4$	1 561	$\{7 \leftrightarrow 223\}$	$\langle 9, 2\rangle^2$	$[(0)^2, 1; (21)^6, (1^3)^4]$
2	**41 509**	$\{31 \to 13 \leftrightarrow 103 \leftarrow 31\}$	$\langle 3^6, 34\rangle^4$	1 339	$\{13 \leftrightarrow 103\}$	$\langle 9, 2\rangle^2$	$[(0)^2, 1; (21)^6, (1^3)^4]$
3	**46 341**	$\{19 \to 9 \leftrightarrow 271 \leftarrow 19\}$	$\langle 3^6, 34\rangle^4$	2 439	$\{9 \leftrightarrow 271\}$	$\langle 27, 4\rangle^2$	$[(0)^2, 1^2; (21)^6, (1^3)^4]$
4	**49 609**	$\{7 \to 19 \leftrightarrow 373 \leftarrow 7\}$	$\langle 3^6, 34\rangle^4$	7 087	$\{19 \leftrightarrow 373\}$	$\langle 27, 4\rangle^2$	$[(0)^2, 1^2; (21)^6, (1^3)^4]$
5	**52 497**	$\{19 \to 9 \leftrightarrow 307 \leftarrow 19\}$	$\langle 3^7, 66 \vert 73\rangle^4$	2 763	$\{9 \leftrightarrow 307\}$	$\langle 9, 2\rangle^2$	$[(0)^2, 1; (2^2)^6, (1^3)^4]$
6	**64 771**	$\{7 \to 19 \leftrightarrow 487 \leftarrow 7\}$	$\langle 3^6, 34\rangle^4$	9 253	$\{19 \leftrightarrow 487\}$	$\langle 27, 4\rangle^2$	$[(0)^2, 1^2; (21)^6, (1^3)^4]$
7	**92 911**	$\{13 \to 7 \leftrightarrow 1021 \leftarrow 13\}$	$\langle 3^7, 66 \vert 73\rangle^4$	7 147	$\{7 \leftrightarrow 1021\}$	$\langle 9, 2\rangle^2$	$[(0)^2, 1; (2^2)^6, (1^3)^4]$

Theorem 8. *Suppose that $u := 10^5$ is an assigned upper bound. Let $c < u$ be a conductor divisible by exactly three primes, $t = 3$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for all four cyclic cubic fields $F_{c,\mu}$, $1 \le \mu \le 4$, with conductor c. If $c = q_1 q_2 q_3$ belongs to the Graphs 5, 8 of Category III, i.e.,*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_i \leftrightarrow q_j; q_k\} & \text{or} \\ \{q_k \to q_i \leftrightarrow q_j \leftarrow q_k\} \end{cases} \tag{6.2.5}$$

with i, j, k pairwise distinct, and the 3-valuation v of the class number h of the 3-genus field F_0^* of the cyclic cubic fields $F_{f,\nu}$, $1 \le \nu \le 2$, with conductor $f = q_i q_j$ is $v \le 2$, that is $\mathrm{G}_3^{(2)} F_{f,\nu} \simeq \langle 9, 2\rangle$ or $\langle 27, 4\rangle$, then the second 3-class groups $\mathrm{G}_3^{(2)} F_{c,\mu}$ of the four fields $F_{c,\mu}$ in dependence on the 3-class groups of the 13 bicyclic bicubic subfields S_i of the 3-genus field F^* are given by $\left(\mathrm{G}_3^{(2)} F_{c,\mu}\right)_{1 \le \mu \le 4} =$

$$\begin{cases} (\langle 81, 7\rangle^4) & \Longleftrightarrow [\mathrm{Cl}_3 S_i]_{1 \le i \le 13} = [(0)^2, 1^2; (1^2)^8, (1^3)^2], \\ (\langle 243, 28\rangle^4) & \Longleftrightarrow [\mathrm{Cl}_3 S_i]_{1 \le i \le 13} = [(0)^2, 1; (1^2)^7, (21)^3], \\ (\langle 243, 27\rangle, \langle 243, 28\rangle^3) & \text{or} \\ (\langle 243, 25\rangle^2, \langle 243, 28\rangle^2) & \Longleftrightarrow [\mathrm{Cl}_3 S_i]_{1 \le i \le 13} = [(0)^2, 1; (1^2)^7, (21)^2, 2^2], \\ (\langle 243, 25\rangle^2, \langle 243, 27\rangle^2) & \Longleftrightarrow [\mathrm{Cl}_3 S_i]_{1 \le i \le 13} = [(0)^2, 1; (1^2)^7, (22)^3]. \end{cases} \tag{6.2.6}$$

If $[q_1, q_2, q_3]_3 = \{q_i \leftrightarrow q_j; q_k\}$ *belongs to Graph* 5 *of Category* III*, then the principal factors of the four fields are* $B(F_{c,1}) = q_i q_j q_k$, $B(F_{c,2}) = q_i^2 q_j q_k$, $B(F_{c,3}) = q_i q_j^2 q_k$, $B(F_{c,4}) = q_i q_j q_k^2$ *(up to the order).*
If $[q_1, q_2, q_3]_3 = \{q_k \to q_i \leftrightarrow q_j \leftarrow q_k\}$ *belongs to Graph* 8 *of Category* III*, then the principal factors of all four fields are* $B(F_{c,\mu}) = q_k$, $1 \le \mu \le 4$.

Proof. See the Tables 6.7, 6.8, which have been computed with the aid of Magma [10, 11, 16, 23]. By $\langle 243, 28\rangle$ we always abbreviate $\langle 243, 28|29|30\rangle$. □

Conjecture 3. Theorem 8 remains true for any upper bound $u > 10^5$.

6.3. The Elementary Tricyclic 3-Group

For the first time, we are going to analyze the capitulation of cyclic fields F with 3-class group $\mathrm{Cl}_3 F$ of elementary abelian type $(3, 3, 3)$ in their $\frac{3^3-1}{3-1} = 13$ unramified cyclic cubic extensions E_i/F with $1 \le i \le 13$. Such a group can be viewed as a *vector space* O with dimension $\dim_{\mathbb{F}_3}(O) = 3$ over the finite field $\mathbb{F}_3$ with three elements. The vector space O possesses $3^2 + 3 + 1 = 13$ *lines*, that is, subgroups L_i of index $(O : L_i) = 3^2$, and 13 *planes*, that is, subgroups P_i of index $(O : P_i) = 3$, where $1 \le i \le 13$. Let x, y, z be fixed generators of $O = \langle x, y, z\rangle$, then we shall arrange the generators of the lines $L_i = \langle g_i\rangle$ in the way shown in Table 6.9.

Table 6.9.: Generators of the 13 Lines L_i in O

i	1	2	3	4	5	6	7	8	9	10	11	12	13
g_i	x	y	z	xy	xy^2	yz	yz^2	zx	zx^2	xyz	xyz^2	xy^2z	x^2yz

Table 6.10.: Identifiers and Generators of the 13 Planes P_i in O

i	1	2	3	4	5	6	7
h_i	y	z	x	x	xy	y	xy
k_i	z	x	y	yz	zx	zx	yz
T_i	2, 3, 6, 7	1, 3, 8, 9	1, 2, 4, 5	1, 6, 10, 13	4, 7, 8, 13	2, 8, 10, 12	4, 6, 9, 12

i	8	9	10	11	12	13
h_i	z	zx	z	zx^2	y	x
k_i	xy	yz	xy^2	yz^2	zx^2	yz^2
T_i	3, 4, 10, 11	5, 6, 8, 11	3, 5, 12, 13	5, 7, 9, 10	2, 9, 11, 13	1, 7, 11, 12

For the sake of brevity, we simply denote g_i by its subscript i, and we introduce identifiers for the planes $P_i = \langle h_i, k_i\rangle$ as shown in Table 6.10. The elements h_i, k_i

can be viewed as generators of a transversal of $L_i = \langle g_i \rangle$, i.e., a system of coset representatives for L_i in O. Each set T_i contains the subscripts of generators g_j contained in P_i. It is useful to list the *bundles* B_i of planes containing an assigned line L_i in Table 6.11.

Table 6.11.: The 13 Bundles of Planes in O

i	1	2	3	4	5
B_i	P_2, P_3, P_4, P_{13}	P_1, P_3, P_6, P_{12}	P_1, P_2, P_8, P_{10}	P_3, P_5, P_7, P_8	P_3, P_9, P_{10}, P_{11}
i	6	7	8	9	10
B_i	P_1, P_4, P_7, P_9	P_1, P_5, P_{11}, P_{13}	P_2, P_5, P_6, P_9	P_2, P_7, P_{11}, P_{12}	P_4, P_6, P_8, P_{11}
i	11	12	13		
B_i	P_8, P_9, P_{12}, P_{13}	P_6, P_7, P_{10}, P_{13}	P_4, P_5, P_{10}, P_{12}		

6.4. Finite 3-Groups of Type (3,3,3)

Since the Categories I and II involve cylic cubic fields F with 3-class group $\mathrm{Cl}_3 F \simeq (3,3,3)$, we supplement Section § 6.1 with finite 3-groups having elementary tricyclic commutator quotient $(3,3,3)$. In Table 6.12, we continue Table 6.1 with metabelian groups $\mathfrak{M}$ of generator rank $d_1(\mathfrak{M}) = 3$. Here, the Shafarevich bound $\mu \le \varrho + r + \theta = 3 + 2 + 0 = 5$ is bigger.

Table 6.12.: Invariants of Metabelian 3-Groups $\mathfrak{M}$ with $\mathfrak{M}/\mathfrak{M}' \simeq (3,3,3)$

$\mathfrak{M}$	cc	$\varkappa$	τ	τ_3	ν	μ	$\pi(\mathfrak{M})$
$\langle 27,5\rangle$	2	(O^{13})	$(1^2)^{13}$	0	6	6	
$\langle 81,12\rangle$	2	(O^{13})	$(1^3)^4,(1^2)^9$	1	2	7	$\langle 27,5\rangle$
$\langle 81,13\rangle$	2	(O^9P^4)	$(1^2)^9,(21)^3,1^3$	1	0	5	$\langle 27,5\rangle$
$\langle 81,14\rangle$	2	(O^9P^4)	$(1^2)^9,(21)^4$	1	0	5	$\langle 27,5\rangle$
$\langle 243,38\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^2	2	6	$\langle 27,5\rangle$
$\langle 729,329\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^3	1	6	$\langle 243,38\rangle$
$\langle 2187,5576\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^4	1	6	$\langle 729,329\rangle$
$\langle 2187,5577\ldots 5579\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^4	1	6	$\langle 729,329\rangle$
$\langle 729,372\rangle$	3	(O^3P^9L)	$(1^3)^3,(21)^9,21^3$	1^3	1	6	$\langle 243,40\rangle$
$\langle 243,42\rangle$	3	$((P_i^3)_{i=1}^4 L)$	$21^2,(21)^{12}$	1^2	1	5	$\langle 27,5\rangle$
$\langle 729,388\ldots 390\rangle$	3	$((P_i^3)_{i=1}^4 L)$	$21^2,(21)^{12}$	1^3	0	4	$\langle 243,42\rangle$
$\langle 243,46\rangle$	3	$((P_i^3)_{i=1}^4 L)$	$1^3,(21)^{11},2^2$	1^2	0	4	$\langle 27,5\rangle$
$\langle 243,47\rangle$	3	$((P_i^3)_{i=1}^4 L)$	$(1^3)^4,(21)^8,2^2$	1^2	0	4	$\langle 27,5\rangle$
$\langle 729,125\rangle$	4	$((P_i)_{i=1}^4 L^9)$	$1^4,(21^2)^6,(2^2)^6$	1^3	3	6	$\langle 27,5\rangle$
$\langle 2187,4595\ldots 4598\rangle$	4	$((P_i)_{i=1}^4 L^9)$	$1^4,(21^2)^6,(2^2)^6$	21^2	0	5	$\langle 729,125\rangle$

Finally, we need non-metabelian groups G of generator rank $d_1(G) = 3$ in the cover of metabelian groups $\mathfrak{M}$ with non-trivial cover $\mathrm{cov}(\mathfrak{M})$. They are given in Table 6.13.

Table 6.13.: Invariants of Non-Metabelian 3-groups G with $G/G' \simeq (3,3,3)$

G	cc	$\varkappa$	τ	τ_3	ν	μ	G/G''
$\langle 6561, 261256 \ldots 261261\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^4	1	6	$\langle 2187, 5576\rangle$
$\langle 6561, 261262 \ldots 261270\rangle$	3	(O^3P^{10})	$1^4,(1^3)^3,(21)^9$	1^4	0	5	$\langle 2187, 5577 \ldots 5579\rangle$

Theorem 9. *Let F be a cyclic cubic number field with elementary bicyclic 3-class group $\mathrm{Cl}_3F \simeq (3,3,3)$. Denote by $\mathfrak{M} = \mathrm{Gal}(F_3^{(2)}/F)$ the second 3-class group of F, and by $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ the 3-class field tower group of F. Then, the Artin pattern $(\tau,\varkappa)$ of F identifies the groups $\mathfrak{M}$ and G, and determines the length ℓ_3F of the 3-class field tower of F, according to the following* ***deterministic laws****.*

1. *If $\tau = [(1^2)^{13}]$, $\varkappa = (O^{13})$, then $G \simeq \langle 27,5\rangle$ and $\ell_3F = 1$.*
2. *If $\tau \sim [(1^2)^9,(21)^3,1^3]$, $\varkappa \sim (O^9P^4)$, then $G \simeq \langle 81,13\rangle$.*
3. *If $\tau \sim [(1^2)^9,(21)^4]$, $\varkappa \sim (O^9P^4)$, then $G \simeq \langle 81,14\rangle$.*
4. *If $\tau \sim [1^3,(21)^{11},2^2]$, $\varkappa \sim ((P_i^3)_{i=1}^4L)$, then $G \simeq \langle 243,46\rangle$.*
5. *If $\tau \sim [(1^3)^4,(21)^8,2^2]$, $\varkappa \sim ((P_i^3)_{i=1}^4L)$, then $G \simeq \langle 243,47\rangle$.*

Proof. The cyclic cubic field F has the torsion free unit rank $r = 2$, does not contain primitive third roots of unity, and thus possesses the maximal admissible relation rank $d_2 \le d_1 + r = 5$ for the group G, when its 3-class rank, i.e. the generator rank of G, is $d_1 = \varrho = 3$. Consequently, $\ell_3F \ge 3$ in the case of $d_2\mathfrak{M} \ge 6$. □

In Algorithm 10, we present a convenient version of the *strategy of pattern recognition via Artin transfers* [38]. Assigned abelian quotient invariants (AQI) $\tau(G)$ of 3-groups G with commutator quotient $G/G' \simeq (3,3,3)$ are selected by a database query in the SmallGroups Library [7, 8] with the aid of Magma [10, 11, 23], extended by the package containing groups of order $6561 = 3^8$ [24]. The number of candidates with orders 3^e, $1 \le e \le 8$, increases very rapidly with the exponent e: $1, 2, 5, 15, 67, 504, 9310, 1396077$. The search is coarse, since only one component τ of the Artin pattern $\mathrm{AP} = (\tau,\varkappa)$ is checked. The given commutator quotient and AQI are hard coded and may be replaced by $G/G' \simeq (3,3)$ and four instead of thirteen components of the AQI. In fact, we seek for candidates of the second 3-class group $\mathfrak{M} = \mathrm{Gal}(F_3^{(2)}/F)$ of the cyclic cubic field F with conductor $c = 82\,327$.

