TWO TREES OF 3-GROUPS WITH COCLASS 2

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With the exception of the final two conjectures, the following information is an excerpt from the presentation [9] and the article [10]. It is intended for visualizing the unique two coclass trees which contain 3-groups with transfer kernel type (TKT) in Scholz and Taussky's Section E [12] and in Nebelung's Section c [11],

The vertices of these two coclass trees in Figure 1 and 2 on the next pages, are classified by using different symbols:

(1) big full discs $\bullet$ represent metabelian groups with bicyclic centre of type $(3,3)$ and defect of commutativity $k = 0$, that is, with maximal commutativity,
(2) small full discs $\bullet$ represent metabelian groups with cyclic centre of order 3 and defect of commutativity $k = 1$,
(3) small contour squares $\square$ represent non-metabelian groups.

A number adjacent to a vertex denotes the multiplicity of a batch of immediate descendants sharing a common parent. The groups of particular importance are labelled by a number in angles, which is the identifier in the SmallGroups library [2] of GAP [3] and MAGMA [4], where the order is omitted since it is given on the left hand scale. The metabelian skeletons were drawn in [11, p. 189 ff], the complete trees were given in [1, p. 76, Fig. 4.8 and p. 123, Fig. 6.1].

The actual distribution of the 2020, resp. 2576, second 3-class groups $G_3^2(K)$ of complex, resp. real, quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ of type $(3,3)$ with discriminant $-10^6 < D < 10^7$ is represented by underlined boldface counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval. See also [5, § 6, tbl. 3–5], [7], and [8, § 6, tbl. 15–18].

The realization of mainline vertices with TKT c.18 and c.21 [6] as $G_3^2(K)$ of real quadratic fields K is no violation of the Weak Leaf Conjecture, since these vertices do not possess metabelian immediate descendants of the same TKT and of higher defect of commutativity.

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Theorem 1. *The structure of the complete coclass tree* $\mathcal{T}(\langle 243,6\rangle)$ *as part of the coclass graph* $\mathcal{G}(3,2)$*, restricted to* 3*-groups* G *with abelianization* $G/G' \simeq (3,3)$*, is globally characterized by the tree invariant* $\varepsilon(G) = 1$ *and is given up to order* $3^{11} = 177\,147$ *by Figure 1. The branches are of depth* 3 *and periodic of length* 2*. The pre-period consists of* $\mathcal{B}_5, \mathcal{B}_6$*, the primitive period of* $\mathcal{B}_7, \mathcal{B}_8$

