Der Wissenschaftsfonds.

Class Towers and Capitulation over Quadratic Fields

Conference:	West Coast Number Theory 2013
Place:	Asilomar Conference Grounds, Pacific Grove
Venue:	Monterey, California, USA
Date:	Wednesday, December 18, 2013
Time:	14:00 – 14:25, p. m.
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Dedication:	to the memory of O. Taussky-Todd (∗ 1906 – † 1995)

The situation around 1930 and today

It was 80 years ago that Emil Artin asked Olga Taussky whether she was still pursuing her investigations of the "hopeless capitulation problem" of algebraic number fields. Artin's pessimistic attitude might seem astonishing, since a few years earlier he had used his reciprocity law (1927) to translate the number theoretic capitulation problem into a purely group theoretic problem concerning the kernels of transfer homomorphisms from finite non-abelian p-groups to their subgroups (1929). Consequently, it should be expected that the classification of all finite p-groups up to a fixed order according to their transfer kernel types would solve the capitulation problem for all number fields up to a certain bound of absolute discriminants. However, the crux at about 1930 was the lack of a systematic approach to classifying finite p-groups. We had to wait for half a century until the idea of visualizing finite p-groups as vertices in rooted digraphs (directed trees of successive class-c quotients of p-groups) was born around 1980 by Charles Leedham-Green and Mike Newman and the closely related p-group generation algorithm was implemented in full generality by Eamonn O'Brien in 1990. Extensive digraphs of p-groups can be constructed nowadays with the aid of the ANUPQ package of GAP and MAGMA. The digraphs can be structured by splitting in branches of isoclinism families or in coclass subtrees. Consequently, we are now able to look forward optimistically to powerful new attacks against the hard-boiled capitulation problem.

Acknowledgement

The author's research is supported by the Austrian Science Fund (FWF): P 26008-N25.

§ 0. Summary of Aims

Section 1

1.1. To summarize facts about
fixed point capitulation and
to provide **arithmetical realizations**.

1.2. To show related facts concerning
abelianizations of p-rank 3.

Section 2

2.1. To present examples of **new capitulation types**
of complex and real quadratic fields
which didn't occur in **Brink**'s Thesis 1984 [2].

2.2. To show how the Galois groups of the
second Hilbert p-class fields
of quadratic fields are distributed.

Section 3

3.1. To disprove incorrect assertions of
Scholz/Taussky [8] and
Heider/Schmithals [5]
concerning some pretended two-stage towers which
actually turned out to be three-stage towers.

3.2. On the one hand,
to underpin the caveats of **Brink/Gold** [3],
who had doubts about Scholz/Taussky's claim,
but on the other hand,
to show that the arguments given by Brink/Gold
are unable to invalidate the Scholz/Taussky claim.

§ 1. Kernels and Targets of Artin Transfers

Definition 1.1.
$p \geq 2$ a prime number,
G a pro-p group of generator rank $d(G) = 2$,
$H_1, \ldots, H_{p+1} \lhd G$ its maximal subgroups,
$T_i = T_{G,H_i} : G/G' \to H_i/H_i'$, $gG' \mapsto$

$$T_i(gG') = \begin{cases} g^p H_i' & \text{if } g \in G \setminus H_i, \\ g^{1+t+\cdots+t^{p-1}} H_i' & \text{if } g \in H_i, \end{cases}$$

for any $t \in G \setminus H_i$ and $1 \leq i \leq p+1$,
the *Artin transfers* from G to the H_i [1].

The family $\varkappa(G) = (\ker(T_i))_{1 \leq i \leq p+1}$
is called the *transfer kernel type* (TKT) of G
[T2], [T4].

The family $\tau(G) = (H_i/H_i')_{1 \leq i \leq p+1}$
is called the *transfer target type* (TTT) of G
[T3], [T4].

[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

Theorem 1.1. (Fixed point capitulation)

(1) $p = 3$: **(negative result)**
The TKT of a pro-3 group G
with $G/G' \simeq (3, 3)$ cannot be the identity

$$\varkappa(G) = (1, 2, 3, 4),$$

that is $\ker(T_i) = H_i/G'$ for each $1 \le i \le 4$.
(A. Scholz and O. Taussky, 1934 [8])

(2) $p \ge 5$: **(positive result)**
The TKT of the metabelian p-group
$G = \Phi_6(221)_a$, which is of coclass $\mathrm{cc}(G) = 2$
and has $G/G' \simeq (p, p)$, is the identity

$$\varkappa(G) = (1, 2, \ldots, p, p + 1).$$

(O. Taussky-Todd, 1970 [9])

(3) $p = 2$: **(positive result)**
The TKT of the quaternion group $G = \langle 8, 4\rangle$,
which is of coclass $\mathrm{cc}(G) = 1$ and thus has
$G/G' \simeq (2, 2)$, is the identity permutation

$$\varkappa(G) = (1, 2, 3).$$

(H. H. Kisilevsky, 1976 [6])

Note that $\Phi_6(221)_a \simeq \langle 3125, 14\rangle$ for $p = 5$, and $\Phi_6(221)_a \simeq \langle 16807, 7\rangle$ for $p = 7$.

In the sequel, $\mathrm{G}_p^2(K)$ denotes the Galois group $\mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ of the maximal metabelian unramified p-extension $\mathrm{F}_p^2(K)$ of an algebraic number field K.

Corollary 1.1. (Actual arithmetical realizations)

(1) $p = 5$:

The minimal absolute discriminant $|D|$ of a complex quadratic field $\mathbb{Q}(\sqrt{D})$ with 5-class group $\mathrm{Cl}_5(K) \simeq (5,5)$ and second 5-class group $\mathrm{G}_5^2(K) \simeq \langle 3125, 14\rangle$ is given by $|D| = 89\,751$. Furthermore, the first five occurrences of discriminants having such a second 5-class group $\mathrm{G}_5^2(K)$ are

$$D \in \{-89\,751,\ -235\,796,\ -1\,006\,931,\ -1\,996\,091,\ -2\,187\,064\}.$$

(D. C. Mayer, T. Bembom and C. Fieker, 2012)

(2) $p = 2$:

The minimal absolute discriminant $|D|$ of a complex quadratic field $\mathbb{Q}(\sqrt{D})$ with 2-class group $\mathrm{Cl}_2(K) \simeq (2,2)$ and second 2-class group $\mathrm{G}_2^2(K) \simeq \langle 8, 4\rangle$ is given by $|D| = 120$ (squarefree radicand -30). (H. H. Kisilevsky, 1976 [6])

Corollary 1.2. (D. C. Mayer, M. R. Bush, 2012) Since $G = \langle 3125, 14\rangle$ is a Schur σ-group, any algebraic number field K with second 5-class group $\mathrm{G}_5^2(K) \simeq G$ has a 5-class field tower of length $\ell_5(K) = 2$.

Theorem 1.2. (Abelianizations of p-rank 3)

(1) $p = 3$: **(negative result)**
The TTT of a pro-3 group G with $G/G' \simeq (3,3)$ cannot consist of four abelianizations with 3-rank 3,

$$\tau(G) = \Big((3,3,3)^4\Big),$$

that is $H_i/H_i' \simeq (3,3,3)$ for each $1 \le i \le 4$.
(D. C. Mayer, 2009 [T3])

(2) $p \ge 5$: **(positive result)**
The TTT of the metabelian p-group $G = \Phi_6(221)_a$, which is of coclass $\mathrm{cc}(G) = 2$ and has $G/G' \simeq (p,p)$, consists of $p+1$ abelianizations with p-rank 3,

$$\tau(G) = \Big((p,p,p)^{p+1}\Big).$$

(D. C. Mayer, 2012 [T4])

(3) $p = 2$: **(negative result)**
The TTT of a non-abelian pro-2 group G with $G/G' \simeq (2,2)$ never contains an abelianization $H_i/H_i' \simeq (2,2,2)$ with 2-rank 3. It is always of the form $\tau(G) = \Big((2^{n-1}),(2,2)^2\Big)$, when $|G| = 2^n$. The quaternion group $G = \langle 8,4\rangle$ is the unique exception with TTT $\tau(G) = \Big((4)^3\Big)$ consisting of three cyclic groups.
(D. C. Mayer, 2009 [T1])

Remark 1.1. Let $p \geq 5$ be a prime.

(1) To the very best of our knowledge, $G = \Phi_6(221)_a$ is the unique p-group with $G/G' \simeq (p,p)$, which has the identity $\varkappa(G) = (1,2,\ldots,p,p+1)$ as its TKT.

(2) However, there are at least two p-groups
$$G \in \{\Phi_6(221)_a,\ \Phi_6(11111)\}$$
with $G/G' \simeq (p,p)$, which have the TTT
$$\tau(G) = \Big((p,p,p)^{p+1}\Big).$$
The group $G = \Phi_6(11111)$ can be distinguished by its TKT $\varkappa(G) = (0^{p+1})$. It cannot occur as the second p-class group $\mathrm{G}_p^2(K)$ of a complex quadratic field.

Note that $\Phi_6(11111) \simeq \langle 3125, 3\rangle$ for $p = 5$, and $\Phi_6(11111) \simeq \langle 16807, 3\rangle$ for $p = 7$.

§ 2. New Capitulation Types

The history of determining 3-principalization types of quadratic fields $K = \mathbb{Q}(\sqrt{D})$ is shown in the following two tables 1 and 2 [T1].

The numbers of fields investigated in 2010 and 2011 clearly beat all earlier numbers by far. Without our principalization algorithm via class group structure [T3] they would have been completely out of reach.