Algorithm 10. (Pattern Recognition via Abelian Quotient Invariants.)
Input: prime `iPrm`, maximal exponent `iMax`, maximal relation rank`iRel`.
Code:

```
intrinsic PatternRecognition(iPrm,iMax,iRel::RngIntElt)
p := iPrm; // prime number
m := iMax; // maximal exponent
u := iRel; // maximal relation rank
for e in [1..m] do // exponent
    o := p^e; // order
    if IsInSmallGroupDatabase(o) then
        N := NumberOfSmallGroups(o);
        z := 0; // counter
        for i in [1..N] do
            G := SmallGroup(o,i);
            AQG := AQInvariants(G);
            // if (3 eq #AQG) then // very general rank 3
            if ([p,p,p] eq AQG) then // particular (3,3,3)
                r := 0; // quadratic p-rank
                for j in [1..#AQG] do
                    if (2 le Valuation(AQG[j],p)) then
                        r := r + 1;
                    end if;
                end for; // j
                nc := NilpotencyClass(G);
                cc := e - nc; // co class
                dl := DerivedLength(G);
                nu := NuclearRank(G);
                mu := pMultiplicatorRank(G);
                // rigorous selection
                s := MaximalSubgroups(G);
                n4 := 0;
                n3 := 0;
                n1 := 0;
                n2 := 0;
                for j in [1..#s] do
                    AQS := AQInvariants(s[j]`subgroup);
                    // <729,372>, c = 82327, Category II, Graph 1
                    if ([p,p,p,p^2] eq AQS) then
                        n4 := n4 + 1;
                    elif ([p,p,p] eq AQS) then
                        n3 := n3 + 1;
                    elif ([p,p^2] eq AQS) then
                        n2 := n2 + 1;
                    end if; // n1 not used
                end for; // j
                if (1 eq n4) and (3 eq n3) and (9 eq n2) then // <729,372>
                    z := z + 1;
                    printf "No=%4o:  Lo=%o, Id=%o:  ",z,e,i;
                    printf "qr=%o, nc=%o, cc=%o, dl=%o, nu=%o, mu=%o\n",r,nc,cc,dl,nu,mu;
                    // additional invariants
                    K := CommutatorSubgroup(G);
                    AQK := AQInvariants(K);
                    printf " AQG=%o, AQK=%o\n",AQG,AQK;
                    s := MaximalSubgroups(G);
                    printf " AQI: ";
                    for j in [1..#s] do
                        printf "%o, ",AQInvariants(s[j]`subgroup);
                    end for; // j
                    printf "\n";
                    l := pCentralSeries(G,p);
                    c := pClass(G);
```

```
                q := G/l[c];
                printf " parent:  %o\n",IdentifyGroup(q);
            end if;
        end if; // (p,p,p)
    end for; // i
  end if;
end for; // e
end intrinsic; // PatternRecognition
```

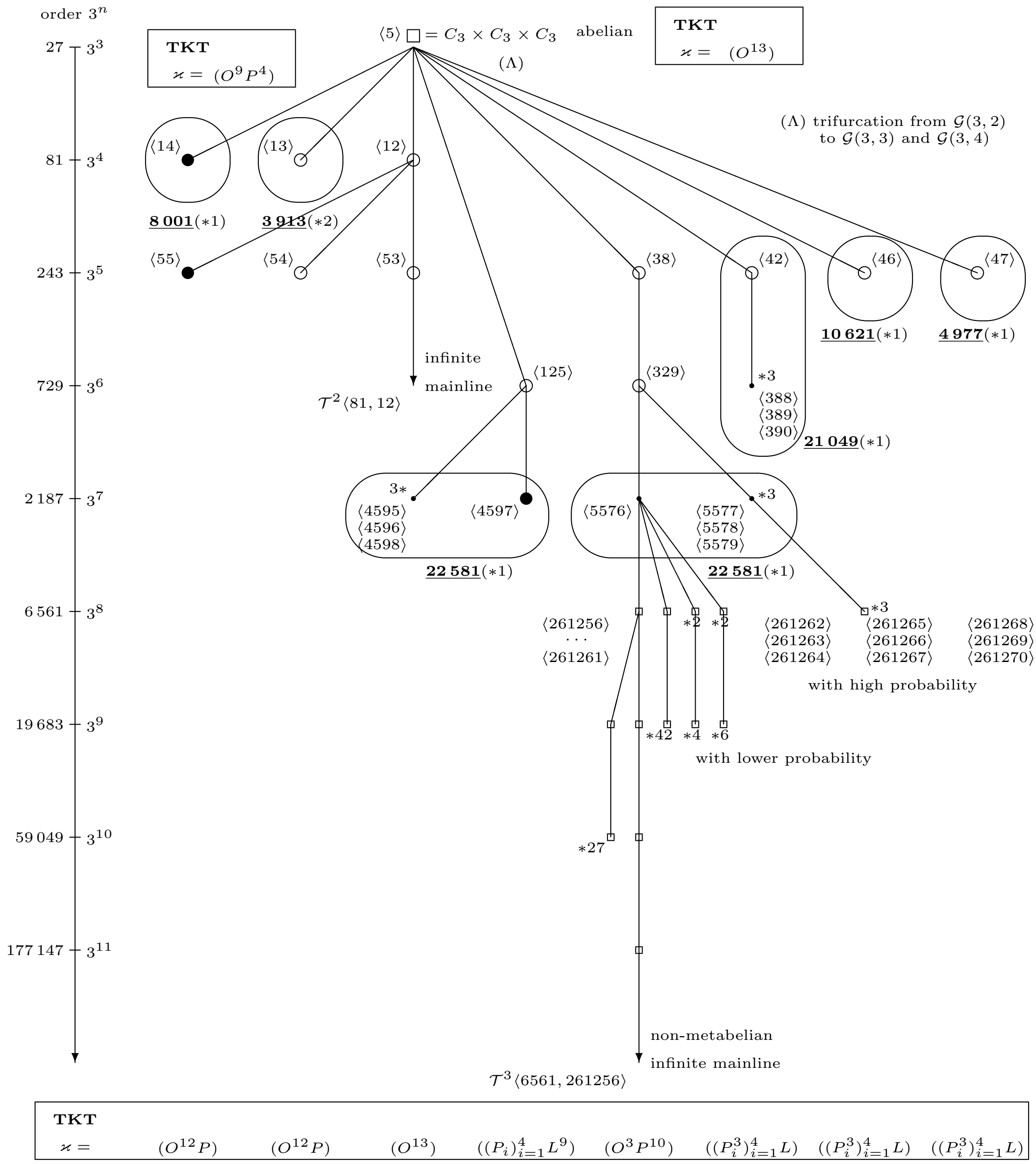
Output: all finite 3-groups G with $\mathrm{ord}(G) \le 3^8$ possessing the assigned AQI.

Example 3. The result set of the database query for abelian quotient invariants $\tau \sim [(1^3)^3, (21)^9, 21^3]$ in Algorithm 10, subject to the bound $\mu = d_2 \le 5$ for the relation rank, is returned by Magma [23] in the following shape:

```
> PatternRecognition(3,8,5);
No= 1:  Lo=6, Id=372:  qr=0, nc=3, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 3 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <243, 40>
No= 1:  Lo=7, Id=5559:  qr=0, nc=4, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 9 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <729, 326>
No= 2:  Lo=7, Id=5560:  qr=0, nc=4, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 9 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <729, 326>
No= 3:  Lo=7, Id=5561:  qr=0, nc=4, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 9 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <729, 326>
No= 4:  Lo=7, Id=5562:  qr=0, nc=4, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 9 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <729, 326>
No= 5:  Lo=7, Id=5563:  qr=0, nc=4, cc=3, dl=2, nu=0, mu=5
    AQG=[ 3, 3, 3 ], AQK=[ 3, 3, 9 ]
    AQI: [3,3,3],[3,9],[3,9],[3,9],[3,9],[3,9],[3,9],[3,3,3],[3,3,3],[3,9],[3,9],[3,9],[3,3,3,9],
    parent:  <729, 327>
```

The result set consists of a single group $\mathfrak{M} = \langle 792, 372\rangle$ of order 3^6 and five groups $\mathfrak{M} = \langle 2187, i\rangle$, $5559 \le i \le 5563$, of order 3^7, which additionally possess the correct transfer kernel type (TKT) $\varkappa \sim [O^3P^9L]$ with a line L contained in the plane P. Several other groups have an inadequate $\varkappa \sim [O^3P^{10}]$ and are eliminated from the listing above.

Figure 6.3.: Distribution of Conductors for $G_3^{(2)}F$ on the Descendant Tree $\mathcal{T}\langle 27, 5\rangle$



6.5. Cyclic Cubic Fields of Type (3,3,3)

6.5.1. Graph 1 of Category II

Since we have completed the discussion of conductors c in the Category III, where all four fields of a quartet share a common 3-class rank $\varrho_3 F_{c,\mu} = 2$, in the sections 6.2.2 and 6.2.3, we must now enter the realm of multiplets $(F_{c,1}, \ldots, F_{c,m})$ with inhomogeneous 3-class rank. We begin with Category II, where two fields have $\varrho_3 F_{c,\mu} = 3$ and the remaining two fields have $\varrho_3 F_{c,\mu} = 2$. In the Tables 6.14, 6.15, 6.16 and 6.17, the second 3-class groups of the two fields with $\varrho_3 F_{c,\mu} = 3$ are given first and separated by a semicolon. Exceptional 3-class groups of type $(9, 3, 3)$, resp. $(9, 9, 3)$, are indicated by asterisks $*$, resp. $**$.

Table 6.14.: Fourty-Seven Examples for Graph 1 of Category II

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1<i<13}$
1	3 913	$\{13 \to 7 \leftarrow 43\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
2	4 123	$\{7 \to 19 \leftarrow 31\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
3	4 921	$\{7 \to 19 \leftarrow 37\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
4	8 827	$\{13 \to 7 \leftarrow 97\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
5	11 557	$\{13 \to 7 \leftarrow 127\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
6	12 649	$\{13 \to 7 \leftarrow 139\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
7	13 699	$\{7 \to 19 \leftarrow 103\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
8	21 679	$\{7 \to 19 \leftarrow 163\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
9	**22 581**	$\{9 \to 193 \leftarrow 13\}$	$\langle 3^7, 5577\rangle, \langle 3^7, 4595\rangle; \langle 3^6, 41\rangle^2$	$[(0)^3; (21)^3, (2^2)^3, (1^3)^3, 1^4]$
10	23 121	$\{7 \to 367 \leftarrow 9\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
11	**25 929**	$\{9 \to 67 \leftarrow 43\}$	$\langle 3^8, 249242\rangle, \langle 3^8, 249293\rangle; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^4]$
12	27 657	$\{7 \to 439 \leftarrow 9\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
13	28 737	$\{9 \to 103 \leftarrow 31\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
14	29 419	$\{31 \to 13 \leftarrow 73\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
15	**30 457**	$\{7 \to 19 \leftarrow 229\}$	$*^2; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 21^3]$
16	31 759	$\{13 \to 7 \leftarrow 349\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
17	31 837	$\{31 \to 13 \leftarrow 79\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
18	**34 029**	$\{19 \to 9 \leftarrow 199\}$	$*^2; \langle 3^7, 248\rangle^2$	$[(0)^3; (21)^3, (32)^3, (1^3)^3, 21^3]$
19	**34 489**	$\{13 \to 7 \leftarrow 379\}$	$\langle 3^7, 5577\rangle, \langle 3^7, 4595\rangle; \langle 3^6, 41\rangle^2$	$[(0)^3; (21)^3, (2^2)^3, (1^3)^3, 1^4]$
20	36 297	$\{37 \to 9 \leftarrow 109\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
21	39 403	$\{13 \to 7 \leftarrow 433\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
22	41 643	$\{7 \to 661 \leftarrow 9\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
23	**41 839**	$\{43 \to 7 \leftarrow 139\}$	$*^2; \langle 6561, 673\rangle^2$	$[(0)^3; (2^2)^6, (1^3)^3, 2^2 1^3]$
24	42 291	$\{37 \to 9 \leftarrow 127\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
25	49 321	$\{31 \to 37 \leftarrow 43\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
26	57 421	$\{13 \to 7 \leftarrow 631\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
27	58 513	$\{13 \to 7 \leftarrow 643\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
28	60 273	$\{37 \to 9 \leftarrow 181\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
29	60 781	$\{7 \to 19 \leftarrow 457\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
30	66 937	$\{13 \to 271 \leftarrow 19\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
31	67 887	$\{19 \to 9 \leftarrow 397\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
32	68 887	$\{13 \to 7 \leftarrow 757\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
33	73 233	$\{9 \to 103 \leftarrow 79\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
34	73 633	$\{7 \to 157 \leftarrow 67\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
35	**74 043**	$\{19 \to 9 \leftarrow 433\}$	$\langle 3^7, 4606\rangle^2; \langle 3^7, 65 \vert 67\rangle^2$	$[(0)^3; (2^2)^6, (1^3)^3, 21^3]$
36	74 971	$\{73 \to 13 \leftarrow 79\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
37	77 337	$\{9 \to 661 \leftarrow 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
38	78 403	$\{13 \to 163 \leftarrow 37\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
39	**82 327**	$\{7 \to 19 \leftarrow 619\}$	$\langle 3^6, 372\rangle^2; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 21^3]$
40	83 327	$\{7 \to 73 \leftarrow 163\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$

Table 6.15.: Graph 1 of Category II Continued

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
41	83 731	$\{31 \to 37 \leftarrow 73\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
42	**83 817**	$\{9 \to 67 \leftarrow 139\}$	$**, *; \langle 6561, 673\rangle^2$	$[(0)^3; (2^2)^6, (1^3)^3, 21^4]$
43	89 053	$\{19 \to 109 \leftarrow 43\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
44	92 407	$\{43 \to 7 \leftarrow 307\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
45	92 511	$\{19 \to 9 \leftarrow 541\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
46	95 641	$\{13 \to 7 \leftarrow 1051\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
47	97 209	$\{7 \to 1543 \leftarrow 9\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$

6.5.2. Graph 2 of Category II

Similarly as for Graph 1, the behaviour of the dominating part of conductors c belonging to Graph 2 seems to be uniform and can be summarized in Theorem 10.