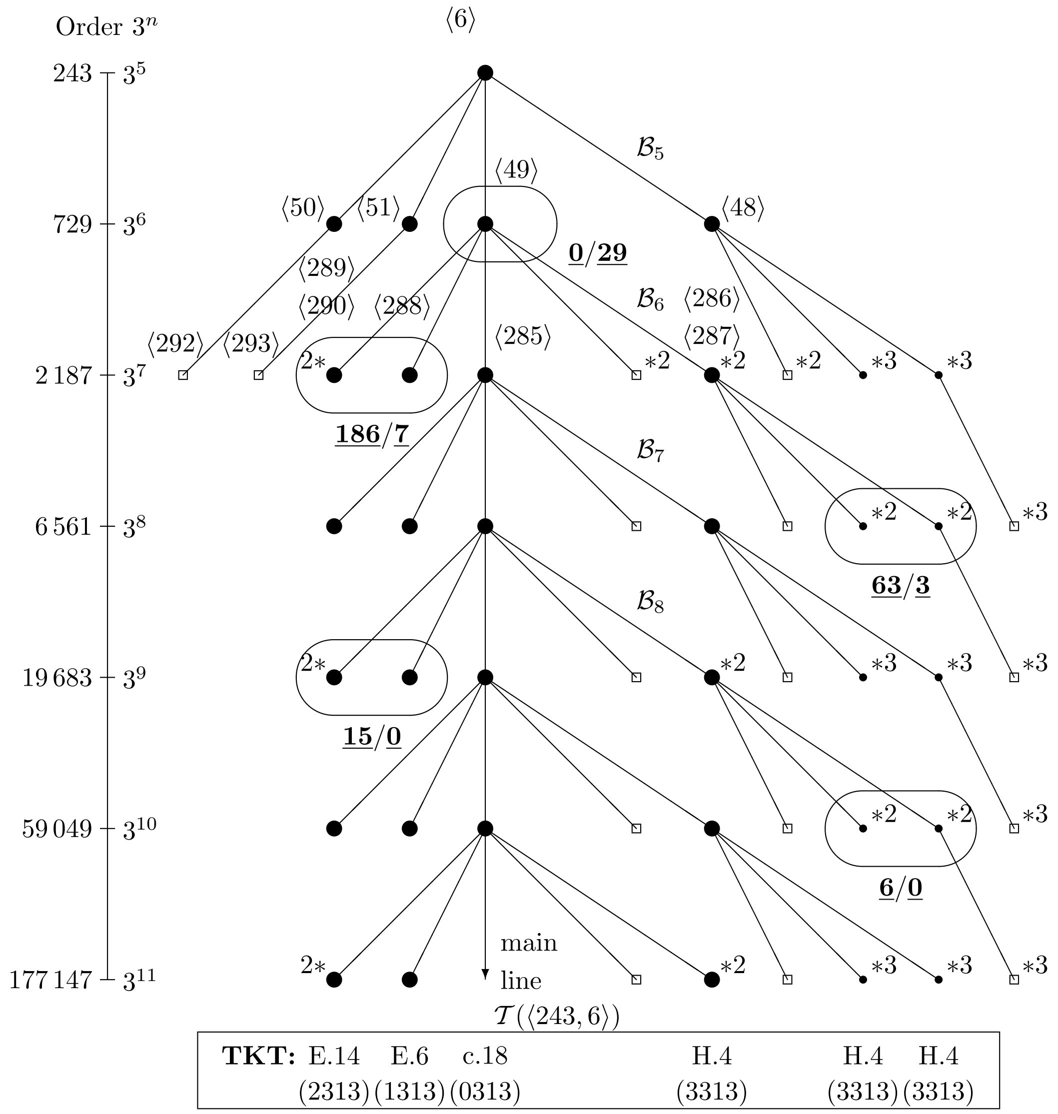
In Figure 1, we have $G_3^2(\mathbb{Q}(\sqrt{D})) \in \mathcal{T}(\langle 243,6\rangle)$ for 270 (13.4%) of the 2020 discriminants $-10^6 < D < 0$ and for 39 (1.5%) of the 2576 discriminants $0 < D < 10^7$, investigated in [5, § 6], [8, § 6].

Since the TKT c.18, $\varkappa = (0313)$, of the mainline is *total* with $\varkappa(1) = 0$, there only occur $G_3^2(K)$ of *real* quadratic fields $K = \mathbb{Q}(\sqrt{D})$, $D > 0$, on the mainline.

Due to the *Selection Rule*, the $G_3^2(K)$ are distributed on *even branches* only, since the second distinguished transfer kernel is partial with $\varkappa(2) \neq 0$.

Underpinning the Weak Leaf Conjecture, there is no actual hit of the vertices at depth 1 with TKT H.4, $\varkappa = (3313)$, e. g., of $\langle 2187, 286\rangle$ and $\langle 2187, 287\rangle$.

FIGURE 1. Distribution of $G_3^2(K)$ on the coclass 2 tree $\mathcal{T}(\langle 243,6\rangle)$



Theorem 2. *The structure of the complete coclass tree $\mathcal{T}(\langle 243,8\rangle)$ as part of the coclass graph $\mathcal{G}(3,2)$, restricted to 3-groups G with abelianization $G/G' \simeq (3,3)$, is globally characterized by the tree invariant $\varepsilon(G)=0$ and is given up to order $3^{11}=177\,147$ by Figure 2. The branches are of depth 3 and periodic of length 2. The pre-period consists of $\mathcal{B}_5,\mathcal{B}_6$, the primitive period of $\mathcal{B}_7,\mathcal{B}_8$*