TABLE 1. History of investigating quadratic fields of type $\mathrm{Cl}_3(K) \simeq (3, 3)$

History		complex		real	
authors	year	range	number	range	number
Scholz, Taussky	1934	$-10\,000 < D$	2		
Heider, Schmithals	1982	$-20\,000 < D$	13	$D < 1 \cdot 10^5$	5
Brink	1984	$-96\,000 < D$	66		
Mayer	1989	$-30\,000 < D$	35		
Mayer	1991			$D < 2 \cdot 10^5$	16
Mayer	2009	$-10^5 < D$	156	$D < 10^6$	149
Mayer	2010	$-10^6 < D$	2 020	$D < 10^7$	2 576

TABLE 2. History of investigating quadratic fields of type $\mathrm{Cl}_3(K) \simeq (9, 3)$

History		complex		real	
authors	year	range	number	range	number
Scholz, Taussky	1934	$-10\,000 < D$	2		
Heider, Schmithals	1982	$-20\,000 < D$	7		
Mayer	1989	$-30\,000 < D$	9		
Mayer	2011	$-10^6 < D$	875	$D < 10^7$	271

§ 2.1. Capitulation of p-Classes

Definition 2.1.
K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa(K) = (\ker(j_i))_{1 \le i \le p+1}$
is called the *p-capitulation type* of K [8], [T2].

The family $\tau(K) = (\mathrm{Cl}_p(L_i))_{1 \le i \le p+1}$
is called the *p-class group type* of K [T4].

Theorem 2.1. (Artin, 1929 [1])
The p-capitulation type $\varkappa(K)$, resp. p-class group type $\tau(K)$, of K coincides with the TKT $\varkappa(G)$, resp. TTT $\tau(G)$, of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $2 \le n \le \infty$.

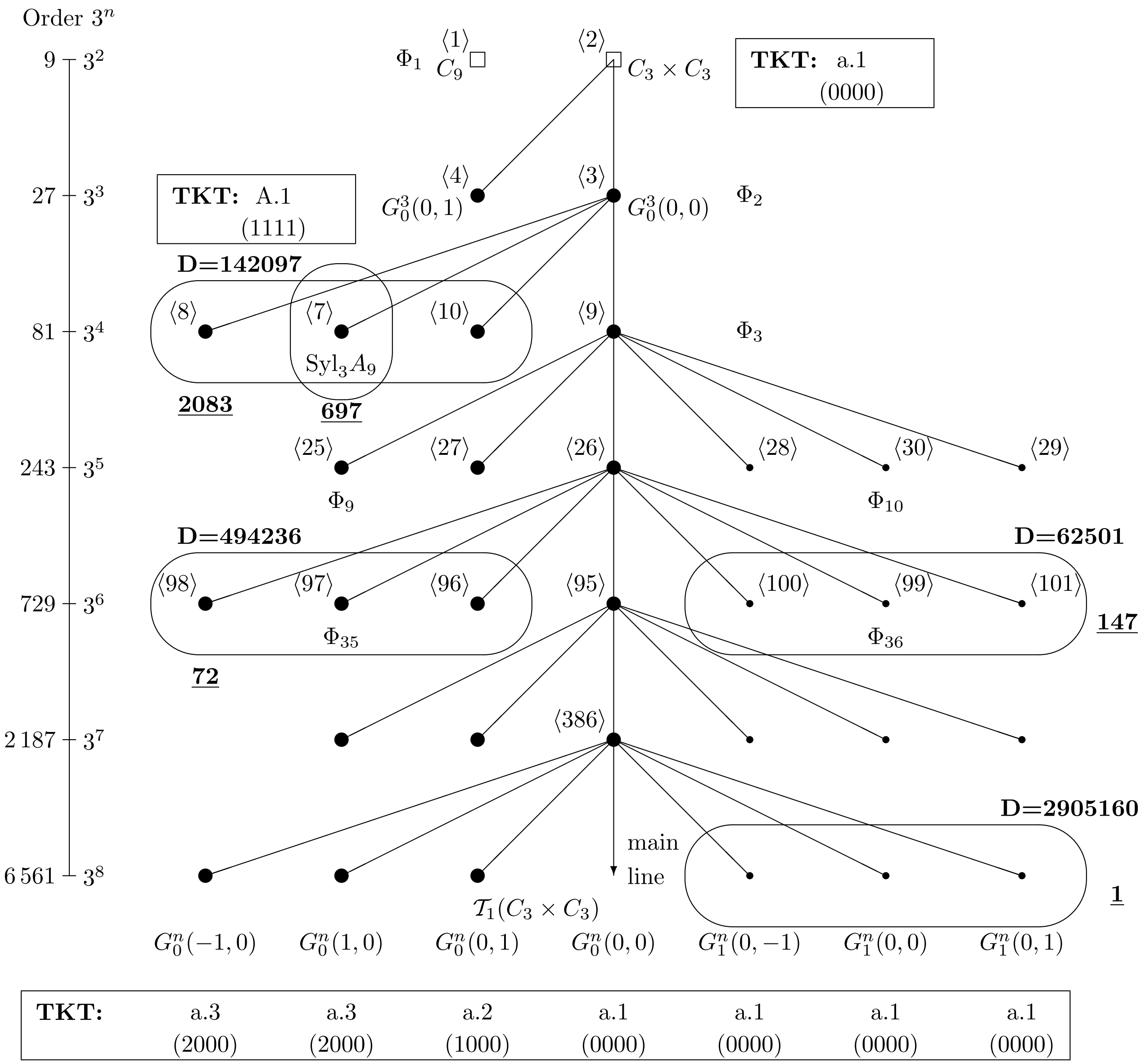
$$\begin{array}{ccccc} & & j_{L_i|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L_i) & \\ \text{Artin} & \downarrow & & \downarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H_i/H_i' & \text{isomorphism} \\ & & T_{G,H_i} & & \end{array}$$

§ 2.2. Distribution of $\mathrm{G}_3^2(K)$ on $\mathcal{G}(3,1)$

Only the second 3-class groups $\mathrm{G}_3^2(K)$ of real quadratic fields $K=\mathbb{Q}(\sqrt{D})$ can occur on the unique coclass tree $\mathcal{T}_1$ with abelian root $C_3 \times C_3 \simeq \langle 9,2\rangle$ of coclass graph $\mathcal{G}(3,1)$. All 3-groups of maximal class are metabelian, and have TKTs in section a or A. The corresponding 3-class towers are of exact length $\ell_3(K)=2$.

The actual distribution of the 2576 second 3-class groups $G_3^2(K)$ of real quadratic number fields $K=\mathbb{Q}(\sqrt{D})$ of type $(3,3)$ with discriminant $0<D<10^7$ is represented by underlined boldface counters of hits of vertices surrounded by the adjacent oval. The results verify the selection rule, that only every other branch is populated by groups $G_3^2(\mathbb{Q}(\sqrt{D}))$, for $D>0$, and underpin the *weak leaf conjecture* that mainline vertices are forbidden for second 3-class groups of quadratic fields, if they have terminal successors of the same TKT. A remarkably different behavior is revealed by certain biquadratic fields $\mathbb{Q}(\sqrt{-3},\sqrt{D})$ of Eisenstein/Scholz/Reichardt type.

FIGURE 1. Distribution of second 3-class groups on coclass graph $\mathcal{G}(3,1)$



Only the second 3-class groups $\mathrm{G}_3^2(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ can occur on the unique coclass tree $\mathcal{T}_1$ with abelian root $C_3 \times C_3 \simeq \langle 9, 2\rangle$ of coclass graph $\mathcal{G}(3,1)$. All 3-groups of maximal class are metabelian, and have TKTs in section a or A. The corresponding 3-class towers are of exact length $\ell_3(K) = 2$.

The actual distribution of the 2576 second 3-class groups $G_3^2(K)$ of real quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ of type $(3,3)$ with discriminant $0 < D < 10^7$ is represented by underlined boldface counters of hits of vertices surrounded by the adjacent oval. The results verify the selection rule, that only every other branch is populated by groups $G_3^2(\mathbb{Q}(\sqrt{D}))$, for $D > 0$, and underpin the *weak leaf conjecture* that mainline vertices are forbidden for second 3-class groups of quadratic fields, if they have terminal successors of the same TKT. A remarkably different behavior is revealed by certain biquadratic fields $\mathbb{Q}(\sqrt{-3}, \sqrt{D})$ of Eisenstein/Scholz/Reichardt type.

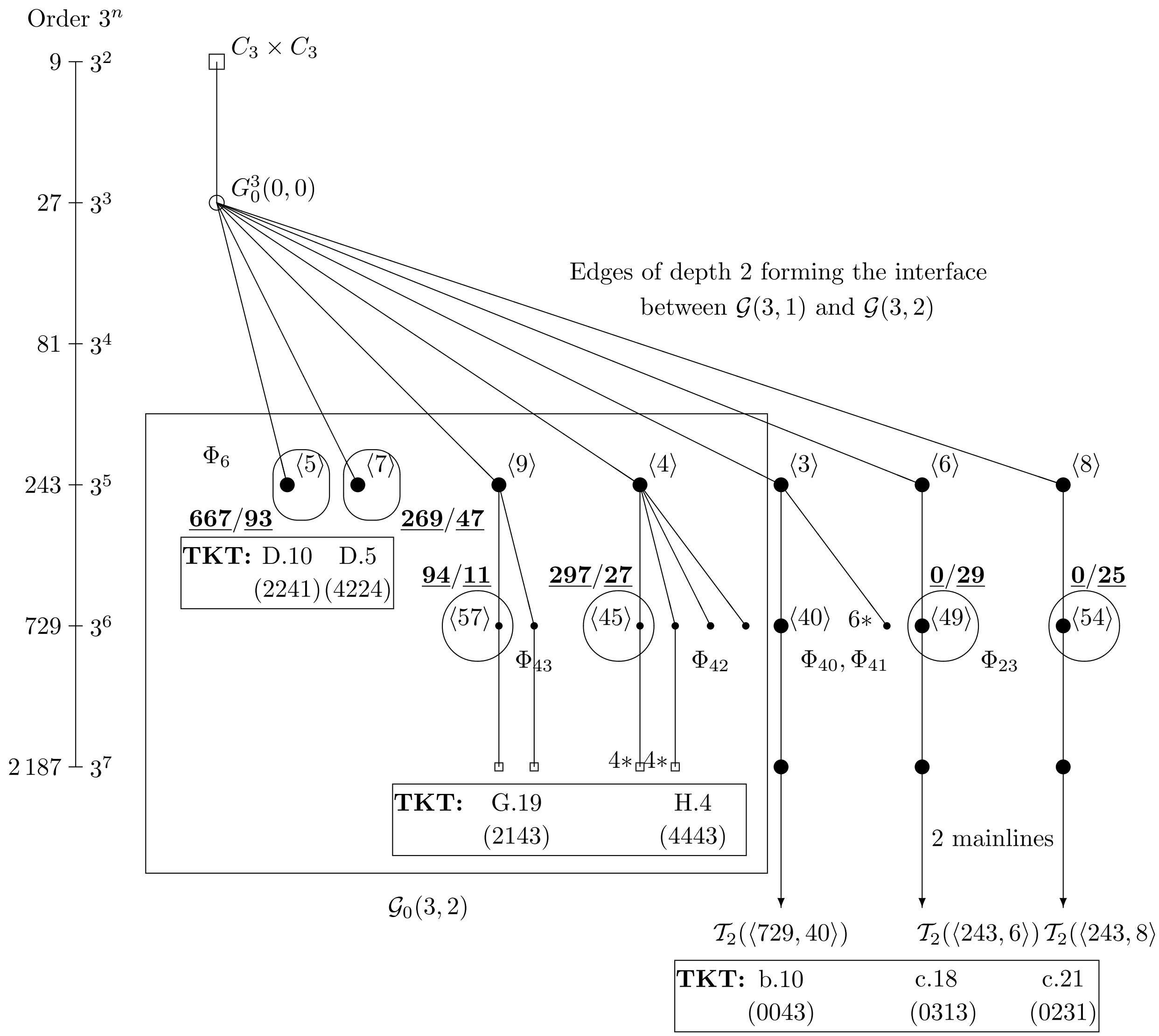
§ 2.3. Distribution of $\mathrm{G}_3^2(K)$ on $\mathcal{G}_0(3,2)$