Table 6.16.: Fourty-Five Examples for Graph 2 of Category II

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
1	6 327	$\{19 \to 9 \leftarrow 37 \to 19\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
2	18 639	$\{109 \to 9 \leftarrow 19 \to 109\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
3	19 201	$\{211 \to 7 \leftarrow 13 \to 211\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
4	20 313	$\{61 \to 9 \leftarrow 37 \to 61\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
5	21 717	$\{9 \to 127 \leftarrow 19 \to 9\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
6	21 793	$\{37 \to 19 \leftarrow 31 \to 37\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
7	21 973	$\{7 \to 73 \leftarrow 43 \to 7\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
8	**27 873**	$\{19 \to 9 \leftarrow 163 \to 19\}$	$*^2; \langle 3^6, 37\ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^5]$
9	27 937	$\{13 \to 7 \leftarrow 307 \to 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
10	**29 197**	$\{43 \to 7 \leftarrow 97 \to 43\}$	$*^2; \langle 3^7, 253\rangle^2$	$[(0)^3; (21)^3, (2^2)^3, (1^3)^3, 21^3]$
11	30 951	$\{181 \to 9 \leftarrow 19 \to 181\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
12	**33 943**	$\{7 \to 373 \leftarrow 13 \to 7\}$	$*^2; \langle 3^6, 37\ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^5]$
13	38 227	$\{43 \to 7 \leftarrow 127 \to 43\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
14	**41 629**	$\{19 \to 313 \leftarrow 7 \to 19\}$	$\langle 3^8, 249232\rangle^2; \langle 3^6, 37\ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^4]$
15	43 927	$\{31 \to 13 \leftarrow 109 \to 31\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
16	44 023	$\{331 \to 19 \leftarrow 7 \to 331\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
17	**46 417**	$\{7 \to 19 \leftarrow 349 \to 7\}$	$*^2; \langle 3^6, 37\ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^5]$
18	49 567	$\{7 \to 73 \leftarrow 97 \to 7\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
19	49 777	$\{13 \to 7 \leftarrow 547 \to 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
20	50 407	$\{7 \to 19 \leftarrow 379 \to 7\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
21	54 279	$\{163 \to 9 \leftarrow 37 \to 163\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
22	54 691	$\{601 \to 7 \leftarrow 13 \to 601\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
23	**56 547**	$\{61 \to 103 \leftarrow 9 \to 61\}$	$\langle 3^7, 5577\rangle, \langle 3^7, 4595\rangle; \langle 3^6, 41\rangle^2$	$[(0)^3; (21)^3, (2^2)^3, (1^3)^3, 1^4]$
24	60 151	$\{7 \to 661 \leftarrow 13 \to 7\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
25	60 667	$\{103 \to 19 \leftarrow 31 \to 103\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$

Table 6.17.: Graph 2 of Category II Continued

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
26	60 853	$\{31 \to 13 \leftarrow 151 \to 31\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
27	63 271	$\{31 \to 13 \leftarrow 157 \to 31\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
28	**63 511**	$\{43 \to 7 \leftarrow 211 \to 43\}$	$*^2; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 21^3]$
29	64 809	$\{19 \to 9 \leftarrow 379 \to 19\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
30	**65 727**	$\{9 \to 67 \leftarrow 109 \to 9\}$	$*^2; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 21^3]$
31	**66 157**	$\{13 \to 7 \leftarrow 727 \to 13\}$	$**, *; \langle 6561, 1989\rangle^2$	$[(0)^3; (21)^3, (3^2)^3, (1^3)^3, 2^2 1^2]$
32	**66 267**	$\{37 \to 9 \leftarrow 199 \to 37\}$	$\langle 3^8, 249242\rangle, \langle 3^8, 249293\rangle; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^4]$
33	71 029	$\{7 \to 73 \leftarrow 139 \to 7\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
34	72 943	$\{181 \to 13 \leftarrow 31 \to 181\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
35	75 943	$\{571 \to 19 \leftarrow 7 \to 571\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
36	**79 933**	$\{7 \to 19 \leftarrow 601 \to 7\}$	$\langle 3^7, 5577\rangle, \langle 3^7, 4595\rangle; \langle 3^6, 41\rangle^2$	$[(0)^3; (21)^3, (2^2)^3, (1^3)^3, 1^4]$
37	**80 731**	$\{607 \to 19 \leftarrow 7 \to 607\}$	$*^2; \langle 3^6, 37 \ldots 39\rangle^2$	$[(0)^3; (21)^6, (1^3)^3, 1^5]$
38	85 267	$\{13 \to 7 \leftarrow 937 \to 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
39	86 233	$\{97 \to 7 \leftarrow 127 \to 97\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
40	87 913	$\{19 \to 661 \leftarrow 7 \to 19\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
41	87 997	$\{13 \to 7 \leftarrow 967 \to 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
42	91 053	$\{67 \to 151 \leftarrow 9 \to 67\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
43	94 381	$\{97 \to 7 \leftarrow 139 \to 97\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
44	96 733	$\{1063 \to 7 \leftarrow 13 \to 1063\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$
45	99 463	$\{13 \to 7 \leftarrow 1093 \to 13\}$	$\langle 81, 13\rangle^2; \langle 81, 7\rangle^2$	$[(0)^3; (1^2)^8, (1^3)^2]$

Theorem 10. *Suppose that $u := 10^5$ is an assigned upper bound. Let $c < u$ be a conductor divisible by exactly three primes, $t = 3$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3, 3, 3)$ for two cyclic cubic fields $F_{c,1}$, $F_{c,2}$ with conductor c, and $\mathrm{Cl}_3 F_{c,\mu} \simeq (3, 3)$ for the other two cyclic cubic fields $F_{c,3}$, $F_{c,4}$ with conductor c. If c belongs either to Graph 1 or to Graph 2 of Category II, and if the 3-class groups of the 13 bicyclic bicubic subfields S_i of the 3-genus field F^* are given by*

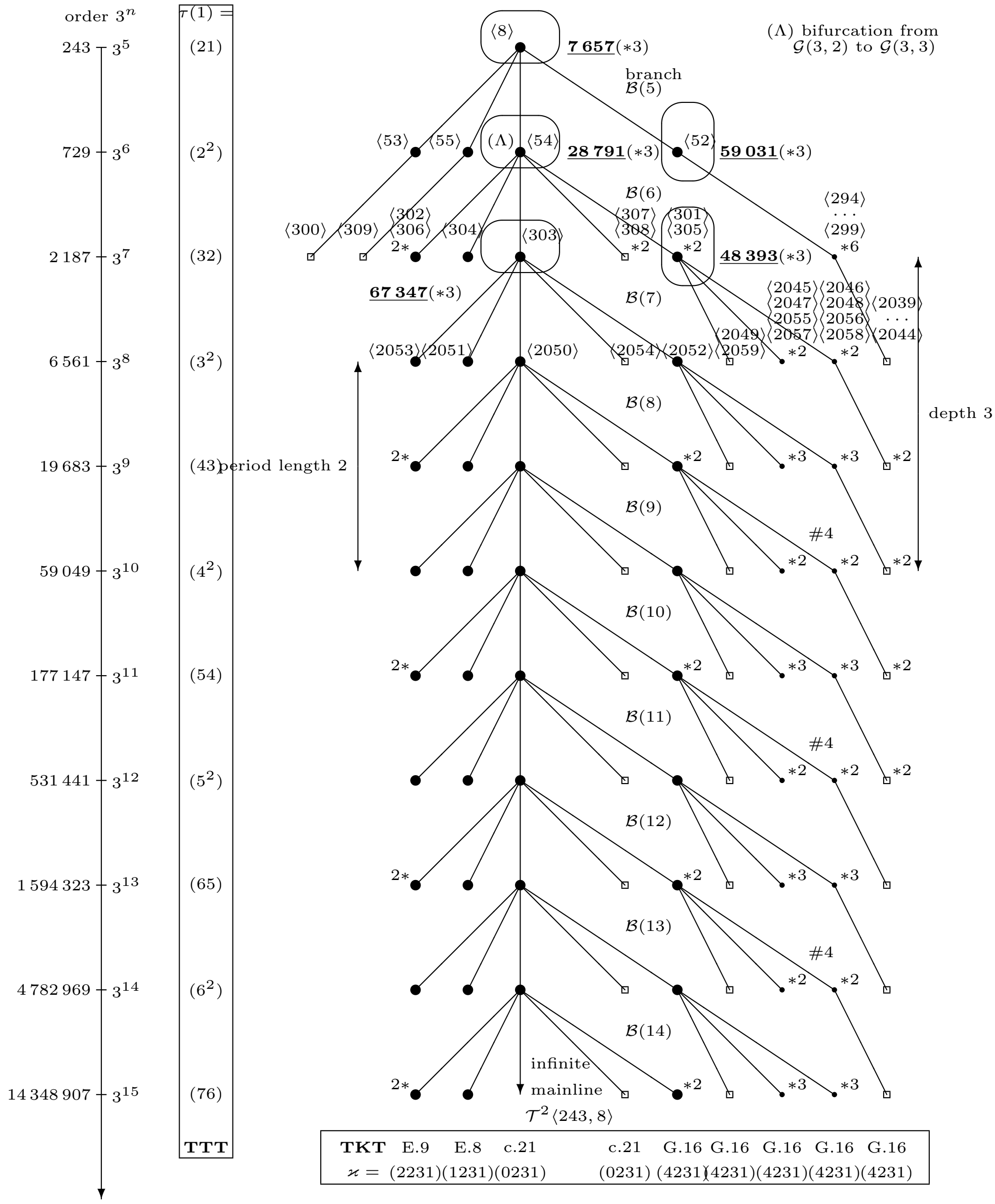
$$[\mathrm{Cl}_3 S_i]_{1 \le i \le 13} = [(0)^3; (11)^8, (111)^2], \qquad (6.5.1)$$

then the second 3-class group $\mathrm{G}_3^{(2)} F_{c,\mu}$ is isomorphic to $\langle 81, 13\rangle$ with 3-capitulation type $\varkappa(F_{c,\mu}) = (O^9 P^4)$ for the fields $F_{c,1}$, $F_{c,2}$, and isomorphic to $\langle 81, 7\rangle \simeq \mathrm{Syl}_3 S_9$ with 3-capitulation type a.3, $\varkappa(F_{c,\mu}) = (2000)$, for the fields $F_{c,3}$, $F_{c,4}$, and the 3-class tower has length $\ell_3 F_{c,\mu} = 2$ for all fields.

Proof. See Tables 6.14, 6.15, 6.16 and 6.17, which have been computed with the aid of Magma [10, 11, 16, 23]. □

Conjecture 4. Theorem 10 remains true for any upper bound $u > 10^5$.

Figure 6.4.: Distribution of Conductors for $G_3^{(2)}F$ on the Coclass Tree $\mathcal{T}^2\langle 243, 8\rangle$



TKT	E.9	E.8	c.21	c.21	G.16	G.16	G.16	G.16	G.16
ϰ =	(2231)	(1231)	(0231)	(0231)	(4231)	(4231)	(4231)	(4231)	(4231)

6.5.3. Graph 1 of Category I

We continue with Category I, where three fields have $\varrho_3 F_{c,\mu} = 2$ and the remaining single field has $\varrho_3 F_{c,\mu} = 3$. In the Tables 6.18, 6.19, 6.20, and 6.21, the second 3-class group of the unique field with $\varrho_3 F_{c,\mu} = 3$ is given first and separated by a semicolon. Again, we denote a 3-class group $\mathrm{Cl}_3 F_{c,\mu} = (9,3,3)$, respectively $(9,9,3)$, by an asterisk $*$, respectively two asterisks $**$.

Table 6.18.: Thirty-Eight Examples for Graph 1 of Category I

No.	c	$[q_1,q_2,q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1\le i\le 13}$
1	4 977	$\{9,7,79;\delta=0\}$	$\langle 243,47\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
2	10 621	$\{13,19,43;\delta=0\}$	$\langle 243,46\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
3	11 349	$\{9,13,97;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
4	16 263	$\{9,13,139;\delta=0\}$	$\langle 243,46\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
5	17 353	$\{7,37,67;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
6	17 829	$\{9,7,283;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
7	22 041	$\{9,31,79;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
8	28 197	$\{9,13,241;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
9	28 609	$\{7,61,67;\delta=0\}$	$\langle 243,46\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
10	**28 791**	$\{9,7,457;\delta=0\}$	$*;\langle 729,54\rangle^3$	$[(0)^3;(21)^6,(2^2)^3,2^21]$
11	**32 227**	$\{13,37,67;\delta=0\}$	$*;\langle 729,54\rangle^3$	$[(0)^3;(21)^6,(2^2)^3,2^21]$
12	34 099	$\{13,43,61;\delta=0\}$	$\langle 243,47\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
13	34 333	$\{13,19,139;\delta=0\}$	$\langle 243,47\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
14	38 727	$\{9,13,331;\delta=0\}$	$*;\langle 243,8\rangle^3$	$[(0)^3;(21)^9,21^2]$
15	40 833	$\{9,13,349;\delta=0\}$	$*;\langle 243,8\rangle^3$	$[(0)^3;(21)^9,21^2]$
16	43 183	$\{7,31,199;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
17	43 533	$\{9,7,691;\delta=0\}$	$*;\langle 243,8\rangle^3$	$[(0)^3;(21)^9,21^2]$
18	46 179	$\{9,7,733;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
19	49 153	$\{13,19,199;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
20	52 297	$\{7,31,241;\delta=0\}$	$\langle 243,46\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$
21	53 793	$\{9,43,139;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
22	58 869	$\{9,31,211;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
23	**61 087**	$\{13,37,127;\delta=0\}$	$*;\langle 729,54\rangle^3$	$[(0)^3;(21)^6,(2^2)^3,2^21]$
24	64 543	$\{19,43,79;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
25	65 457	$\{9,7,1039;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
26	67 239	$\{9,31,241;\delta=0\}$	$*;\langle 243,8\rangle^3$	$[(0)^3;(21)^9,21^2]$
27	**67 347**	$\{9,7,1069;\delta=0\}$	$**;\langle 2187,303\rangle^3$	$[(0)^3;(21)^6,(32)^3,2^3]$
28	77 931	$\{9,7,1237;\delta=0\}$	$\langle 81,14\rangle;\langle 81,8\rangle,\langle 81,10\rangle^2$	$[(0)^3;(1^2)^7,(21)^3]$
29	78 589	$\{7,103,109;\delta=0\}$	$*;\langle 243,8\rangle^3$	$[(0)^3;(21)^9,21^2]$
30	80 847	$\{9,13,691;\delta=0\}$	$\langle 243,46\rangle;\langle 243,25\rangle,\langle 243,28\rangle^2$	$[(0)^3;(1^2)^6,(21)^2,2^2,1^3]$