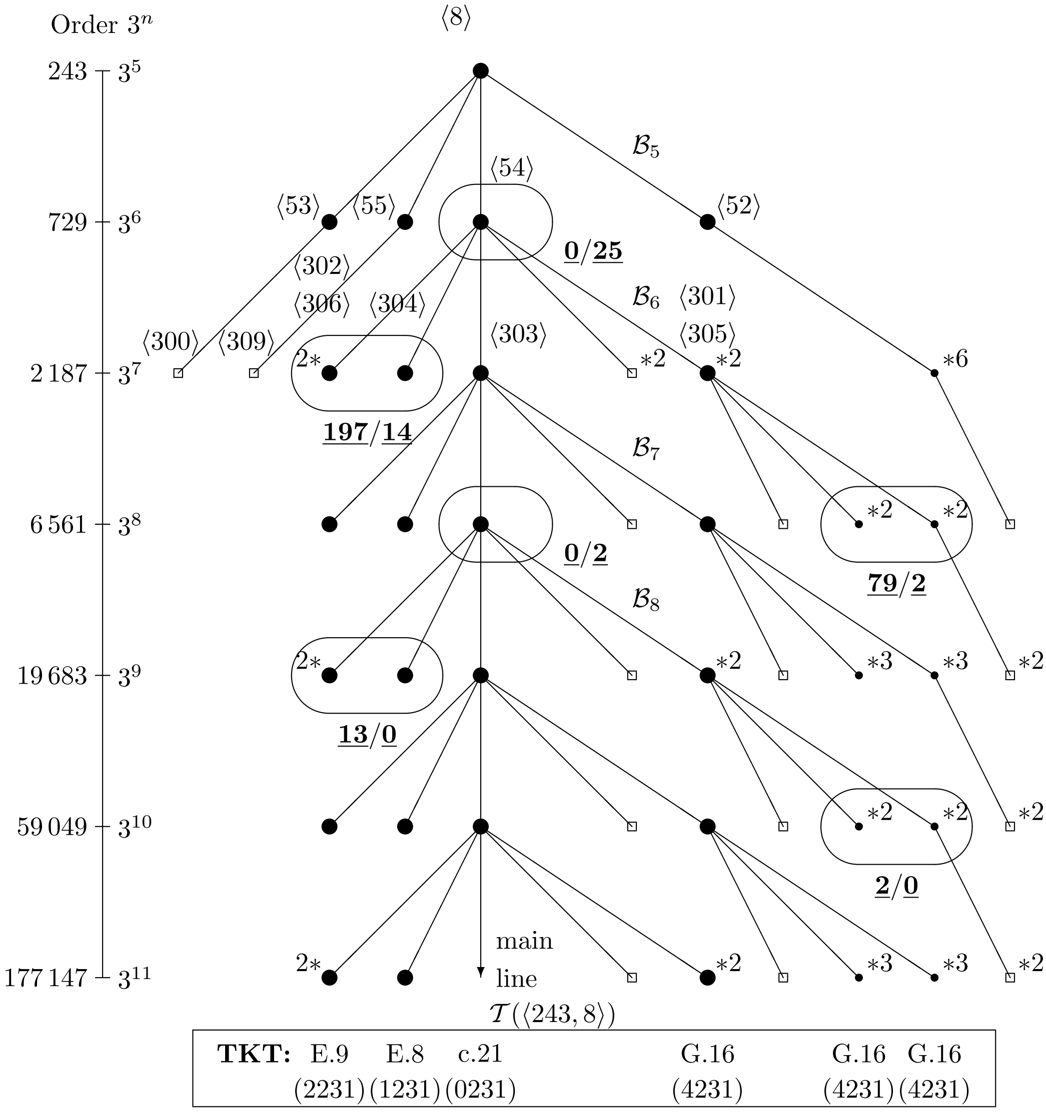
In Figure 2, we have $G_3^2(\mathbb{Q}(\sqrt{D}))\in\mathcal{T}(\langle 243,8\rangle)$ for 291 (14.4%) of the 2020 discriminants $-10^6 < D < 0$ and for 43 (1.7%) of the 2576 discriminants $0 < D < 10^7$, investigated in [5, § 6], [8, § 6].