Until 2006 it was unknown that second 3-class groups $\mathrm{G}_3^2(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ can occur on the sporadic part $\mathcal{G}_0(3,2)$ of coclass graph $\mathcal{G}(3,2)$. These 3-groups of second maximal class have TKTs in section D or G or H. Exactly the two groups with TKT D.10 and D.5 are Schur σ-groups and the corresponding 3-class towers have exact length $\ell_3(K) = 2$.

Until 2008 it was unknown whether second 3-class groups $\mathrm{G}_3^2(K)$ of quadratic fields $K = \mathbb{Q}(\sqrt{D})$ are always terminal leaves of the metabelian skeleton graph. Then it turned out that there exist real quadratic fields whose second 3-class groups are mainline vertices with TKT c.18 and c.21.

The actual distribution of the 2020, resp. 2576, second 3-class groups $G_3^2(K)$ of complex, resp. real, quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ of type $(3,3)$ with discriminant $-10^6 < D < 10^7$ is represented by underlined boldface counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval.

FIGURE 2. Distribution of second 3-class groups on coclass graph $\mathcal{G}(3,2)$



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The actual distribution of the 2020, resp. 2576, second 3-class groups $G_3^2(K)$ of complex, resp. real, quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ of type $(3,3)$ with discriminant $-10^6 < D < 10^7$ is represented by underlined boldface counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval.

§ 2.4. Distribution of $\mathrm{G}_3^2(K)$ on $\mathcal{T}_2(\langle 243, 8\rangle)$

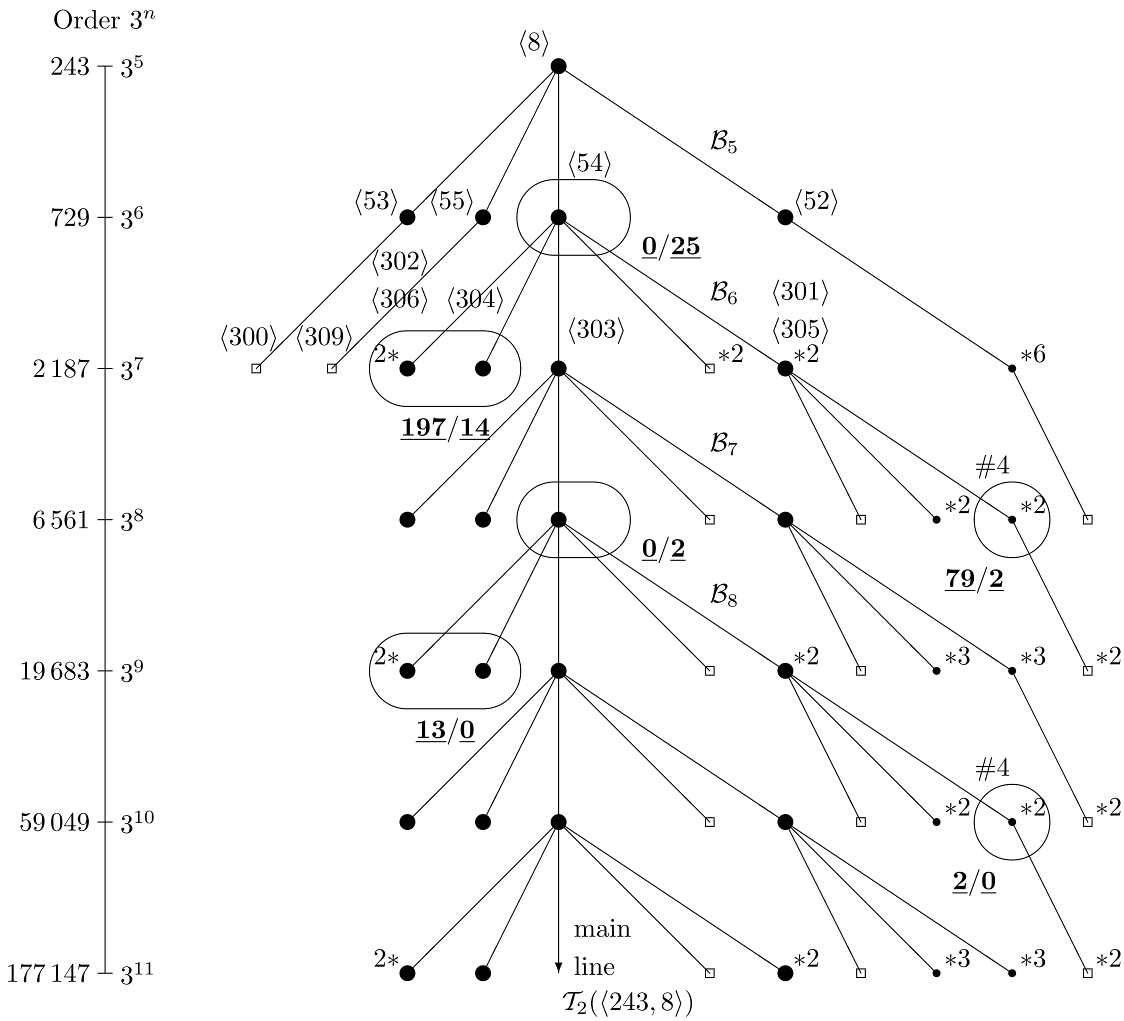
The structure of the complete coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$ as part of the coclass graph $\mathcal{G}(3, 2)$ is given up to order $3^{11} = 177\,147$ by the next figure. The branches are of depth 3 and periodic of length 2. The pre-period consists of $\mathcal{B}_5, \mathcal{B}_6$, the primitive period of $\mathcal{B}_7, \mathcal{B}_8$.

We have $G_3^2(\mathbb{Q}(\sqrt{D})) \in \mathcal{T}_2(\langle 243, 8\rangle)$ for 291 (14.4%) of the 2020 discriminants $-10^6 < D < 0$ and for 43 (1.7%) of the 2576 discriminants $0 < D < 10^7$. Since the TKT c.21, $\varkappa = (0231)$, of the mainline is *total* with $\varkappa(1) = 0$, there only occur $G_3^2(K)$ of *real* quadratic fields $K = \mathbb{Q}(\sqrt{D})$, $D > 0$, on the mainline.

The actual distribution of the 2020, resp. 2576, second 3-class groups $G_3^2(K)$ of complex, resp. real, quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ of type $(3, 3)$ with discriminant $-10^6 < D < 10^7$ is represented by underlined boldface counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval.

The realization of mainline vertices with TKT c.18 and c.21 as $G_3^2(K)$ is no violation of the weak leaf conjecture, since these vertices do not possess metabelian immediate descendants of the same TKT and higher defect of commutativity.

FIGURE 3. Distribution of second 3-class groups on coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$



TKT:	E.9	E.8	c.21	G.16	G.16	G.16
	(2231)	(1231)	(0231)	(4231)	(4231)	(4231)

§ 3. Exact Length of 3-Class Towers

Theorem 3.1. (Scholz & Taussky, 1934 [8])
The Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ has transfer kernel type E

$$\varkappa(G) = (2,3,3,4) \sim (2,4,3,4)$$

and the 3-class numbers of the non-Galois absolute cubic subfields $K_1,\ldots,K_4$ of the unramified cyclic cubic extension fields $L_1,\ldots,L_4$ of K are given by

$$(\mathrm{h}_3(K_i))_{1\le i\le 4} = (9,3,3,3)\,.$$

Corollary 3.1. (Mayer, 2009 [T3])
The Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ has transfer target type

$$\tau(G) = [(9,27),(3,9)^3].$$

[8] A. Scholz und O. Taussky, Die Hauptideale der kubischen Klassenkörper imaginär quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluß auf den Klassenkörperturm, *J. Reine Angew. Math.* **171** (1934), 19–41.

Definition 3.1. For a finite metabelian p-group $G = \langle x, y \rangle$ with generator rank $d(G) = 2$ and main commutator $s_2 = [y, x]$, the ideal

$$\mathfrak{A}(G) = \{f(X,Y) \in \mathbb{Z}[X,Y] \mid s_2^{f(x-1,y-1)} = 1\}$$

is called the *annihilator* of G.

Theorem 3.2. (Scholz & Taussky, 1934 [8])
The annihilator $\mathfrak{A}(G)$ of the Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over any quadratic field $K = \mathbb{Q}(\sqrt{D})$ with transfer kernel type E

$$\varkappa(G) = (2,3,3,4) \sim (2,4,3,4)$$

is one of the ideals

$$\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3 \rangle$$

with even $\alpha \geq 4$.