Table 6.19.: Graph 1 of Category I Continued

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1 \le i \le 13}$
31	81 333	$\{9, 7, 1291; \delta = 0\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
32	82 411	$\{7, 61, 193; \delta = 0\}$	$\langle 243, 47\rangle; \langle 243, 25\rangle, \langle 243, 28\rangle^2$	$[(0)^3; (1^2)^6, (21)^2, 2^2, 1^3]$
33	84 607	$\{19, 61, 73; \delta = 0\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
34	87 327	$\{9, 31, 313; \delta = 0\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
35	87 867	$\{9, 13, 751; \delta = 0\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
36	90 649	$\{13, 19, 367; \delta = 0\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
37	92 349	$\{9, 31, 331; \delta = 0\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
38	96 291	$\{9, 13, 823; \delta = 0\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$

6.5.4. Graph 2 of Category I

Theorem 11. *Suppose that $u := 10^5$ is an assigned upper bound. Let $c < u$ be a conductor divisible by exactly three primes, $t = 3$, such that $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3,3)$ for the single cyclic cubic field $\mu = 1$, and $\mathrm{Cl}_3 F_{c,\mu} \simeq (3,3)$ for the other three cyclic cubic fields $2 \le \mu \le 4$ with conductor c. If $c = q_1 q_2 q_3$ belongs to Graph 1 or 2 of Category I, i.e.,*

$$[q_1, q_2, q_3]_3 = \begin{cases} \{q_1, q_2, q_3; \delta = 0\} & \text{or} \\ \{q_i \leftarrow q_j \rightarrow q_k\} & \text{with } i, j, k \text{ pairwise distinct,} \end{cases} \tag{6.5.2}$$

then the second 3-class groups $\mathrm{G}_3^{(2)} F_{c,\mu}$ of the four fields $1 \le \mu \le 4$ in dependence on the 3-class groups of the 13 bicyclic bicubic subfields S_i, $1 \le i \le 13$, of the 3-genus field F^* are given by $\left(\mathrm{G}_3^{(2)} F_{c,\mu}\right)_{1 \le \mu \le 4} =$

$$\begin{cases} \langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2 & \iff [\mathrm{Cl}_3 S_i] = [(0)^3; (1^2)^7, (21)^3], \\ \langle 243, 46\rangle; \langle 243, 25\rangle, \langle 243, 28\rangle^2 & \iff [\mathrm{Cl}_3 S_i] = [(0)^3; (1^2)^6, (21)^2, 2^2, 1^3], \\ \langle 243, 42\rangle; \langle 243, 8\rangle^3 & \iff [\mathrm{Cl}_3 S_i] = [(0)^3; (21)^9, 21^2]. \end{cases} \tag{6.5.3}$$

Proof. Theorem 11 is a consequence of the Tables 6.18, 6.19, 6.20, and 6.21, which have been computed with the aid of Magma [10, 11, 16, 23]. □

Conjecture 5. Theorem 11 remains true for any upper bound $u > 10^5$.

Table 6.20.: Sixty Examples for Graph 2 of Category I

No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1 \le i \le 13}$
1	7 657	$\{13 \leftarrow 31 \rightarrow 19\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
2	8 001	$\{9 \leftarrow 127 \rightarrow 7\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
3	9 709	$\{19 \leftarrow 7 \rightarrow 73\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
4	11 137	$\{7 \leftarrow 43 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
5	12 753	$\{9 \leftarrow 109 \rightarrow 13\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
6	14 833	$\{7 \leftarrow 13 \rightarrow 163\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
7	14 911	$\{13 \leftarrow 31 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
8	16 587	$\{9 \leftarrow 19 \rightarrow 97\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
9	17 563	$\{7 \leftarrow 13 \rightarrow 193\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
10	20 167	$\{7 \leftarrow 43 \rightarrow 67\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
11	20 881	$\{19 \leftarrow 7 \rightarrow 157\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
12	21 049	$\{7 \leftarrow 97 \rightarrow 31\}$	$\langle 243, 42\rangle; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
13	21 177	$\{9 \leftarrow 181 \rightarrow 13\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
14	23 877	$\{7 \leftarrow 379 \rightarrow 9\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
15	24 661	$\{7 \leftarrow 13 \rightarrow 271\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
16	25 123	$\{7 \leftarrow 97 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
17	25 207	$\{7 \leftarrow 13 \rightarrow 277\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
18	27 279	$\{7 \leftarrow 433 \rightarrow 9\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
19	30 163	$\{7 \leftarrow 139 \rightarrow 31\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
20	30 411	$\{9 \leftarrow 109 \rightarrow 31\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
21	32 809	$\{7 \leftarrow 43 \rightarrow 109\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
22	35 113	$\{13 \leftarrow 73 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
23	36 783	$\{61 \leftarrow 9 \rightarrow 67\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
24	37 219	$\{7 \leftarrow 13 \rightarrow 409\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
25	39 753	$\{7 \leftarrow 631 \rightarrow 9\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
26	42 883	$\{19 \leftarrow 37 \rightarrow 61\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
27	**48 393**	$\{9 \leftarrow 19 \rightarrow 283\}$	$*; \langle 3^7, 301 \mid 305\rangle^3$	$[(0)^3; (21)^6, (32)^3, 2^2 1]$
28	48 811	$\{19 \leftarrow 7 \rightarrow 367\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
29	49 149	$\{9 \leftarrow 127 \rightarrow 43\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
30	53 523	$\{9 \leftarrow 19 \rightarrow 313\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$

Table 6.21.: Graph 2 of Category I Continued

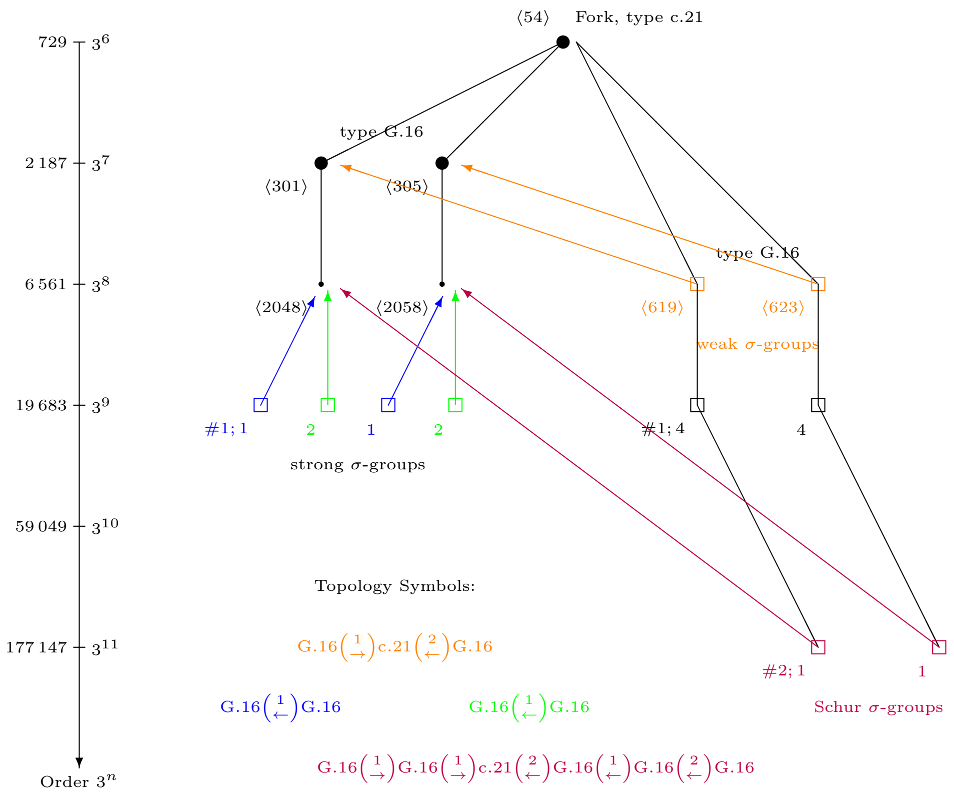
No.	c	$[q_1, q_2, q_3]_3$	$\mathrm{G}_3^{(2)} F_{c,\mu}$	$[\mathrm{Cl}_3 S_i]_{1 \le i \le 13}$
31	54 649	$\{7 \leftarrow 211 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
32	58 387	$\{19 \leftarrow 7 \rightarrow 439\}$	$\langle 243, 42\rangle; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
33	**59 031**	$\{9 \leftarrow 937 \rightarrow 7\}$	$\langle 6561, 217701\rangle; \langle 3^6, 52\rangle^3$	$[(0)^3; (21)^6, (2^2)^3, 21^2]$
34	62 109	$\{67 \leftarrow 9 \rightarrow 103\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
35	63 297	$\{9 \leftarrow 541 \rightarrow 13\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
36	64 519	$\{7 \leftarrow 13 \rightarrow 709\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
37	65 191	$\{7 \leftarrow 139 \rightarrow 67\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
38	66 969	$\{7 \leftarrow 1063 \rightarrow 9\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
39	67 249	$\{7 \leftarrow 13 \rightarrow 739\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
40	68 929	$\{7 \leftarrow 43 \rightarrow 229\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
41	69 939	$\{9 \leftarrow 19 \rightarrow 409\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
42	71 953	$\{19 \leftarrow 7 \rightarrow 541\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
43	71 991	$\{9 \leftarrow 19 \rightarrow 421\}$	$\langle 243, 42\rangle; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
44	72 541	$\{7 \leftarrow 43 \rightarrow 241\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
45	77 013	$\{9 \leftarrow 199 \rightarrow 43\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
46	79 513	$\{7 \leftarrow 307 \rightarrow 37\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
47	80 227	$\{73 \leftarrow 7 \rightarrow 157\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
48	82 899	$\{61 \leftarrow 9 \rightarrow 151\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
49	83 629	$\{7 \leftarrow 13 \rightarrow 919\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
50	84 409	$\{13 \leftarrow 151 \rightarrow 43\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
51	85 183	$\{7 \leftarrow 43 \rightarrow 283\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
52	90 097	$\{7 \leftarrow 211 \rightarrow 61\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
53	91 567	$\{7 \leftarrow 127 \rightarrow 103\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
54	92 241	$\{9 \leftarrow 37 \rightarrow 277\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
55	93 961	$\{7 \leftarrow 433 \rightarrow 31\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
56	94 877	$\{9 \leftarrow 811 \rightarrow 13\}$	$*; \langle 243, 8\rangle^3$	$[(0)^3; (21)^9, 21^2]$
57	94 939	$\{13 \leftarrow 109 \rightarrow 67\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
58	95 157	$\{9 \leftarrow 109 \rightarrow 97\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
59	98 721	$\{9 \leftarrow 1567 \rightarrow 7\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$
60	99 883	$\{19 \leftarrow 7 \rightarrow 751\}$	$\langle 81, 14\rangle; \langle 81, 8\rangle, \langle 81, 10\rangle^2$	$[(0)^3; (1^2)^7, (21)^3]$

6.6. 3-Towers of Length 3 over Quadratic Fields and Cyclic Cubic Fields

It is very illuminating to compare three kinds of algebraic number fields with distinct signatures, which share a common Artin pattern. By a lucky coincidence, our computations yielded a complex quadratic field $F = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -17\,131$ and signature $(0, 1)$, two real quadratic fields $F_i = \mathbb{Q}(\sqrt{d_i})$ with discriminants $d_1 = +8\,711\,453$, $d_2 = +9\,448\,265$ and signature $(2, 0)$, and four cyclic cubic fields $F_{c,\mu}$ with conductor $c = 48\,393$ and signature $(3, 0)$. The conductor c belongs to Graph 2 of Category I, since the combined cubic residue symbol is given by $[q_1, q_2, q_3]_3 = \{9 \leftarrow 19 \rightarrow 283\}$. All quadratic fields and three of the cyclic

cubic fields, having 3-class groups $\mathrm{Cl}_3 F$ of type $(3,3)$, possess the Artin pattern $\mathrm{AP} = (\tau, \varkappa)$ with $\varkappa \sim (4231)$, of type G.16, and $\tau \sim [32, 21, 21, 21]$.
Notation. We are going to use *logarithmic type invariants* of abelian 3-groups, for instance $(321) \hat{=} (27, 9, 3)$, $(2^2 1) \hat{=} (9, 9, 3)$, and $(41^2) \hat{=} (81, 3, 3)$.

Figure 6.5.: Various Kinds of Fields with Tree Topology of Type G.16



Let F be a number field with $\mathrm{Cl}_3 \simeq (3,3)$ and Artin pattern $\mathrm{AP}(F) = (\tau(F), \varkappa(F))$, where $\varkappa(F) \sim (4231)$ is of type G.16 and $\tau(F) \sim [32, 21, 21, 21]$ indicates the ground state. Denote by $\mathfrak{M} = \mathrm{Gal}(F_3^{(2)}/F)$ the second 3-class group and by $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ the 3-tower group of F.

Theorem 12. ***(Imaginary quadratic field with fork topology).***
If $\tau^{(2)} F = [(32; 321, (\mathbf{41}^2)^3), (21; 321, (\mathbf{31})^3)^3]$ for an imaginary quadratic field $F = \mathbb{Q}(\sqrt{d})$, $d < 0$, then $\mathfrak{M} \simeq \langle 3^8, 2048|2058 \rangle$, and $G \simeq \langle 3^8, 619|623 \rangle - \#1;4 - \#2;1$ is a ***Schur*** *σ-group.*

Theorem 13. ***(Real quadratic field with child topology).***
If $F = \mathbb{Q}(\sqrt{d})$, $d > 0$, is a real quadratic field, and

$\tau^{(2)}F = [(32; 321, (\mathbf{411})^3), (21; 321, (\mathbf{21})^3)^3]$, *respectively*
$\tau^{(2)}F = [(32; 321, (\mathbf{311})^3), (21; 321, (\mathbf{21})^3)^3]$,
then $\mathfrak{M} \simeq \langle 3^8, i\rangle$ *with* $i = 2048$, *respectively* $i = 2058$, *and* $G \simeq \mathfrak{M} - \#1; j$ *with* $j = 1$, *respectively* $j = 2$, *is a* ***strong*** *σ-group.*

Theorem 14. ***(Cyclic cubic field with fork topology).***
If $\tau^{(2)}F = [(32; \mathbf{2^2}1, (\mathbf{31^2})^3), (21; \mathbf{2^2}1, (\mathbf{31})^3)^3]$ *for a cyclic cubic field* F, *then* $\mathfrak{M} \simeq \langle 3^7, 301|305\rangle$, *and* $G \simeq \langle 3^8, 619|623\rangle$ *is a* ***weak*** *σ-group.*

Proof. (Simultaneous proof of the Theorems 12, 13, and 14.) Due to the various signatures of the different kinds of fields, the torsionfree Dirichlet unit ranks are given by $r = 0$, $r = 1$, $r = 2$, respectively. According to the Shafarevich Theorem [43], the relation rank of the 3-tower group is therefore bounded by $d_2 \le 2$, $d_2 \le 3$, $d_2 \le 4$, respectively. Exclusively the Schur σ-groups $\langle 3^8, i\rangle - \#1; 4 - \#2; 1$, $i \in \{619, 623\}$, have a balanced presentation with $d_2 = 2$. Thus they provide the unique possibility for the fastidious imaginary quadratic fields. The real quadratic fields are happy with a strong σ-group, like $\langle 3^8, i\rangle - \#1; 1$ or $\langle 3^8, i\rangle - \#1; 2$, $i \in \{2048, 2058\}$, in dependence on the IPAD of second order $\tau^{(2)}F$. The cover of $\langle 3^7, i\rangle$, $i \in \{301, 305\}$, does not contain any strong σ-group, let alone a Schur σ-group. It is thus forbidden as second 3-class group for both, real and imaginary quadratic fields. Only the frugal cyclic cubic fields are satisfied with a metabelianization of their 3-tower group $\langle 3^8, 619|623\rangle$, which is only a weak σ-group (even a non-σ group would do it). □

Example 4. The smallest concrete realizations of the fields in the Theorems 12, 13, and 14, whose tree topologies are drawn in Figure 6.5, have the following discriminants, conductors:
$F = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -17\,131$,
$F = \mathbb{Q}(\sqrt{d})$ with discriminant $d = +8\,711\,453$,
$F = \mathbb{Q}(\sqrt{d})$ with discriminant $d = +9\,448\,265$,
F cyclic cubic field with conductor $c = 48\,393$.