Since the TKT c.21, $\varkappa=(0231)$, of the mainline is *total* with $\varkappa(1)=0$, there only occur $G_3^2(K)$ of *real* quadratic fields $K=\mathbb{Q}(\sqrt{D})$, $D>0$, on the mainline.

Due to the *Selection Rule*, the $G_3^2(K)$ are distributed on *even branches* only, since the second distinguished transfer kernel is partial with $\varkappa(2)\neq 0$.

Underpinning the Weak Leaf Conjecture, there is no actual hit of the vertices at depth 1 with TKT G.16, $\varkappa=(4231)$, e. g., of $\langle 2187,301\rangle$ and $\langle 2187,305\rangle$.

FIGURE 2. Distribution of $G_3^2(K)$ on the coclass 2 tree $\mathcal{T}(\langle 243,8\rangle)$



Finally, it should be pointed out that the following two conjectures provide a *complete solution* for the problem of finding the non-metabelian *covers* G of soluble length 3 of all metabelian 3-groups H with TKTs in *Scholz and Taussky's Section E* [12], that is, groups satisfying $G/G'' = H$.

Concerning *even* branches of the trees $\mathcal{T}(\langle 243,6\rangle)$ and $\mathcal{T}(\langle 243,8\rangle)$, which are *admissible* as second 3-class groups $G_3^2(K)$ of quadratic number fields $K = \mathbb{Q}(\sqrt{D})$, we have:

Conjecture 1. Let $n \geq 2$ be a positive integer. There exist exactly 6 groups G of order 3^{3n+2}, class $2n+1$, coclass $n+1$, having fixed derived length 3, such that

(1) the factors of their upper central series are given by

$$\zeta_{j+1}(G)/\zeta_j(G) \simeq \begin{cases} (3,3) & \text{for } j = 2n, \\ (3) & \text{for } 1 \leq j \leq 2n-1, \\ (3,3^n) & \text{for } j = 0, \end{cases}$$

(2) their second derived group G'' is cyclic of order 3^{n-1}.

Furthermore,

- they are Schur σ-groups with automorphism group $\mathrm{Aut}(G)$ of order $2 \cdot 3^{4n+2}$,
- the factors of their lower central series are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3,3) & \text{for odd } 1 \leq j \leq 2n+1, \\ (3) & \text{for even } 2 \leq j \leq 2n, \end{cases}$$

- their metabelianization G/G'' is of order 3^{2n+3} and of fixed coclass 2,
- their generalized parent $G/\gamma_{2n+1}(G)$ is of order 3^{3n}, class $2n$, coclass n, and still of derived length 3, except for $n = 2$,
- their biggest metabelian generalized predecessor, that is the $(2n-3)$rd generalized parent, is given by either $\langle 729,49\rangle$ or $\langle 729,54\rangle$.

Concerning *odd* branches of the trees $\mathcal{T}(\langle 243,6\rangle)$ and $\mathcal{T}(\langle 243,8\rangle)$, which are *forbidden* as second 3-class groups $G_3^2(K)$ of quadratic number fields $K = \mathbb{Q}(\sqrt{D})$, we have:

Conjecture 2. Let $n \geq 2$ be a positive integer. There exist exactly 4 groups G of order 3^{3n}, class $2n$, coclass n, having fixed derived length 3, except for $n = 2$, such that

(1) the factors of their upper central series are given by

$$\zeta_{j+1}(G)/\zeta_j(G) \simeq \begin{cases} (3,3) & \text{for } j = 2n-1, \\ (3) & \text{for } 1 \leq j \leq 2n-2, \\ (3,3^n) & \text{for } j = 0, \end{cases}$$

(2) their second derived group G'' is cyclic of order 3^{n-2}.

Furthermore,

- they neither possess balanced presentation nor a Schur σ-automorphism acting as inversion on G/G', and their automorphism group $\mathrm{Aut}(G)$ is of order 3^{4n},
- the factors of their lower central series are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3,3) & \text{for odd } 1 \leq j \leq 2n-1, \\ (3) & \text{for even } 2 \leq j \leq 2n, \end{cases}$$

- their metabelianization G/G'' is of order 3^{2n+2} and of fixed coclass 2.

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