A Deep Mystery since 80 Years

Claim 3.1. (Scholz & Taussky, 1934 [8])
The 3-class field tower over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ terminates at the second stage,

$$\mathrm{F}_3^3(K) = \mathrm{F}_3^2(K),$$

resp. has length $\ell_3(K) = 2$.

Claim 3.2. (Heider & Schmithals, 1982 [5])
The 3-class field tower over any complex quadratic field $K = \mathbb{Q}(\sqrt{D})$ with 3-capitulation type E

$$\varkappa(K) = (2,3,3,4) \sim (2,4,3,4)$$

has length $\ell_3(K) = 2$.

Claim 3.2 on p. 20 of Heider and Schmithals [5] has been used in the table on p. 84 of our paper [7], where the rows Nr. 6, 8, 9, and 14 are marked by the symbol $\times$ to indicate a two-stage tower.

[5] F.-P. Heider und B. Schmithals, Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen, *J. Reine Angew. Math.* **336** (1982), 1–25.

A Caveat by Brink and Gold

Theorem 3.3. (Brink and Gold, 1987 [2,3])
The 3-groups with parametrized presentation
$M_2(\beta) = \langle x, y, s_2, s_3, t_3, u \mid$
$[y, x] = s_2, [s_2, x] = s_3, [s_2, y] = t_3,$
$[s_3, x] = s_2^{-3} s_3^{-3} t_3^6,\ [s_3, y] = u^2,\ [s_3, s_2] = u,$
$[t_3, x] = [t_3, y] = [t_3, s_2] = [t_3, s_3] = 1,\ t_3^3 = u,$
$x^3 = t_3^{-1},\ y^3 = s_2^{-3} s_3^{-1},\ s_2^{3^\beta} = s_3^{3^\beta} = u^3 = 1 \rangle$
have cyclic second derived subgroup $M_2(\beta)''$ of order 3, for all parameter values $\beta \geq 2$. Hence, they are non-metabelian with derived length

$$\mathrm{dl}(M_2(\beta)) = 3.$$

The annihilator ideal $\mathfrak{A}(G)$ of the metabelianization $G = M_2(\beta)/M_2(\beta)''$ is given by

$$\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3 \rangle$$

with even $\alpha = 2\beta \geq 4$.

Claim 3.3. (Brink and Gold, 1987 [2,3])
The groups $M_2(\beta)$ with $\beta \geq 2$ are possible candidates for Galois groups $\mathrm{Gal}(M|K)$ of unramified cyclic cubic extensions $M|\mathrm{F}_3^2(K)$ within the third Hilbert 3-class field $\mathrm{F}_3^3(K)$ over complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with 3-capitulation type E

$$\varkappa(K) = (2, 3, 3, 4) \sim (2, 4, 3, 4).$$

Crucial Ingredients for the Disproof

Definition 3.2. $p \geq 3$ an odd prime.
A pro-p group G is called a *σ-group*,
if it admits an automorphism $\sigma \in \mathrm{Aut}(G)$ acting as inversion $x \mapsto x^{-1}$ on the abelianization G/G'.

Theorem 3.4. (Artin, 1928 [4])
For any *quadratic* field $K = \mathbb{Q}(\sqrt{D})$,
the p-tower group $\mathrm{G}_p^{\infty}(K)$ and the higher p-class groups $\mathrm{G}_p^{n}(K)$, for $n \geq 2$, are σ-groups.

$p \geq 3$ an odd prime,
G a pro-p group,
$d(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(G, \mathbb{F}_p))$ the *generator rank* of G,
$r(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^2(G, \mathbb{F}_p))$ the *relation rank* of G.

Definition 3.3. A pro-p group G which satisfies the equation $r(G) = d(G)$ is said to have a *balanced presentation*, or to be a *Schur group*.

Theorem 3.5. (Shafarevich, 1963 [10])
The p-tower group $\mathrm{G}_p^{\infty}(K)$ of a *complex quadratic* field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, is a Schur group.

Theorem 3.6. (Mayer, Boston & Bush, 2012) There are exactly two non-isomorphic metabelian 3-groups G_1 and G_2 with transfer kernel type E

$$\varkappa(G_i) = (2,3,3,4) \sim (2,4,3,4)$$

and transfer target type

$$\tau(G_i) = [(9,27),(3,9)^3].$$

G_1 and G_2 do not have a balanced presentation. Further, there are exactly two non-isomorphic non-metabelian 3-groups H_1 and H_2 such that $G_i \simeq H_i/H_i''$. H_1 and H_2 are Schur σ-groups of derived length $\mathrm{dl}(H_i) = 3$.

Remark 3.1. The identifiers of these 3-groups are

$$G_1 \simeq \langle 2187, 302\rangle,$$
$$G_2 \simeq \langle 2187, 306\rangle$$

in the SmallGroups library, resp.

$$H_1 \simeq \langle 729, 54\rangle - \#2; 2,$$
$$H_2 \simeq \langle 729, 54\rangle - \#2; 6$$

in the ANUPQ package of GAP and MAGMA.

Corollary 3.6. (Mayer, Boston & Bush, 2012) The 3-class field tower over the complex quadratic field $\mathbb{Q}(\sqrt{-9748})$ terminates at the third stage,

$$\mathrm{F}_3^4(K) = \mathrm{F}_3^3(K) > \mathrm{F}_3^2(K),$$

resp. has exact length $\ell_3(K) = 3$.

Brink and Gold — Tidy !!!

Theorem 3.7. (Mayer and Newman, 2013) Brink and Gold's 3-groups $G = M_2(\beta)$ with parameter values $\beta \geq 2$ are of order $3^{2\beta+4}$, class $2\beta + 1$, and fixed coclass 3.

None of these groups has a balanced presentation and further they are all of transfer kernel type c

$$\varkappa(G) = (2, 0, 3, 4).$$

Their metabelianizations $M_2(\beta)/M_2(\beta)''$ are the mainline groups of order $3^{2\beta+3}$ on the tree $\mathcal{T}_2^*(\langle 243, 8\rangle)$.

Corollary 3.7. (Mayer and Newman, 2013) None of Brink and Gold's 3-groups $M_2(\beta)$, $\beta \geq 2$, can be the Galois group $\mathrm{Gal}(M|K)$ of an unramified cyclic cubic extension $M|\mathrm{F}_3^2(K)$ within the third Hilbert 3-class field $\mathrm{F}_3^3(K)$ over any complex quadratic field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with 3-capitulation type E

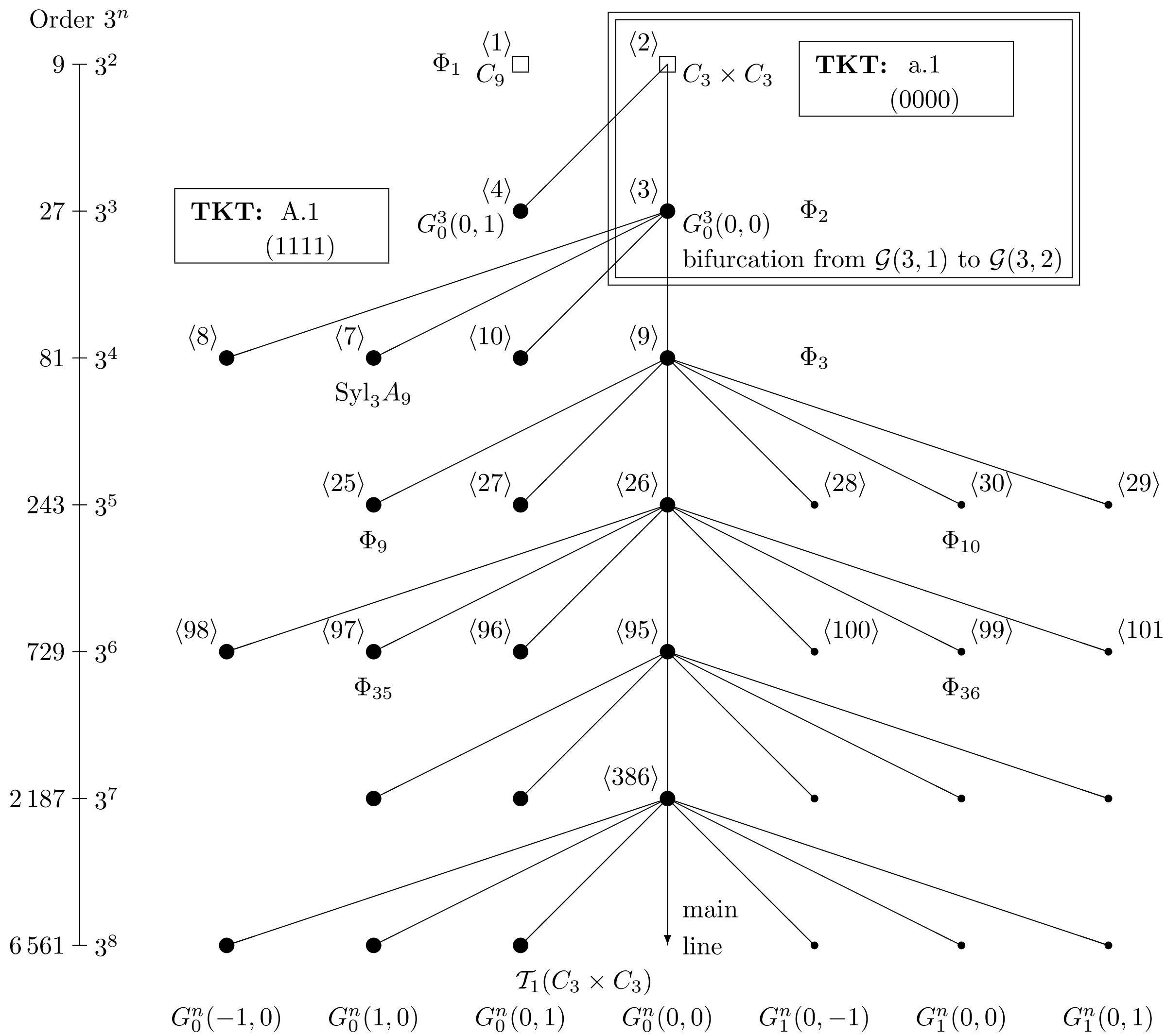
$$\varkappa(K) = (2, 3, 3, 4) \sim (2, 4, 3, 4).$$

§ 4. Proof of Theorem 3.6

§ 4.1. Starting Generation of 3-Groups

We start our search for 3-groups with TKT in section E at the abelian root $C_3 \times C_3 \simeq \langle 9, 2\rangle$ of the unique coclass tree $\mathcal{T}_1$ in coclass graph $\mathcal{G}(3,1)$. However, we leave this graph very quickly, since all 3-groups of maximal class have TKTs in sections a,A. The immediate descendant $G_0^3(0,0) \simeq \langle 27, 3\rangle$ gives rise to a bifurcation from $\mathcal{G}(3,1)$ to $\mathcal{G}(3,2)$, but the following mainline vertex $G_0^4(0,0) \simeq \langle 81, 9\rangle$ is coclass-settled and no further bifurcations can occur.