Theorem 15. ***(Three stage tower* c.21*).*** *Let* F *be a cyclic cubic field with 3-class group* $\mathrm{Cl}_3F \simeq (3,3)$, *capitulation type* c.21, $\varkappa \sim (0231)$, *and (logarithmic) abelian type invariants of* ***second order***

$$\tau^{(2)} = \left(11; [22; (211)^4], [21; 211, (\mathbf{31})^3], [21; 211, (21)^3]^2\right).$$

Then F *has a 3-class field tower of precise length* $\ell_3F = 3$ *with group* $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ *isomorphic to either* $\langle 2187, 307\rangle$ *or* $\langle 2187, 308\rangle$, *the latter with double probability.*

Proof. The isomorphism class of the metabelianization $\mathfrak{M} = G/G''$ of G is determined uniquely as $\langle 729, 54\rangle = U$ by the Artin pattern $\mathrm{AP} = (\tau, \varkappa)$ with $\tau = [22, 21, 21, 21]$ and $\varkappa = (0231)$. The Shafarevich cover [34] with respect to F,

$$\mathrm{cov}_F(\mathfrak{M}) = \{\langle 729, 54\rangle, \langle 2187, 307\rangle, \langle 2187, 308\rangle\},$$

is non-trivial, finite and *inhomogeneous*. Since all members have admissible relation ranks $3 \le d_2 \le 4$, the decision between $\ell_3 F = 2$ and $\ell_3 F = 3$ requires abelian type invariants of second order $\tau^{(2)}$, because $\langle 729, 54\rangle$ has

$$\tau^{(2)} = \left(11; [22; (211)^4], [21; 211, (21)^3]^3\right).$$

Finally the order of the automorphism group of $\langle 2187, 307\rangle$ is bigger by a factor of 2 than that of $\langle 2187, 308\rangle$. □

Example 5. All three fields of type $(3, 3)$ in the quartets for each of the conductors $c = \mathbf{28\,791}$, $\mathbf{32\,227}$, $\mathbf{61\,087}$, i.e. nine fields, which belong to graph 1 of category I, satisfy the conditions of Theorem 15. (The fourth field is of type $(9, 3, 3)$.)

Theorem 16. ***(Three stage tower* d.19*).*** *Let F be a cyclic cubic field with 3-class group $\mathrm{Cl}_3 F \simeq (3, 3)$, capitulation type* d.19*, $\varkappa \sim (4043)$, and abelian type invariants of* ***second order***

$$\tau^{(2)} = \left(11; [22; (211)^4], [21; 211, (21)^3], [111; (\mathbf{211})^{\mathbf{4}}, (11)^9], [111; 211, (111)^3, (11)^9]\right).$$

Then F has a 3-class field tower of precise length $\ell_3 F = 3$ with group $G = \mathrm{Gal}(F_3^{(\infty)}/F)$ isomorphic to $\langle 2187, 265\rangle$.

Proof. The isomorphism class of the metabelianization $\mathfrak{M} = G/G''$ of G is determined uniquely as $\langle 729, 41\rangle = D$ by the Artin pattern $\mathrm{AP} = (\tau, \varkappa)$ with $\tau = [22, 21, 111, 111]$ and $\varkappa = (4043)$. The Shafarevich cover [34] with respect to F,

$$\mathrm{cov}_F(\mathfrak{M}) = \{\langle 729, 41\rangle, \langle 2187, 263\rangle, \langle 2187, 264\rangle, \langle 2187, 265\rangle\},$$

is non-trivial, finite and *inhomogeneous*. Since all members have admissible relation ranks $3 \le d_2 \le 4$, the decision between $\ell_3 F = 2$ and $\ell_3 F = 3$ requires abelian type invariants of second order $\tau^{(2)}$ and is possible only for $\langle 2187, 265\rangle$, since $\langle 729, 41\rangle$, $\langle 2187, 263\rangle$, $\langle 2187, 264\rangle$ have the same pattern

$$\tau^{(2)} = \left(11; [22; (211)^4], [21; 211, (21)^3], [111; 211, (111)^3, (11)^9]^2\right).$$ □

Example 6. A single field for each of the conductors $c = \mathbf{22\,581}$, $\mathbf{34\,489}$, $\mathbf{56\,547}$, $\mathbf{79\,933}$, i.e. four fields, which belong to graph $1, 1, 2, 2$ of category II, respectively, satisfy the conditions of Theorem 16.

Theorem 17. ***(World record three stage tower of type*** $(\mathbf{3},\mathbf{3},\mathbf{3})$***).*** *Let* F *be a cyclic cubic field with* 3*-class group* $\mathrm{Cl}_3F \simeq (3,3,3)$*, capitulation type* $\varkappa \sim (O^3P^{10})$*, and abelian type invariants of first order* $\tau^{(1)} = [111; 1111, (111)^3, (21)^9]$*. Then* F *has a* 3*-class field tower of exact length* $\ell_3F = 3$ *with group* $\mathrm{Gal}(F_3^{(\infty)}/F)$ *isomorphic to one of the nine groups* $\langle 6561, n\rangle$ *with* $n \in \{261262, \ldots, 261270\}$*.*

Proof. There are four possible isomorphism classes of the metabelianization $\mathfrak{M} = G/G''$ of G, namely $\langle 2187, m\rangle$ with $m \in \{5576, \ldots, 5579\}$ sharing the common Artin pattern $\mathrm{AP} = (\tau, \varkappa)$. The cover [34] of the latter three groups is given by

$$\mathrm{cov}(\mathfrak{M}) = \{\langle 2187, 5577\rangle, \langle 6561, 261262\rangle, \langle 6561, 261263\rangle, \langle 6561, 261264\rangle\}, \text{ resp.}$$

$$\mathrm{cov}(\mathfrak{M}) = \{\langle 2187, 5578\rangle, \langle 6561, 261265\rangle, \langle 6561, 261266\rangle, \langle 6561, 261267\rangle\}, \text{ resp.}$$

$$\mathrm{cov}(\mathfrak{M}) = \{\langle 2187, 5579\rangle, \langle 6561, 261268\rangle, \langle 6561, 261269\rangle, \langle 6561, 261270\rangle\}.$$

However, the Shafarevich cover $\mathrm{cov}_F(\mathfrak{M})$ with respect to F consists of the non-metabelian groups with order 6561 only. The reason for this fact is that the relation rank of the metabelian group $\mathfrak{M}$ takes the forbidden value 6. So, the Shafarevich cover is non-trivial, finite and *homogeneous*, which dispenses us with calculating abelian type invariants of second order, $\tau^{(2)}$. □

Example 7. A single field of the conductor $c = \mathbf{22\,581}$, which belongs to graph 1 of category II, satisfies the conditions of Theorem 17.

Part III.

Applications

7. Galois Action of Cyclic Fields

7.1. p-Capitulation Enforced by Galois Action

The generating automorphism σ of a cyclic number field $F/\mathbb{Q}$ of degree d with Galois group $\mathrm{Gal}(F/\mathbb{Q}) = \langle\sigma\rangle$ acts on the class group Cl_F of F and thus also on the higher p-class groups $\mathrm{G}_p^{(n)}F$ with $n \in \mathbb{N} \cup \{\infty\}$ [28], for a fixed prime number p. When d and p are coprime, a remarkable restriction of the possibilities for the metabelian second p-class group $\mathfrak{M} = \mathrm{G}_p^{(2)}F$, and consequently for the transfer kernel type $\varkappa(F)$ of F, is due to the fact that the trace $T_\sigma = \sum_{i=0}^{d-1} \sigma^i$ of σ annihilates the commutator quotient of all the groups $\mathrm{G}_p^{(n)}F$.

Definition 5. Let p be a prime number and G be a pro-p group with finite abelianization G/G'. Suppose that $d \geq 2$ is a fixed integer. G is said to be a **σ-group of degree** d, if G possesses an automorphism σ of order d whose trace $T_\sigma = \sum_{j=0}^{d-1} \sigma^j \in \mathbb{Z}[\mathrm{Aut}(G)]$ annihilates G modulo G', that is, if there exists $\sigma \in \mathrm{Aut}(G)$ such that $\mathrm{ord}(\sigma) = d$ and $x^{T_\sigma} = \prod_{j=0}^{d-1} \sigma^j(x) \in G'$ for all $x \in G$.

The following theorem is the cubic analogue of [6, Thm. 2.2].

Theorem 18. *The p-class tower group $G_p^{(\infty)}F$ and all higher p-class groups $G_p^{(n)}F$, $n \geq 2$, of a cyclic cubic number field F are σ-groups of degree 3, for any prime p.*

Proof. The generating automorphism σ of $F/\mathbb{Q}$ annihilates the class group Cl_F when it acts by its trace $T_\sigma = \sum_{i=0}^{2} \sigma^i \in \mathbb{Z}[\langle\sigma\rangle]$, since $x^{T_\sigma} = \prod_{i=0}^{2} \sigma^i(x) = \mathrm{Norm}_{F/\mathbb{Q}}(x) \in \mathrm{Cl}_{\mathbb{Q}} = 1$, for all $x \in \mathrm{Cl}_F$. Of course, the same is true for all p-class groups Cl_pF with primes p. Finally, we have isomorphisms $\mathrm{G}_p^{(n)}F/\left(\mathrm{G}_p^{(n)}F\right)' \simeq \mathrm{Cl}_pF$, for any $n \in \mathbb{N} \cup \{\infty\}$ [28]. □

In Table 7.1, we denote the SmallGroups identifier [8] by Id, the nuclear rank by ν, the p-multiplicator rank by μ [29]. Further we give the order #Aut of the automorphism group $\mathrm{Aut}(G)$ and its prime power factors. o denotes the cardinality of the set $\mathrm{Aut}_3(G) := \{\sigma \in \mathrm{Aut}(G) \mid \mathrm{ord}(\sigma) = 3\}$, $w := \#\{\sigma \in \mathrm{Aut}_3(G) \mid (\forall x \in$

Table 7.1.: Number of Automorphisms with Order 3 of p-Groups G

No.	p	Order	Id	ν	μ	#Aut	Factors	o	w	s
1	2	4	2	3	3	6	$2,3$	2	2	2
2	2	8	3	1	3	8	2^3	0	0	0
3	2	8	4	0	2	24	$2^3,3$	8	8	2
4	2	8	5	1	3	168	$2^3,3,7$	56	0	0
1	5	25	2	3	3	480	$2^5,3,5$	20	20	20
2	5	125	3	2	4	12000	$2^5,3,5^3$	500	500	20
3	5	125	4	0	2	500	$2^2,5^3$	0	0	0
4	5	625	7	1	4	50000	$2^4,5^5$	0	0	0
5	5	625	8	0	3	12500	$2^2,5^5$	0	0	0
6	5	625	9	0	3	5000	$2^3,5^4$	0	0	0
7	5	625	10	0	3	5000	$2^3,5^4$	0	0	0
8	5	3125	3	3	5	7500000	$2^5,3,5^7$	12500	12500	500
9	5	3125	4	1	3	62500	$2^2,5^6$	0	0	0
10	5	3125	5	1	3	625000	$2^3,5^7$	0	0	0
11	5	3125	6	1	3	625000	$2^3,5^7$	0	0	0
12	5	3125	7	1	3	125000	$2^3,5^6$	0	0	0
13	5	3125	8	0	2	156250	$2,5^7$	0	0	0
14	5	3125	9	0	2	93750	$2,3,5^6$	1250	1250	50
15	5	3125	10	1	3	187500	$2^2,3,5^6$	1250	1250	50
16	5	3125	11	0	2	62500	$2^2,5^6$	0	0	0
17	5	3125	12	0	2	93750	$2,3,5^6$	1250	1250	50
18	5	3125	13	0	2	156250	$2,5^7$	0	0	0
19	5	3125	14	0	2	1875000	$2^3,3,5^7$	12500	12500	500

$G)\,x^{1+\sigma+\sigma^2} \in G'\}$ the number of *weak* σ-automorphisms of degree 3, and $s := \#\{\sigma \in \mathrm{Aut}_3(G) \mid (\forall x \in G)\,x^{1+\sigma+\sigma^2} = 1\}$ the number of *strong* σ-automorphisms of degree 3.

For the dominating part of the finite p-groups in Table 7.1, the failure of being a σ-group of degree 3 is a consequence of $\gcd(3, \#\mathrm{Aut}(G)) = 1$ already. For the elementary abelian 2-group $\langle 8, 5\rangle$ of rank 3, however, the computation of w is required for the decision, since $v_3(\#\mathrm{Aut}(G)) = 1$.

In view of our special situation with $p \in \{2, 5\}$ and $\mathrm{Cl}_pF = (p, p)$, we tested finite metabelian p-groups G with $G/G' \simeq (p, p)$ for the property of being a σ-group of degree 3. The following theorem is the analogue of [6, Thm. 2.3] for degree 3.

Theorem 19. *We characterize finite p-groups by their identifier in the SmallGroups Library [7, 8], which gives the order,* ord*, of the group and a counting number,* id*, enclosed in angle brackets* $\langle \mathrm{ord}, \mathrm{id}\rangle$.

1. *A finite 2-group G with $G/G' \simeq (2, 2)$ which is a σ-group of degree 3 is isomorphic to either the **abelian group** $\langle 4, 2\rangle$ of type $(2, 2)$ or the **quaternion group** $\langle 8, 4\rangle$.*

2. *There are no finite* 2*-groups* G *with* $G/G' \simeq (2,2,2)$ *which are* σ*-groups of degree* 3*. (Thus, there are no cyclic cubic fields of type* $(2,2,2)$*.)*

3. *A finite* 5*-group* G *with* $G/G' \simeq (5,5)$ *which is a* σ*-group of degree* 3 *is isomorphic to either the* ***abelian group*** $\langle 25,2\rangle$ *of type* $(5,5)$ *or the* ***extra special group*** $\langle 125,3\rangle$ *or one of the three* ***Schur*** σ***-groups*** $\langle 3125,i\rangle$ *with* $i \in \{9,12,14\}$ *(and TKT two* 3*-cycles or one* 6*-cycle or identity permutation) or a descendant of one of the two groups* $\langle 3125,i\rangle$ *with* $i \in \{3,10\}$*.*

Proof. Using permutation representations, we compiled a program script in Magma [23] in order to test whether an assigned p-group G with $G/G' \simeq (p,p)$ is a σ-group of degree 3, for any prime number $p \neq 3$. See Table 7.1. According to [6, Cor. 2.1], all descendants of a group R which is not a σ-group of degree 3 share this property with their ancestor R. Consequently, the descendant tree $\mathcal{T}^{(1)}\langle 8,3\rangle$ [28, Fig. 3.1, p. 419] is entirely forbidden for $\mathrm{G}_2^{(2)}F$ of cyclic cubic fields F, and only the abelian root $\langle 4,2\rangle$ and its terminal immediate descendant $\langle 8,4\rangle$ are admissible. Similarly, the complete coclass tree $\mathcal{T}^{(1)}\langle 625,7\rangle$ is forbidden for $\mathrm{G}_5^{(2)}F$ of cyclic cubic fields F. Table 7.1 was computed by Algorithm 11, rather than Algorithm 12. □

Algorithm 11. (Action of the Trace of an Automorphism.)
Input: degree `d`, compact presentation `sP` of group `G`.
Code: implemented as an intrinsic procedure `TraceAut()`.

```
intrinsic TraceAut(d::RngIntElt,sP::[RngIntElt]) {}
G := PCGroup(sP); // G as group with power-conjugate presentation
Id := ""; // identifier of G
if CanIdentifyGroup(Order(G)) then
    Id := IdentifyGroup(G);
end if;
n := Ngens(G); // number of generators of G
IG := Identity(G); // neutral element of G
DG := DerivedSubgroup(G); // commutator subgroup of G
A := AutomorphismGroup(G);
T,AP,R := PermutationRepresentation(A); // much CPU time (slow)
c := 0;
o := 0;
w := 0;
s := 0;
for a in AP do
    c := c+1; // counter of all automorphisms
    if (Order(a) eq d) then
        o := o+1; // sigma-aut of assigned degree
        bw := true;
        bs := true;
        for i in [1..n] do // let trace act on all generators
            tr := G.i^0;
            for j in [0..d-1] do
```

```
                tr := tr*(G.i@((a@@T)^j));
            end for; // j
            bw := bw and (tr in DG); // weak
            bs := bs and (tr eq IG); // strong
        end for; // i
        if (bw eq true) then
            w := w+1; // weak sigma-aut
        end if; // bw
        if (bs eq true) then
            s := s+1; // strong sigma-aut
        end if; // bs
    end if; // d
end for; // a
printf"%o c=%o, o=%o, w=%o, s=%o\n",Id,c,o,w,s;
end intrinsic; // TraceAut
```

Output: Identifier `Id` and all counters `c`, `o`, `w`, `s` of various σ-automorphisms.

The following algorithm checks whether the Frattini quotient $G/\Phi(G)$ of an assigned p-group G has an action by the cyclic group C_3 of order 3. It can easily be modified by replacing $C_3 \simeq \langle 3, 1\rangle$ with another critical group, for instance, the symmetric group $S_3 \simeq \langle 6, 1\rangle$ of order 6 or the metacyclic group $M_5 \simeq \langle 20, 3\rangle$ of order 20. The Magma function `pQuotient()` with parameters $G, p, 1$ returns the Frattini quotient $Q = G/\Phi(G)$ of G together with the natural projection $qp : G \to Q$. The composition of mappings runs from the left to the right in Magma, so `A.j*qp` means $qp \circ A_j$, and the inverse image $qp^{-1}(Q_k)$ is denoted by `Q.k@@qp`, so $B = (qp \circ A_j \circ qp^{-1})(Q_k)$ is the image of Q_k under the composite mapping $qp \circ A_j \circ qp^{-1} : Q \to Q$. In the (rare) case that the subgroup $U \leq \mathrm{Aut}(Q)$ cannot be identified, due to a huge order, the group G is considered admissible, in order to avoid an erroneous elimination.

Algorithm 12. (Action on the Frattini Quotient.)
Input: group `G`, prime `p`.
Code: implemented as a function `IsAdmissible()` with boolean return value.

```
IsAdmissible := function(G,p);
    A := AutomorphismGroup(G);
    Q,qp := pQuotient(G,p,1);
    Lj := [];
    for j in [1..Ngens(A)] do
        Lk := [];
        for k in [1..Ngens(Q)] do
            B := (A.j*qp)(Q.k@@qp);
            Append(~Lk,<Q.k,B>);
        end for; // k
        h := hom<Q->Q|Lk>;
        Append(~Lj,h);
```

```
    end for; // j
    AQ := AutomorphismGroup(Q);
    U := sub<AQ|Lj>;
    o := Order(U);
    sN := [o,0];
    Boole := true;
    if CanIdentifyGroup(o) then
        sN := IdentifyGroup(U);
        PCU := SmallGroup(sN[1],sN[2]);
        Critical := SmallGroup(3,1); // cyclic group of order 3
        SU := Subgroups(PCU);
        Boole := false;
        for i in [1..#SU] do
            if IsIsomorphic(Critical,SU[i]`subgroup) then
                Boole := true;
                break;
            end if;
        end for; // i
    end if;
    return Boole;
end function; // IsAdmissible
```

Output: `true`, if there is an action of `Critical` on the Frattini quotient of `G`.

7.2. Cyclic Cubic Fields of Type (2,2)

We are now in the position to give a new proof of Derhem's result on cyclic cubic fields F with $\mathrm{Cl}_2F \simeq (2,2)$, and to add a trivial supplement on the case $\mathrm{Cl}_2F \simeq (2,2,2)$.

Theorem 20. *(**Derhem, 1988**, [14].)*
*For a cyclic cubic field F with 2-class group $\mathrm{Cl}_2F \simeq (2,2)$, there are only two possibilities for the second 2-class group $\mathrm{G}_2^{(2)}F$, either the **abelian group** $\langle 4,2\rangle$ with $\varkappa(F) = (000)$, $\tau(F) = [(1)^3]$, $\ell_2F = 1$ or the **quaternion group** $\langle 8,4\rangle$ with $\varkappa(F) = (123)$, $\tau(F) = [(2)^3]$, $\ell_2F = 2$.*

A cyclic cubic field F can never have a 2-class group $\mathrm{Cl}_2F \simeq (2,2,2)$.

Proof. These are immediate consequences of items 1. and 2. in Theorem 19. □

Table 7.2 shows that cyclic cubic fields F with elementary 2-class group Cl_2F of type $(2,2)$ arise randomly without regard to the prime decomposition of the conductor c of $F/\mathbb{Q}$. There even occur cases where several, $n = 2$, members of a multiplet are simultaneously of type $(2,2)$.

Table 7.2.: Various Multiplets of Cyclic Cubic Fields of Type $(2,2)$

v_3c	c	Factors	t	m	n	$\mathrm{G}_2^{(2)}F_{c,\mu}$
0	163	prime	1	1	1	$\langle 4,2\rangle$
0	277	prime	1	1	1	$\langle 8,4\rangle$
0	679	$7,97$	2	2	1	$\langle 4,2\rangle$
0	703	$19,37$	2	2	1	$\langle 8,4\rangle$
2	1 413	$3^2,157$	2	2	1	$\langle 4,2\rangle$
2	711	$3^2,79$	2	2	1	$\langle 8,4\rangle$
2	2 169	$3^2,241$	2	2	2	$\langle 4,2\rangle,\langle 4,2\rangle$
0	6 349	$7,907$	2	2	2	$\langle 4,2\rangle,\langle 4,2\rangle$
2	1 899	$3^2,211$	2	2	2	$\langle 8,4\rangle,\langle 4,2\rangle$
0	3 667	$19,193$	2	2	2	$\langle 8,4\rangle,\langle 4,2\rangle$
2	1 197	$3^2,7,19$	3	4	1	$\langle 4,2\rangle$
0	3 913	$7,13,43$	3	4	1	$\langle 4,2\rangle$
2	6 489	$3^2,7,103$	3	4	2	$\langle 4,2\rangle,\langle 4,2\rangle$
0	6 643	$7,13,73$	3	4	2	$\langle 4,2\rangle,\langle 4,2\rangle$

7.3. Cyclic Cubic Fields of Type (5,5)

Example 8. In the range $1 < c < 1\,000\,000$ of conductors, there are 481 occurrences of $\mathrm{Cl}_5F \simeq (5,5)$. The dominating part of 463 fields (96%) has $\mathrm{G}_5^{(2)}F \simeq \langle 125,3\rangle$ with six total transfer kernels $\varkappa(F) = (000000)$. The leading, respectively trailing, example of the dominant part is

$$c = 6\,901 = 67\cdot 103, \text{ respectively } c = 96\,733 = 7\cdot 13\cdot 1063.$$

Exceptions occur for the following 18 conductors only, confirming Corollary 3.

All these 481 cyclic cubic fields have 5-class towers of length $\ell_5F = 2$. For the group $\mathrm{G}_5^{(2)}F \simeq \langle 125,3\rangle$ of coclass 1 with cyclic commutator subgroup of order 5, this follows from a theorem by Blackburn [9]. In the exceptional cases with groups $\mathrm{G}_5^{(2)}F \simeq \langle 3125,i\rangle$, $i \in \{9,12,14\}$, it is a consequence of the fact that a metabelian Schur σ-group cannot be the second derived quotient of a non-metabelian 5-group. Therefore, we always have $\mathrm{G}_5^{(2)}F = \mathrm{G}_5^{(\infty)}F$ and $\ell_5F = 2$.

The fixed point capitulation problem by Olga Taussky 1970 [44, Rem. 1, p. 438], which has first been solved in 2011 by five imaginary quadratic fields [28, § 3.5.2, p. 448], and later in 2015 by eleven cyclic quartic fields [6, Thm. 4.4, Tbl. 4–5], now also has nine solutions with cyclic cubic fields possessing the identity permutation type $\varkappa = (123456)$ and the group $\mathrm{G}_5^{(2)}F \simeq \langle 3125,14\rangle$.

The numerical results are in perfect accordance with Theorem 19. However, it should be pointed out that the abelian group $\langle 25,2\rangle$ with $\ell_5F = 1$ and the unbalanced groups $\langle 3125,i\rangle$, $i \in \{3,10\}$, did not occur as $\mathrm{G}_5^{(2)}F$, up to now.

Table 7.3.: Exceptional Singlets of Cyclic Cubic Fields of Type $(5,5)$

No.	c	Factors	t	m	n	$\mathrm{G}_5^{(2)}F$	$\varkappa(F)$
1	66 313	13, 5101	2	2	1	$\langle 3125, 12\rangle$	a 6-cycle
2	68 791	prime	1	1	1	$\langle 3125, 14\rangle$	the identity
3	77 971	103, 757	2	2	1	$\langle 3125, 14\rangle$	the identity
4	87 409	7, 12487	2	2	1	$\langle 3125, 12\rangle$	a 6-cycle
5	199 621	prime	1	1	1	$\langle 3125, 9\rangle$	two 3-cycles
6	317 853	9, 35317	2	2	1	$\langle 3125, 12\rangle$	a 6-cycle
7	425 257	7, 79, 769	3	4	1	$\langle 3125, 14\rangle$	the identity
8	464 191	7, 13, 5101	3	4	1	$\langle 3125, 12\rangle$	a 6-cycle
9	481 537	7, 68791	2	2	1	$\langle 3125, 14\rangle$	the identity
10	545 797	7, 103, 757	3	4	1	$\langle 3125, 14\rangle$	the identity
11	596 817	9, 13, 5101	3	4	1	$\langle 3125, 12\rangle$	a 6-cycle
12	619 119	9, 68791	2	2	1	$\langle 3125, 14\rangle$	the identity
13	678 303	9, 75367	2	2	1	$\langle 3125, 12\rangle$	a 6-cycle
14	701 739	9, 103, 757	3	4	1	$\langle 3125, 14\rangle$	the identity
15	767 623	prime	1	1	1	$\langle 3125, 9\rangle$	two 3-cycles
16	786 681	7, 9, 12487	3	4	1	$\langle 3125, 12\rangle$	a 6-cycle
17	894 283	13, 68791	2	2	1	$\langle 3125, 14\rangle$	the identity
18	909 229	487, 1867	2	2	1	$\langle 3125, 14\rangle$	the identity

Theorem 21. *(σ-groups of degree 3 with type $(5,5)$)*
The transfer kernel type of a pro-5 group $\mathfrak{G}$ with abelianization $\mathfrak{G}/\mathfrak{G}' \simeq (5,5)$ which is a σ-group of degree 3 is restricted to the following admissible types
- (000000),
- *two 3-cycles,*
- *a 6-cycle,*
- *the identity,*
- *three 2-cycles,*

and descendant types of (000000).