FIGURE 4. Starting 3-group generation at the top of coclass graph $\mathcal{G}(3,1)$



TKT:	a.3	a.3	a.2	a.1	a.1	a.1	a.1
	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

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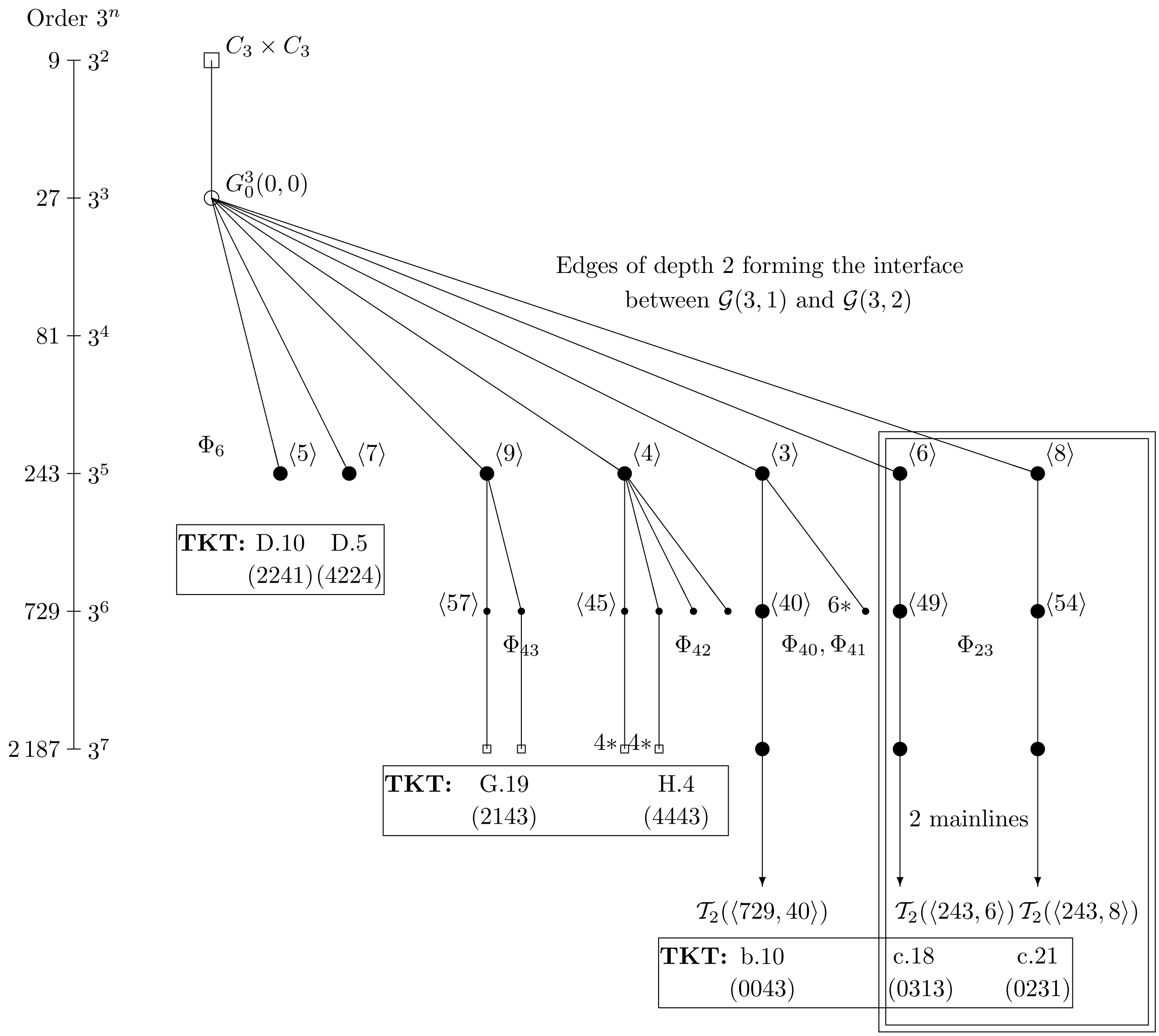
§ 4.2. TKT-Pruning $\mathcal{G}(3,2)$

The top vertices $\langle 243,5\rangle$ and $\langle 243,7\rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243,9\rangle$, resp. $\langle 243,4\rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243,3\rangle$ must contain a transposition, which is not the case for TKTs in sections c and E. Therefore, only descendants of $\langle 243,6\rangle$ and $\langle 243,8\rangle$ can have TKTs in sections c and E.

Figure 5. TKT-pruning the top of coclass graph $\mathcal{G}(3,2)$



The top vertices $\langle 243, 5\rangle$ and $\langle 243, 7\rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243, 9\rangle$, resp. $\langle 243, 4\rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243, 3\rangle$ must contain a transposition, which is not the case for TKTs in sections c and E. Therefore, only descendants of $\langle 243, 6\rangle$ and $\langle 243, 8\rangle$ can have TKTs in sections c and E.

§ 4.3. TKT-Pruning $\mathcal{T}_2(\langle 243, 8\rangle)$

Definition 4.1.
The *TKT-pruned descendant tree* $\mathcal{T}^*(\langle 243, 8\rangle)$ consists of all descendants G of the root $\langle 243, 8\rangle$ such that

(1) $\varkappa(G)$ is one of the TKTs c.21 or E.8 or E.9 (that is, we cancel all the trash with TKT G.16),

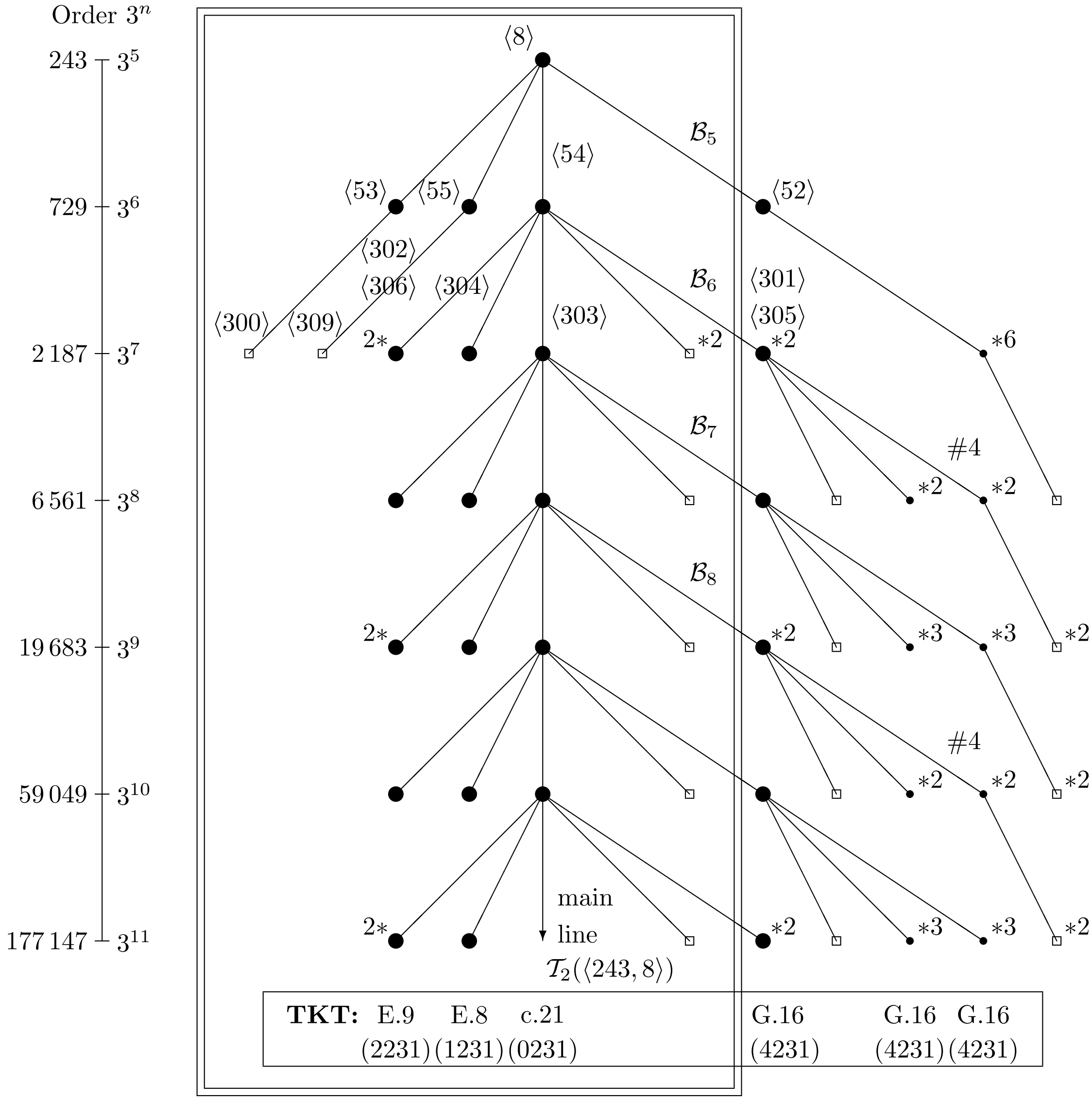
(2) if $\varkappa(G)$ is of TKT c.21 then G has descendants, (i.e., we omit terminal vertices with TKT c.21),

(3) G is a σ-group.