Proof. These are immediate consequences of item 3. in Theorem 19. □

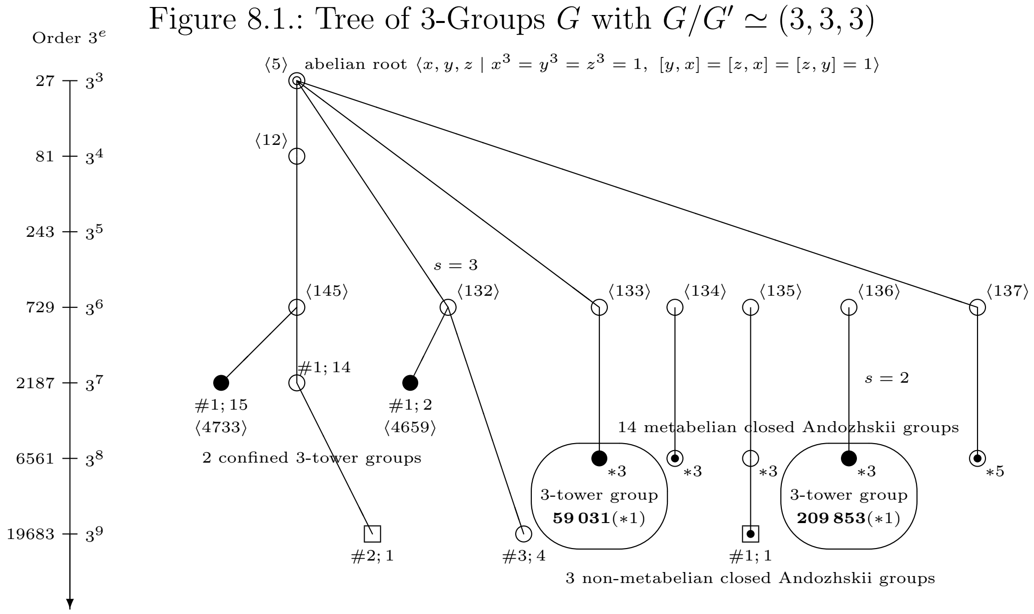
Corollary 3. *(Cyclic cubic fields of type $(5,5)$)*
The second 5-class group $\mathrm{G}_5^{(2)}F$ of a cyclic cubic field F with 5-class group $\mathrm{Cl}_5F \simeq (5,5)$ is restricted to the isomorphism classes of the following groups
- $\langle 25, 2\rangle$,
- $\langle 125, 3\rangle$,
- *a descendant of* $\langle 3125, 3\rangle$,
- $\langle 3125, 9\rangle$,
- $\langle 3125, 12\rangle$,
- $\langle 3125, 14\rangle$,
- *a descendant of* $\langle 3125, 10\rangle$.

All these finite 5-groups are σ-groups of degree 3.

8. Closed Andozhskii Groups

According to Koch and Venkov [22], *Schur σ-groups S* are known to be mandatory for realizations $S \simeq \mathrm{Gal}(\mathrm{F}_p^\infty k/k)$ by p-class field towers of *imaginary* quadratic fields k, with an odd prime p. They possess a balanced presentation $d(S) = r(S)$ with coinciding generator rank $d(S) = \dim_{\mathbb{F}_p} H^1(S, \mathbb{F}_p)$ and relation rank $r(S) = \dim_{\mathbb{F}_p} H^2(S, \mathbb{F}_p)$, and an automorphism $\sigma \in \mathrm{Aut}(S)$ acting as inversion $x \mapsto x^{-1}$ on the commutator quotient $S/[S,S]$. However, in the older literature, for instance Shafarevich [43, § 6, pp. 88–91], there also appear Schur groups with balanced presentation, but without a generator inverting σ-automorphism, and they are called *closed*, according to the original terminology by Schur. In the present chapter, we are interested in finite closed 3-groups given by Andozhskii [1, 2], whose position in the descendant tree of 3-groups G with elementary tricyclic commutator quotient $G/[G,G]$ is illuminated in Figure 8.1.

Figure 8.1.: Tree of 3-Groups G with $G/G' \simeq (3,3,3)$



In § 8.1, we identify the 17 *closed Andozhskii groups*. In § 8.2, we show that two of them can be realized as Galois groups of 3-class field towers of *cyclic cubic fields*. Incidentally, we point out that related but not closed 3-groups (see $j = 2$

in Table 8.1) are realized by lots of 3-class field towers over *totally complex S_3-fields K*, which are unramified extensions of imaginary quadratic fields $k = \mathrm{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 k \simeq (3,3)$, capitulation type H.4, $\varkappa(k) \sim (4111)$, and three abelian type invariants $\alpha(k) \sim (111, 111, 111, 21)$ of rank 3.

8.1. Identification of Closed Andozhskii Groups

Theorem 22. *Among the finite* 3*-groups G with commutator quotient $G/G' \simeq (3,3,3)$, there exist precisely* 14 *metabelian closed groups S of order $\#(S) = 3^8$ with identifiers*

$$S \simeq \langle 6561, 217700 + i\rangle \text{ where } 1 \le i \le 6 \text{ or } 10 \le i \le 17, \qquad (8.1.1)$$

and 3 *non-metabelian closed groups S of order $\#(S) = 3^9$ with identifiers*

$$S \simeq \langle 6561, 217700 + i\rangle - \#1; 1 \text{ where } 7 \le i \le 9. \qquad (8.1.2)$$

They possess a trivial Schur multiplier $M(S) = H_2(S, \mathbb{Q}/\mathbb{Z}) = 0$ and a balanced presentation $d(S) = r(S)$ with coinciding generator rank $d(S) = \dim_{\mathbb{F}_p} H^1(S, \mathbb{F}_p)$ and relation rank $r(S) = \dim_{\mathbb{F}_p} H^2(S, \mathbb{F}_p)$. The class is $\mathrm{cl}(S) = 3$ for soluble length $\mathrm{sl}(S) = 2$ and $\mathrm{cl}(S) = 4$ for $\mathrm{sl}(S) = 3$.

Proof. By a search in the SmallGroups database [8], extended to order 3^9 by the p-group generation algorithm [40, 41], the finite closed Andozhskii 3-groups S are identified. There are 14 hits of order $\#(S) = 3^8$ and only three hits of order $\#(S) = 3^9$. The non-metabelian groups are characterized by their identifiers defined by the ANUPQ package [17]. □

Corollary 4. *Each of the* 17 *closed Andozhskii groups S in Theorem 22 shares a common Artin pattern $(\varkappa, \alpha)$ with its unique ancestor $A \simeq \langle 729, 130 + j\rangle$, as shown in Table 8.1, where*

$$j = \begin{cases} 3 & \text{for } 1 \le i \le 3, \\ 4 & \text{for } 4 \le i \le 6, \\ 5 & \text{for } 7 \le i \le 9, \\ 6 & \text{for } 10 \le i \le 12, \\ 7 & \text{for } 13 \le i \le 17. \end{cases} \qquad (8.1.3)$$

Proof. According to the theorem on the antitony of the Artin pattern [32, §§ 5.1–5.4, pp. 78–87], it suffices to calculate the *stable* transfer kernels of the five ancestors A of the 17 closed groups in Theorem 22. They are of order $\#(A) = 3^6$ and have much simpler presentations. It turns out that the transfer kernels are *harmonically balanced*, that is, permutations in the symmetric group S_{13}. □

Table 8.1.: TKT $\varkappa$ and AQI α of Closed Andozhskii 3-Groups S, $S/S' \simeq (3,3,3)$

j	$\varkappa$ respectively α													Action
2	1	2	3	6	11	9	10	4	13	5	12	8	7	$\langle 24,12\rangle$
	22	22	22	211	22	22	22	211	22	211	22	22	211	
3	9	2	3	6	10	8	4	11	12	13	5	1	7	$\langle 6,2\rangle$
	22	22	22	211	22	211	211	211	22	211	211	22	211	
4	1	7	3	2	10	9	4	11	5	12	13	8	6	$\langle 4,1\rangle$
	22	22	22	22	22	22	211	211	211	211	22	22	22	
5	9	7	3	2	4	8	11	10	13	5	12	1	6	$\langle 3,1\rangle$
	22	22	22	22	211	211	22	211	22	211	22	22	22	
6	12	7	3	2	9	5	8	1	10	11	4	13	6	$\langle 6,2\rangle$
	22	22	22	22	22	22	22	22	22	22	22	211	22	
7	10	7	3	6	8	4	1	9	13	5	12	11	2	$\langle 24,3\rangle$
	22	22	22	211	211	22	22	22	22	211	22	211	22	

We recall that the transfer kernel (capitulation kernel) $\ker(T_i)$ of an Artin transfer homomorphism $T_i : G/G' \to M_i/M_i'$ [32] from a 3-group G with $G/G' \simeq (3,3,3)$ to one of its 13 maximal subgroups M_i, $1 \le i \le 13$, is called of *Taussky type* A, if the meet $\ker(T_i) \cap M_i > 1$ is non-trivial, and of *Taussky type* B, if $\ker(T_i) \cap M_i = 1$.

Corollary 5. *The Artin patterns* $(\varkappa(A), \alpha(A))$ *of the six groups* $A = \langle 729, 130+j\rangle$ *with* $2 \le j \le 7$ *share the common property that the Taussky type of their transfer kernels* $\ker(T_i)$ *is determined uniquely by the AQI* M_i/M_i' *of the corresponding maximal subgroup* M_i*:*

$$\begin{aligned} \alpha(A)_i = M_i/M_i' \simeq (211) &\iff \varkappa(A)_i = \ker(T_i) \cap M_i > 1, \ \textit{Taussky type } \mathrm{A}, \\ \alpha(A)_i = M_i/M_i' \simeq (22) &\iff \varkappa(A)_i = \ker(T_i) \cap M_i = 1, \ \textit{Taussky type } \mathrm{B}, \end{aligned} \tag{8.1.4}$$

for all $1 \le i \le 13$*. Here the abelian quotient invariants are logarithmic.*

Proof. This follows by comparing the 1-dimensional transfer kernels in Table 8.1 to the planes in Table 6.10.

□

8.2. Realization as 3-Class Field Tower Groups

Since the groups in Theorem 22 are non-σ groups, they cannot be realized by any quadratic field, neither imaginary nor real. Therefore, we investigated the possible Galois actions (Table 8.1) on the five ancestors $A = \mathrm{SmallGroup}(729, 130+j)$. It turned out that the unique non-metabelian case $j = 5$ can only be realized by *cyclic cubic* fields, $j = 4$ by cyclic quartic fields, and $j \in \{3,6,7\}$ by *cyclic cubic or sextic* fields. We show that two certain metabelian descendants S for $j \in \{3,6\}$

can actually be realized as Galois groups $\mathrm{Gal}(\mathrm{F}_3^\infty K/K) \simeq S$ of maximal unramified pro-3 extensions of two cyclic cubic fields F with conductors $c \in \{\mathbf{59031}, \mathbf{209853}\}$ and 3-class group $\mathrm{Cl}_3 F \simeq (3,3,3)$.

Theorem 23. *If a number field $K/\mathbb{Q}$ with elementary tricyclic 3-class group $\mathrm{Cl}_3 K \simeq (3,3,3)$ possesses the Artin pattern $(\varkappa(K), \alpha(K))$ with harmonically balanced capitulation type $\varkappa(K) \sim (9,2,3,6,10,8,4,11,12,13,5,1,7)$ and abelian type invariants $\alpha(K) \sim ((22)^3, 211, 22, (211)^3, 22, (211)^2, 22, 211)$, then $K/\mathbb{Q}$ must be cyclic cubic or sextic, and has a metabelian 3-class field tower with automorphism group*

$$\mathrm{Gal}(\mathrm{F}_3^\infty K/K) \simeq \langle 6561, 217700 + i \rangle, \ 1 \le i \le 3. \tag{8.2.1}$$

Theorem 24. *If a number field $K/\mathbb{Q}$ with elementary tricyclic 3-class group $\mathrm{Cl}_3 K \simeq (3,3,3)$ possesses the Artin pattern $(\varkappa(K), \alpha(K))$ with harmonically balanced capitulation type $\varkappa(K) \sim (12,7,3,2,9,5,8,1,10,11,4,13,6)$ and abelian type invariants $\alpha(K) \sim ((22)^{11}, 211, 22)$, then $K/\mathbb{Q}$ must be cyclic cubic or sextic, and has a metabelian 3-class field tower with automorphism group*

$$\mathrm{Gal}(\mathrm{F}_3^\infty K/K) \simeq \langle 6561, 217700 + i \rangle, \ 10 \le i \le 12. \tag{8.2.2}$$

Proof. Theorems 23 and 24 are immediate consequences of Table 8.1. □

In April 2002, we used the Voronoi algorithm and the Euler product method in order to compute the 15851 cyclic cubic fields F with conductors $c_{F/\mathbb{Q}} < 10^5$ and their class numbers $1 \le h_F \le 1953$. Among the fields, 4785 occur as singlets, 7726 in doublets, 3132 in quartets, and 208 in octets. Twenty years later, in July 2022, we have confirmed these results, extended by the class group structures, as reported in detail in § 1.3. The cyclic cubic fields F were constructed as ray class fields over the rational number field, using the class field theoretic routines of Magma [23]. Additionally, we constructed the 13 unramified cyclic cubic relative extensions E_i/F, whenever the class group of F was $\mathrm{Cl}_3 F \simeq (3,3,3)$, which was the primary goal for the reconstruction in view of the intended realization of Andozhskii groups.

Example 9. Indeed, there are 15851 fields in 9457 multiplets: 4785 singlets, 3863 doublets, 783 quartets, and 26 octets, as shown in Table 1.2. One of the four fields with $c = \mathbf{59\,031} = 3^2 \cdot 7 \cdot 937$, respectively $c = \mathbf{209\,853} = 3^2 \cdot 7 \cdot 3331$, both quartets of Category I and Graph 2, was the unique lucky hit of the Artin pattern in the Main Theorem 23, respectively 24. The density of such fields is horrificly sparse. The latter conductor $c = 209\,853$ lies beyond our systematic investigations.

Part IV.

Future Research

Among the *remaining challenges* for future activities of the young researchers we mention the following open problems.

- Theorems 7 – 11 have been obtained by thorough observations and analysis of computational results. It would be desirable to prove these theorems by exploiting results in the later chapters of Ayadi's Thesis [3] on principal factors (Parry invariants) and bicyclic bicubic subfields of the 3-genus field.

- An interesting supplement would be the determination of the 3-class field tower group of cyclic cubic fields in quartets of Category IV, Graphs 1–3, and in octets with $t = 4$.

- It should still be within the reach of Magma to investigate the 6, respectively 31, unramified cyclic quintic extensions E/F of **cyclic quintic fields** F with elementary 5-class group of bicyclic type $\mathrm{Cl}_5 F \simeq (5,5)$, respectively tricyclic type $\mathrm{Cl}_5 F \simeq (5,5,5)$, and to establish theoretical foundations for the capitulation $\ker(T_{E/F})$ and 5-class field tower F_5^∞.

Conclusion

1. Construction Process and Statistics of Cyclic Cubic Fields

1.1. Computational Techniques

In 2002, we computed the regulators $\mathrm{Reg}(F)$ and class numbers $h(F)$ of all 15851 cyclic cubic fields F with conductor $c < 10^5$, based on generating polynomials in $\mathbb{Z}[X]$ by Marie Nicole Gras [20, p. 90], that is, $P(X) = X^3+X^2+\frac{1-c}{3}X-\frac{c(3+a)-1}{27}$ for $\gcd(3,c) = 1$, simplified by a Tschirnhausen transformation, respectively $P(X) = X^3 - \frac{c}{3}X - \frac{ac}{27}$ for $9 \mid c$, where $c = \frac{a^2+27b^2}{4}$. For the *regulator*, we used the two-dimensional Voronoi algorithm [45], implemented in Delphi (i.e. object oriented Pascal), for the *class number*, we used the analytic Euler product formula.