(See Figures 7,8.)

Remark 4.1.
The motivation for defining $\mathcal{T}^*(\langle 243, 8\rangle)$ is that Brink and Gold indicated a possible length $\ell_3(K) \geq 3$ for the field $K = \mathbb{Q}(\sqrt{-9748})$ with TKT E.9 for which Scholz and Taussky had claimed $\ell_3(K) = 2$.
(See [2], [3], and page 41 in [8].)

FIGURE 6. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$

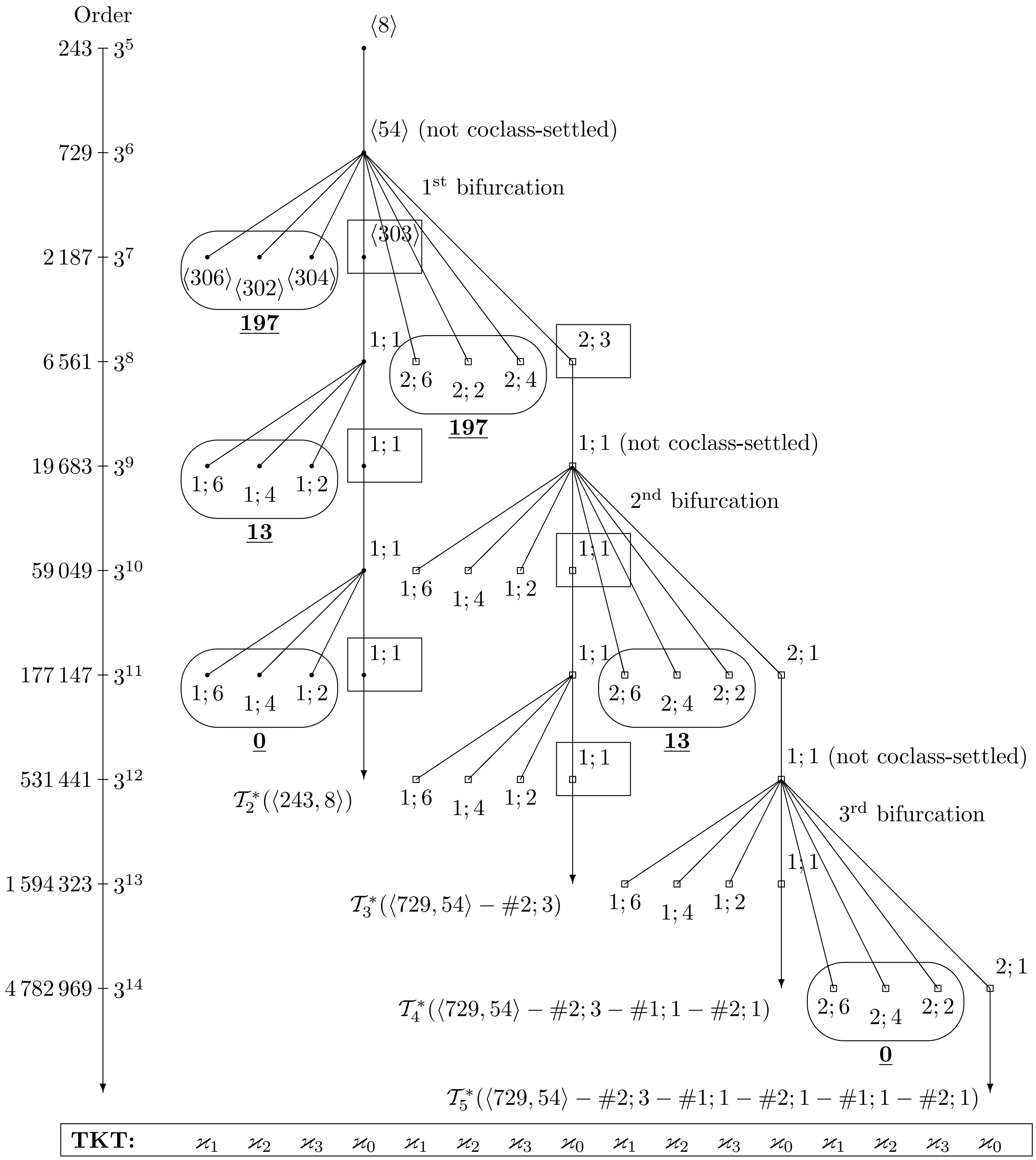


§ 4.4. Construction of $\mathcal{T}^*(\langle 243, 8\rangle)$

Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

The bifurcation at $\langle 729, 54\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

FIGURE 7. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, and identifiers of SmallGroups and ANUPQ



Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

§ 4.5. Biperiodic Structure of $\mathcal{T}^*(\langle 243, 8\rangle)$

We consider the intersections of $\mathcal{T}^*(\langle 243, 8\rangle)$ with coclass graphs $\mathcal{G}(3, r)$. We put

$$\mathcal{T}_2^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, 2)$$

and, for all $r \geq 3$,

$$\mathcal{G}_r^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, r).$$

Theorem 4.1. (*First periodicity*).
(See Figures 6 and 7,8.)

(1) $\mathcal{T}_2^*(\langle 243, 8\rangle)$ is a subtree of $\mathcal{T}^*(\langle 243, 8\rangle)$.
(2) All vertices are metabelian and unbalanced.
(3) Vertices of TKT c.21 form an infinite mainline with unique group $M_n^{(2)}$ of each order 3^n, $n \geq 5$.
(4) Every branch is of depth 1 and contains two groups $G_{n,1}^{(2)}, G_{n,2}^{(2)}$ of TKT E.9 and a single group $G_{n,3}^{(2)}$ of TKT E.8, each of order 3^n with odd $n \geq 7$.

Theorem 4.2. (*Second periodicity*).
(See Figures 7,8.)
For $3 \le r \le 5$,

(1) the graph $\mathcal{G}_r^*(\langle 243, 8\rangle)$ consists of
3 isolated vertices $S_k^{(r)}$, $1 \le k \le 3$,
and a subtree $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ of $\mathcal{T}^*(\langle 243, 8\rangle)$,

(2) $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ is isomorphic to $\mathcal{T}_2^*(\langle 243, 8\rangle)$ as a graph, and additionally, the two trees share the same distribution of TKTs,

(3) all vertices G of $\mathcal{G}_r^*(\langle 243, 8\rangle)$ are
non-metabelian of derived length $\mathrm{dl}(G) = 3$ with
cyclic second derived subgroup G'' of order 3^{r-2} contained in the centre $\zeta_1(G)$ of type $(3, 3^{r-1})$,

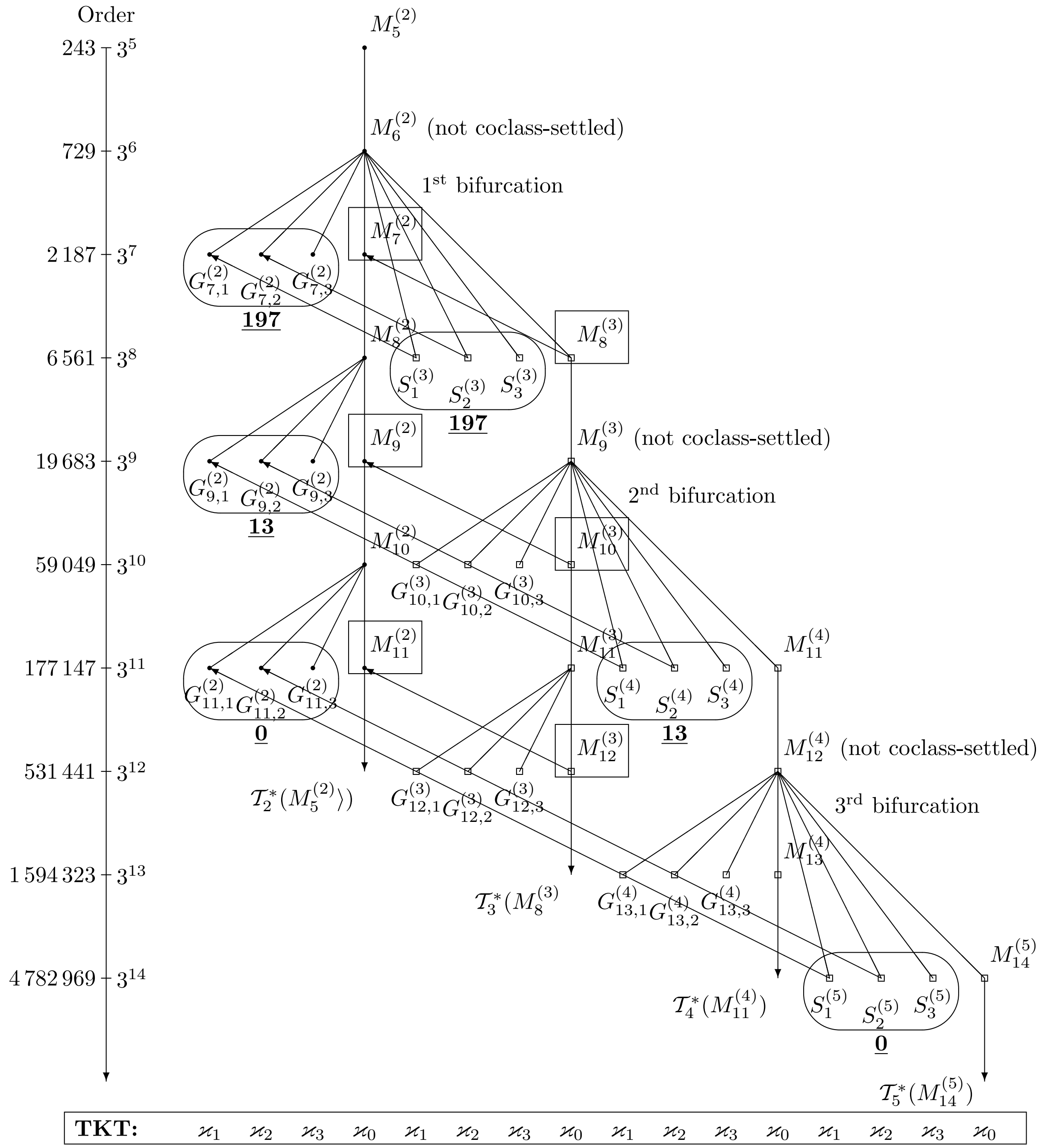
(4) the tree root $M_{3r-1}^{(r)}$ and the isolated vertices $S_k^{(r)}$ are of order 3^{3r-1},

(5) only the isolated vertices $S_k^{(r)}$ are Schur σ-groups,
two of them $S_1^{(r)}, S_2^{(r)}$ have TKT E.9,
and the remaining one $S_3^{(r)}$ has TKT E.8,

(6) each $S_k^{(r)}$ is the unique element in the balanced cover $\mathrm{cov}_*(G_{2r+1,k}^{(2)})$ of the branch group $G_{2r+1,k}^{(2)}$ of order 3^{2r+1} on the tree $\mathcal{T}_2^*(\langle 243, 8\rangle)$.

Conjecture 4.2.
Theorem 4.2 is also correct for any $r \ge 6$.

FIGURE 8. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, projections to the metabelianizations, and formal identifiers



Remark 4.2.
The root of the subtree $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ of $\mathcal{G}_r^*(\langle 243, 8\rangle)$ is given by $M_{3r-1}^{(r)} =$

$$= \begin{cases} \langle 729, 54\rangle\#2;3 & \text{for } r = 3, \\ \langle 729, 54\rangle\#2;3\#1;1\#2;1 & \text{for } r = 4, \\ \langle 729, 54\rangle\#2;3\#1;1\#2;1\#1;1\#2;1 & \text{for } r = 5. \end{cases}$$

§ 4.6. Two Conclusive Main Results

Group Theoretic Main Result 4.1.

All metabelian 3-groups with TKT in section E share the common coclass 2.

They are vertices $G_{n,k}^{(2)}$ with $n \geq 6$, $1 \leq k \leq 3$ for odd n, $2 \leq k \leq 3$ for even n, of depth 1 on the coclass tree $\mathcal{T}_2^*(M_5^{(2)})$ with root $M_5^{(2)}$ either $\langle 243, 6\rangle$ or $\langle 243, 8\rangle$.

None of these groups has a balanced presentation. The cardinalities of the covers and balanced covers of these groups are finite. They are given by

(1) $\#\mathrm{cov}(G_{2j+1,k}^{(2)}) = j+1$ and $\#\mathrm{cov}_*(G_{2j+1,k}^{(2)}) = 1$, for $j \geq 3$ and $1 \leq k \leq 3$, that is, σ-groups have a unique Schur σ-group as their balanced cover,

(2) $\#\mathrm{cov}(G_{2\ell,k}^{(2)}) = j+1$ and $\#\mathrm{cov}_*(G_{2\ell,k}^{(2)}) = 0$, for $j \geq 3$ and $2 \leq k \leq 3$, that is, the balanced cover of non-σ groups is empty.

Remark. The Result is a Theorem for $j \leq 10$ and a Conjecture for $j \geq 11$.

Number Theoretic Main Result 4.2.
If the TKT $\varkappa(G)$ of the second 3-class group $G = \mathrm{G}_3^2(K)$ of a *complex quadratic* number field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with $\mathrm{Cl}_3(K) \simeq (3, 3)$ belongs to the four types of section E,
then the 3-tower of K has exactly three stages, that is, $\ell_3(K) = 3$.

Remark.
The Result is a Theorem for odd $\mathrm{cl}(G) \le 19$ and a Conjecture for odd $\mathrm{cl}(G) \ge 21$.
This is more than enough to disprove the claims of Scholz and Taussky on page 41 in [8] and of Heider and Schmithals on page 20 in [5], since for this purpose $\mathrm{cl}(G) = 5$ suffices already.

Among complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3, 3)$, those with TKT in section E occur with relative frequency $\frac{411}{2020} \approx 20.3\%$.

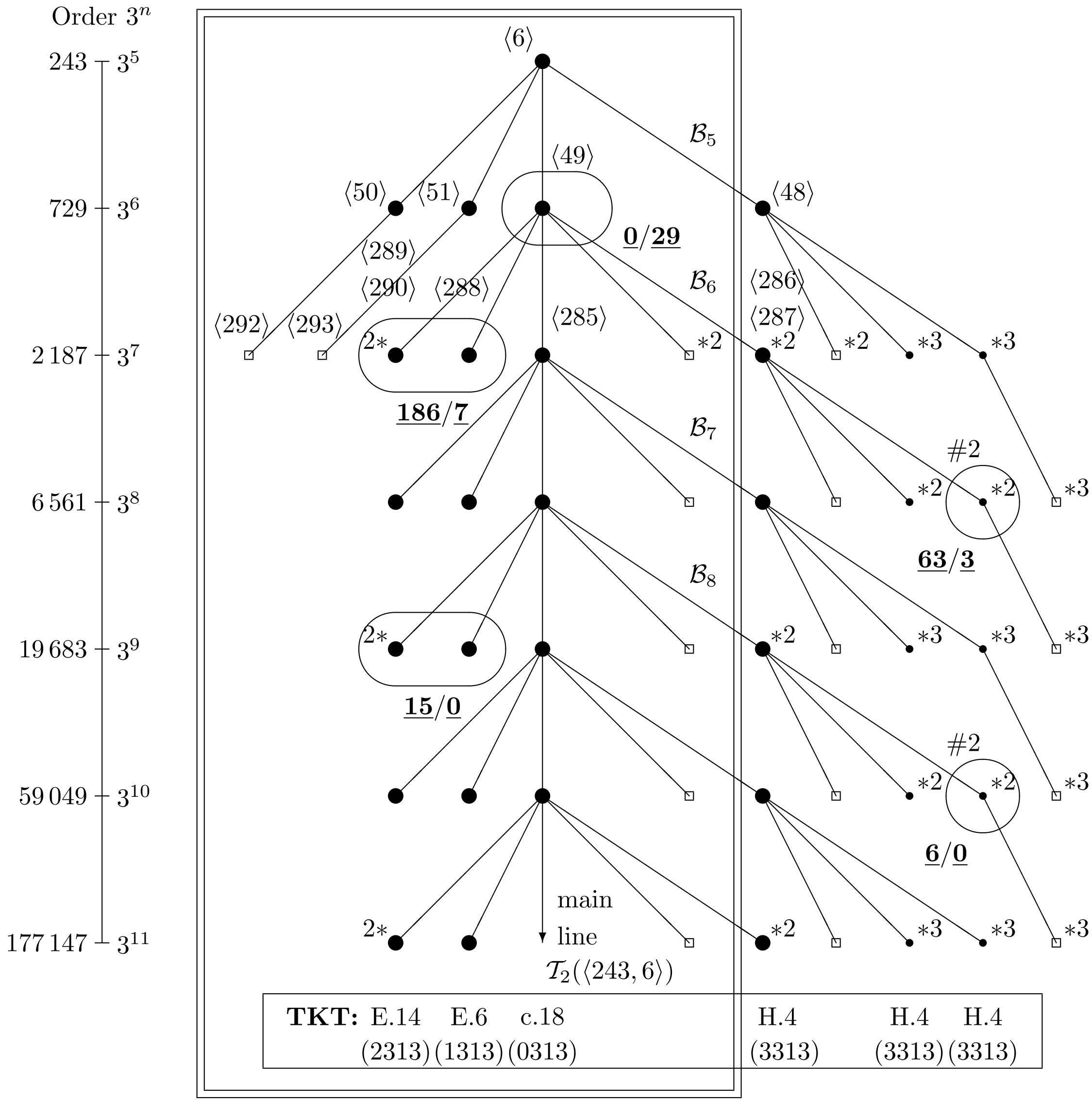
Corresponding complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3, 9)$ and TKT in sections C and D occur with relative frequency $\frac{182}{875} \approx 20.8\%$.

§ 5. Appendix

In Figures 9 and 10 we indicate how the TKT-pruned descendant tree for $\langle 243, 6\rangle$ can be constructed in a completely similar manner as for $\langle 243, 8\rangle$ in Figures 6 and 7,8.