In 2013, we computed a list of the first 251 conductors c of cyclic cubic number fields $F = F_c$, having a 3-class group $\mathrm{Cl}_3 F$ of type $(3,3)$. These conductors occur in the range $657 \le c \le 26\,523$. They are divisible by two or three primes, $2 \le t \le 3$. Inspired by Amandine Leriche and Henri Cohen [12, pp. 337–339], we used a very convenient parametrization of generating polynomials $P(X) = X^3 - 3cX - cu \in \mathbb{Z}[X]$ for the cyclic cubic fields F. These results were the basis of preliminary statistics in § 1.2.

In 2016, we extended the list of conductors up to $c \le 10^5$ without strict book keeping of exact numbers. The construction process was completely different: Instead of using generating polynomials, we employed the class field package of Magma [10, 11, 23], as implemented by Claus Fieker [16], and obtained cyclic cubic fields $F = F_c$ as subfields of *ray class fields* $\mathbb{Q}_c$ modulo admissible conductors c over the rational number field $\mathbb{Q}$. Then, we constructed the *unramified cyclic cubic extensions* E/F and determined the 3-capitulation kernel of F in each superfield E (the transfer kernel type, TKT, $\varkappa$), and the structure of the 3-class group $\mathrm{Cl}_3 E$ of every extension E (the transfer target type, TTT, τ). Finally, we applied our new algorithm, the strategy of *pattern recognition* via Artin transfers [38], in order to determine the second 3-class group $\mathfrak{M} = \mathrm{G}_3^2 F$ of the number field F by seeking the *Artin pattern* $\mathrm{AP} = (\tau, \varkappa)$, which consists of targets τ and kernels $\varkappa$ of transfer homomorphisms, in the descendant tree $\mathcal{T}(R)$ with root $R = \langle 9, 2\rangle$. The tree was constructed by successive applications of the p-group generation algorithm by Mike

F. Newman [40] and Eamonn A. O'Brien [41], which is also described in [21].

In 2022, we repeated and double checked the computations of 2002 and 2016 with modern techniques, very thorough book keeping and extensive statistics. Some metabelian 3-groups in Table 6.1 could only be identified unambiguously by means of the commutator subgroup. This required the computation of the 3-class group $\mathrm{Cl}_3 F_3^{(1)}$ of the first Hilbert 3-class field of many cyclic cubic fields F, which is of absolute degree $[F_3^{(1)} : \mathbb{Q}] = 3 \cdot 9 = 27$. In this manner, we had to distinguish between $\mathfrak{M} = \langle 729, 34|35|36\rangle$ with $\mathfrak{M}' = (3,3,3,3)$ and $\mathfrak{M} = \langle 729, 37|38|39\rangle$ with $\mathfrak{M}' = (3,3,9)$. On the other hand, for extensive classes of cyclic cubic fields F, initially we did not know whether $\mathfrak{M} = \langle 81, 9\rangle$, which was known to arise for certain bicyclic biquadratic fields [5], or $\mathfrak{M} = \langle 243, 28|29|30\rangle$, known for real quadratic fields [26]. Since $\mathrm{Cl}_3 F_3^{(1)}$ turned out to be $(3,9)$, rather than $(3,3)$, the group $\langle 81, 9\rangle$ could be eliminated (cannot occur for cyclic cubic fields).

1.2. Statistics

For the calculation of relative frequencies (percentages), we need some absolute frequencies as references. We restrict to 3-class group $\mathrm{Cl}_3 F$ of type $(3,3)$. Due to the explanations given in Remark 1 and Definition 4, the 251 conductors in the range $657 \le c \le 26\,523$ of our computations in 2013 consist of
• 19 with $t = 2$ and $9 \mid c$, corresponding to 38 fields, and
69 with $t = 2$ and $\gcd(c, 3) = 1$, corresponding to 138 fields,
since fields with $t = 2$ arise in doublets, and
• 85 with $t = 3$ and $9 \mid c$, corresponding to $10 \cdot 3 + 7 \cdot 2 + 68 \cdot 4 = 316$ fields, and
78 with $t = 3$ and $\gcd(c, 3) = 1$, corresponding to $14 \cdot 3 + 11 \cdot 2 + 53 \cdot 4 = 276$ fields,
since fields with $t = 3$ may arise in triplets (Category I), doublets (Category II) or quartets (Category III) — always with respect to type $(3,3)$ alone.

There is a single instance with *dominating two-stage towers*: For $t = 2$, $9 \mid c$, the extraspecial group $\langle 27, 4\rangle$ is populated most densely with $20/38 = 53\%$.

However, *abelian towers* are most frequent in all other situations: For $t = 2$, $\gcd(c, 3) = 1$, the single stage 3-class towers are dominating with $96/138 = 70\%$. For $t = 3$, $9 \mid c$, the single stage 3-class towers are dominating with $220/316 = 70\%$. For $t = 3$, $\gcd(c, 3) = 1$, the single stage 3-class towers are dominating with $184/276 = 67\%$. Note that the 3-class tower of F consists of a single stage if and only if $\mathrm{G}_3^2 F \simeq \langle 9, 2\rangle$ is abelian.

This *preliminary legacy statistics* is superseded by our most recent computations in 2022, presented in the *final statistics* of the Tables 1 and 2.

Table 1.: Statistics of Second 3-Class Group $\mathfrak{M}$ and Tower Length $\ell_3 F$

Abs.	Rel.	Ref.	$\mathfrak{M}$	cc	$\ell_3 F$	t	m	v	Remark
4785	30.2%	15851	$\langle 1,1\rangle$	0	0	1	1	0	Singlets
7726	48.7%	15851				2	2		Doublets
									regular
6910	**89.4**%	7726	$\langle \mathbf{3},\mathbf{1}\rangle$	0	1			0	$h_3 = 3$
704	9.1%	7726							$h_3 = 9$
420	**59.7**%	704	$\langle \mathbf{9},\mathbf{2}\rangle$	1	1			1	$h_3 = 9$
284	40.3%	704	$\langle 27,4\rangle$	1	2			2	$h_3 = 9$
112	1.4%	7726							$h_3 > 9$
									singular
74	**66.1**%	112	$\langle \mathbf{81},\mathbf{3}\rangle$	2	2			3	$h_3 = 27$
									super-singular
18	**16.1**%	112	$\langle \mathbf{243},\mathbf{14}\rangle$	2	2			≥ 4	$h_3 = 27$
5	4.5%	112	$\langle 243,13\rangle$	2	≥ 2			4	$h_3 = 27$
3	2.7%	112	$\langle 729,12\rangle$	3	≥ 2			4	$h_3 = 27$
3	2.7%	112	$\langle 729,17\vert 20\rangle$	3	≥ 2			4	$h_3 = 27$
2	1.8%	112	$\langle 243,15\rangle$	2	2			4	$h_3 = 27$
7	6.3%	112					≥ 4	$h_3 > 27$	
3132	19.8%	15851				3	4		Quartets
2316	73.9%	3132							Cat. III
1820	**78.6**%	2316	$\langle \mathbf{9},\mathbf{2}\rangle$	1	1				III/1–4
148	6.4%	2316							III/5
82	**55.4**%	148	$\langle \mathbf{243},\mathbf{28}\vert\mathbf{29}\vert\mathbf{30}\rangle$	1	2				III/5
32	21.6%	148	$\langle 81,7\rangle$	1	2				III/5
16	10.8%	148	$\langle 243,27\rangle$	1	2				III/5
10	6.8%	148	$\langle 243,25\rangle$	1	2				III/5
8	5.4%	148	$\langle 729,37\vert 38\vert 39\rangle$	2	≥ 2				III/5
124	5.4%	2316							III/6
120	**96.8**%	124	$\langle \mathbf{81},\mathbf{7}\rangle$	1	2				III/6
2	1.6%	124	$\langle 2187,250\rangle$	2	≥ 2				III/6
2	1.6%	124	$\langle 2187,251\vert 252\rangle$	2	≥ 2				III/6
136	5.9%	2316							III/7
108	**79.4**%	136	$\langle \mathbf{81},\mathbf{7}\rangle$	1	2				III/7
12	8.8%	136	$\langle 729,41\rangle$	2	≥ 2				III/7
8	5.9%	136	$\langle 2187,65\vert 67\rangle$	3	≥ 3				III/7
4	2.9%	136	$\langle 729,37\vert 38\vert 39\rangle$	2	≥ 2				III/7
4	2.9%	136	$\langle 2187,253\vert 254\rangle$	2	≥ 2				III/7
28	1.2%	2316							III/8
20	**71.4**%	28	$\langle \mathbf{729},\mathbf{34}\vert\mathbf{35}\vert\mathbf{36}\rangle$	2	≥ 2				III/8
60	2.6%	2316							III/9
56	**93.3**%	60	$\langle \mathbf{81},\mathbf{7}\rangle$	1	2				III/9

2. Main results

The 2-class tower of cyclic cubic fields F with 2-class group $\mathrm{Cl}_2 F \simeq (2,2)$ is completely settled by our Theorems 18, 19, 20, and by the detailed criteria distinguishing the two cases $\ell_2 F \in \{1,2\}$ in [14, 13]. The second fruitful application of restrictions enforced by Galois action in Theorem 21 and Corollary 3 justifies the concrete numerical results for cyclic cubic fields F with 5-class group $\mathrm{Cl}_5 F \simeq (5,5)$ in Table 7.3. In particular, Olga Taussky's famous 1970 *fixed point capitulation* problem [28, § 3.5.2, p. 448] which has first been solved by five *imaginary quadratic* fields [28], and later by eleven *cyclic quartic* fields [6, Thm. 4.4, Tbl. 4–5], now also has nine solutions with *cyclic cubic* fields having the identity permutation type $\varkappa = (123456)$. There only remains the open question whether the abelian group $\langle 25,2\rangle$ or descendants of the groups $\langle 3125,i\rangle$ with $i \in \{3,10\}$ will be realized

Table 2.: Second 3-Class Group $\mathfrak{M}$ and Tower Length $\ell_3 F$ Continued

Abs.	Rel.	Ref.	$\mathfrak{M}$	cc	$\ell_3 F$	t	m	v	Remark
392	12.5%	3132							Cat. I
152	38.8%	392							I/1
36	**23.7**%	152	$\langle \mathbf{81}, \mathbf{10} \rangle$	1	2				I/1
21	13.8%	152	$\langle 243, 8 \rangle$	2	2				I/1
18	11.8%	152	$\langle 81, 8 \rangle$	1	2				I/1
18	11.8%	152	$\langle 81, 14 \rangle$	2	2				I/1
18	11.8%	152	$\langle 243, 28 \vert 29 \vert 30 \rangle$	1	2				I/1
9	5.9%	152	$\langle 243, 25 \rangle$	1	2				I/1
9	5.9%	152	$\langle 729, 54 \rangle$	2	3				I/1
5	3.3%	152	$\langle 243, 46 \rangle$	3	2				I/1
4	2.6%	152	$\langle 243, 47 \rangle$	3	2				I/1
3	2.0%	152	$\langle 2187, 303 \rangle$	2	≥ 2				I/1
240	61.2%	392							I/2
96	**40.0**%	240	$\langle \mathbf{81}, \mathbf{10} \rangle$	1	2				I/2
48	20.0%	240	$\langle 81, 8 \rangle$	1	2				I/2
48	20.0%	240	$\langle 81, 14 \rangle$	2	2				I/2
30	12.5%	240	$\langle 243, 8 \rangle$	2	2				I/2
3	1.3%	240	$\langle 243, 42 \rangle$	3	2				I/2
3	1.3%	240	$\langle 729, 52 \rangle$	2	≥ 2				I/2
3	1.3%	240	$\langle 2187, 301 \vert 305 \rangle$	2	3				I/2
1	0.4%	240	$\langle 6561, 217701 \rangle$	5	2				I/2
368	11.7%	3132							Cat. II
188	51.1%	368							II/1
76	**40.4**%	188	$\langle \mathbf{81}, \mathbf{7} \rangle$	1	2				II/1
76	**40.4**%	188	$\langle \mathbf{81}, \mathbf{13} \rangle$	2	2				II/1
6	3.2%	188	$\langle 729, 37 \vert 38 \vert 39 \rangle$	2	≥ 2				II/1
4	2.1%	188	$\langle 729, 41 \rangle$	2	3				II/1
2	1.1%	188	$\langle 729, 372 \rangle$	3	≥ 2				II/1
2	1.1%	188	$\langle 2187, 248 \vert 249 \rangle$	2	≥ 2				II/1
2	1.1%	188	$\langle 2187, 65 \vert 67 \rangle$	3	≥ 3				II/1
2	1.1%	188	$\langle 2187, 4595 \rangle$	4	≥ 2				II/1
2	1.1%	188	$\langle 2187, 4606 \rangle$	4	≥ 2				II/1
2	1.1%	188	$\langle 2187, 5577 \rangle$	3	3				II/1
180	48.9%	368							II/2
66	**36.7**%	180	$\langle \mathbf{81}, \mathbf{7} \rangle$	1	2				II/2
66	**36.7**%	180	$\langle \mathbf{81}, \mathbf{13} \rangle$	2	2				II/2
16	8.9%	180	$\langle 729, 37 \vert 38 \vert 39 \rangle$	2	≥ 2				II/2
4	2.2%	180	$\langle 729, 41 \rangle$	2	3				II/2
2	1.1%	180	$\langle 2187, 253 \vert 254 \rangle$	2	≥ 2				II/2
2	1.1%	180	$\langle 2187, 4595 \rangle$	4	≥ 2				II/2
2	1.1%	180	$\langle 2187, 5577 \rangle$	3	3				II/2
56	1.8%	3132							Cat. IV
208	1.3%	15851				4	8		Octets

as $\mathrm{G}_5^{(2)} F$ by cyclic cubic fields F with bigger conductors $c > 10^6$.

Our investigation of the 3-class tower of cyclic cubic fields F with *elementary tricyclic* 3-class group $\mathrm{Cl}_3 F \simeq (3,3,3)$ is a striking novelty. Similar attempts with imaginary quadratic fields of type $(3,3,3)$, where all capitulation kernels are of order $\#L = 3$ (lines), successfully yielded the Artin pattern $\mathrm{AP} = (\tau, \varkappa)$ by means of arithmetic computations [30, § 7.2, Tbl. 2–4, pp. 308–311] but were doomed to failure group theoretically, since the complexity of descendant trees became unmanageable [30, § 7.4, p. 312], [31, § 10, p. 54], [36, Thm. 8.2, p. 174], [34, § 8, pp. 98–99], [35, § 2, Example, p. 6]. Therefore, we were delighted that cyclic cubic fields of type $(3,3,3)$ impose much less severe requirements on the second 3-class group $\mathfrak{M} = \mathrm{G}_3^{(2)} F$, since capitulation kernels of order $\#P = 9$ (planes) and even $\#O = 27$ (full space) are admissible.

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