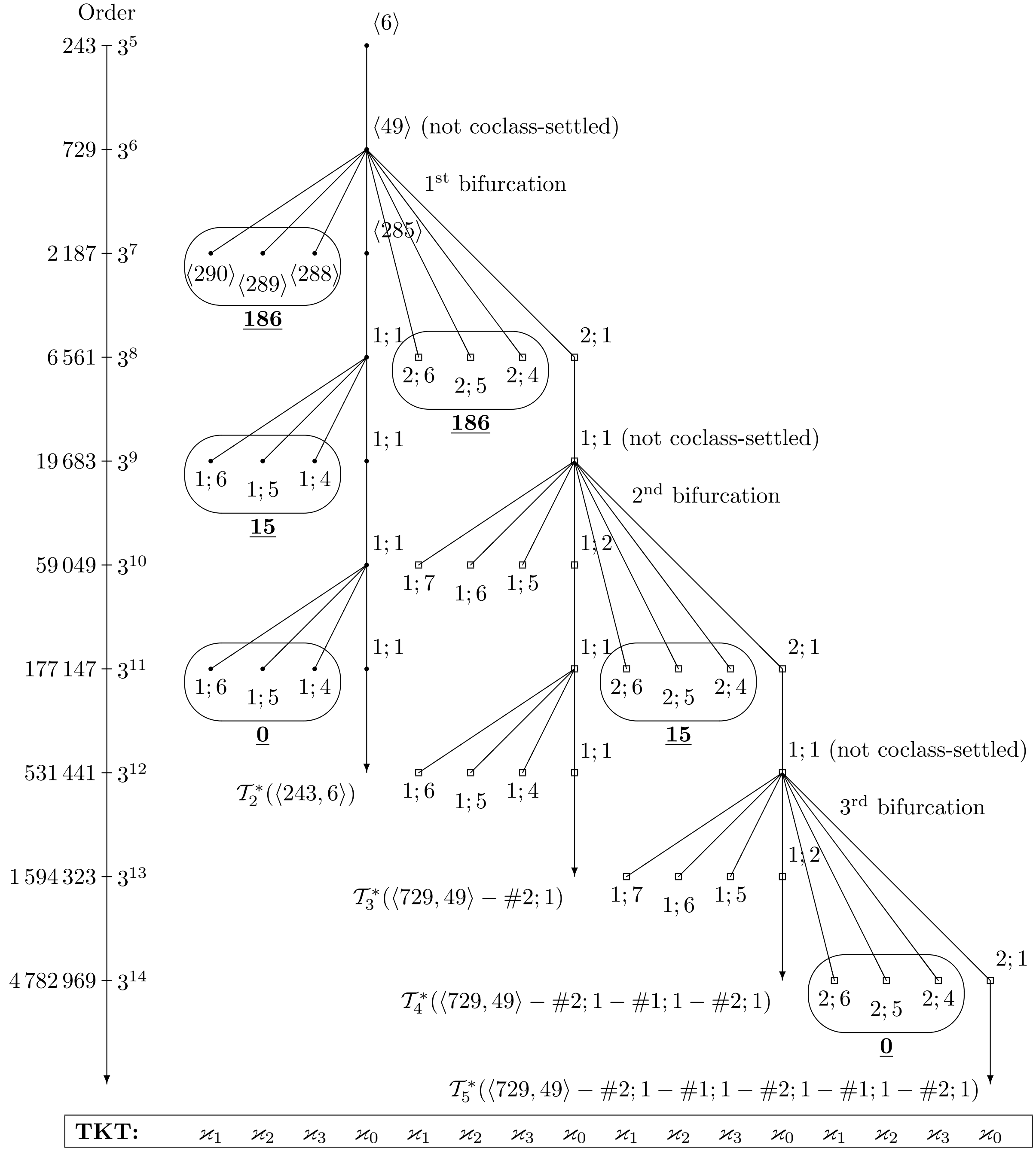
The structure of the smallest non-metabelian Schur σ-groups with TKT in section E is shown for class 5 in Figure 11 and for class 7 in Figure 12.
The smallest group $M_2(\beta)$ with $\beta = 2$ constructed by Brink and Gold has also the structure in Figure 11 although it is not a Schur group and has different TKT in section c.

FIGURE 9. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$



The bifurcation at $\langle 729, 49\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

Figure 10. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243,6\rangle)$ restricted to σ-groups



Here we also prune the tree from vertices with TKT c.18 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (3122) \sim \varkappa_2 = (4122)$ E.14,
$\varkappa_3 = (1122)$ E.6,
$\varkappa_0 = (0122)$ c.18.

FIGURE 11. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(3)}$, with TKT E, class 5.

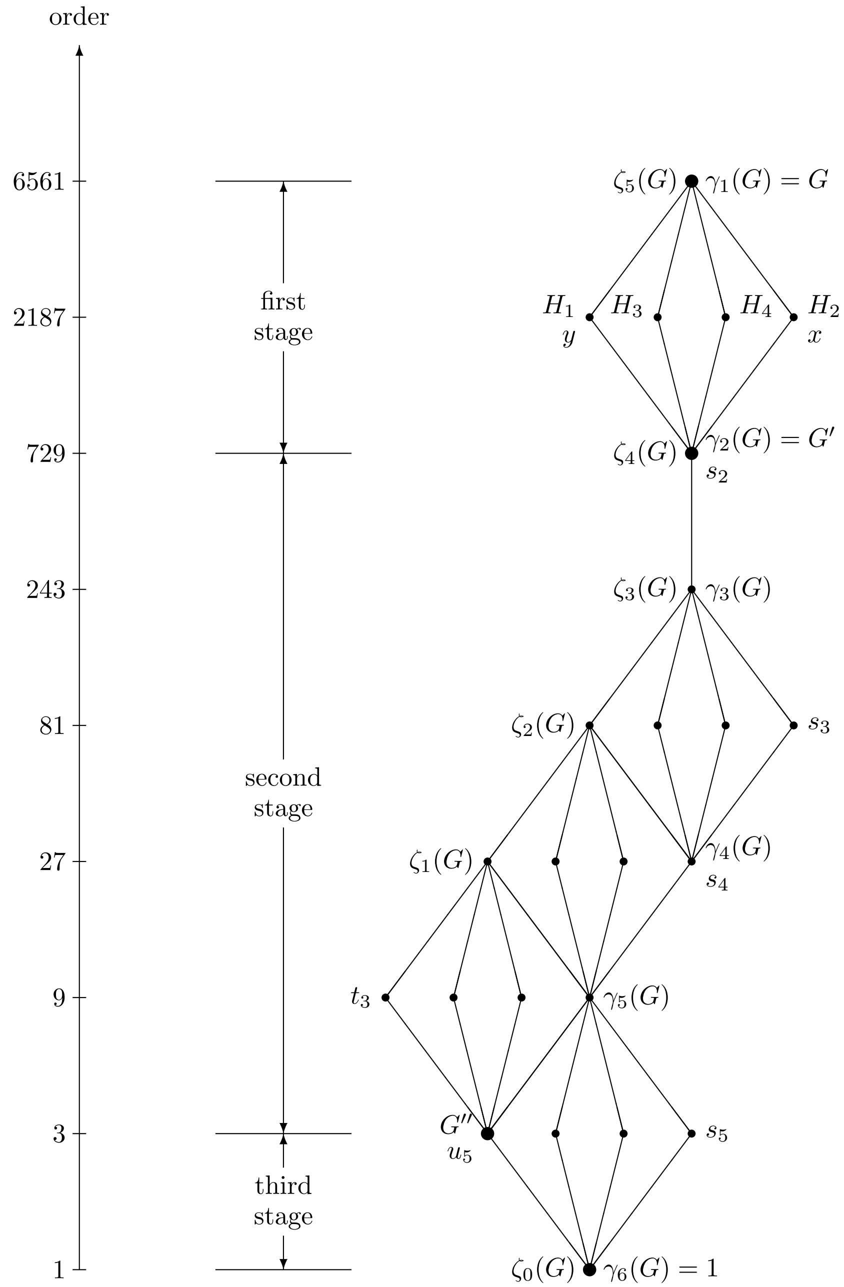
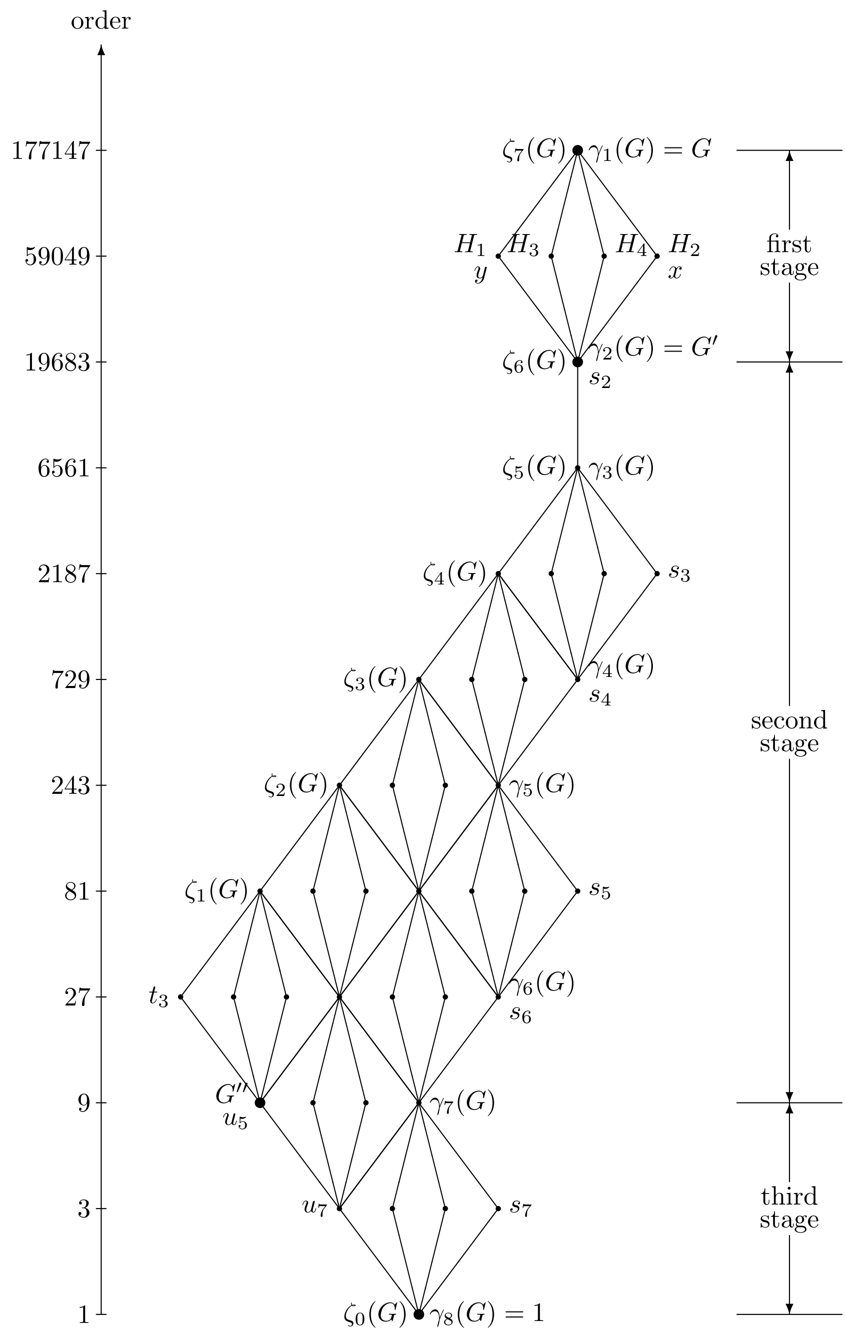


FIGURE 12. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(4)}$, with TKT E, class 7.



The techniques for reaching the targets of this presentation are based on the results of

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