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Recent Progress in Determining p-Class Field Towers

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Towers of p-Class Fields over Algebraic Number Fields

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INTRODUCTION.

• The key for determining the Galois group

$$G := \mathrm{G}_p^{(\infty)} F = \mathrm{Gal}(F_p^{(\infty)}/F)$$

of the unramified Hilbert p-class field tower $F_p^{(\infty)}$, i.e. the maximal unramified pro-p extension, of an algebraic number field F is the **Artin pattern** $\mathrm{AP}(G)$ combined with bounds for the **relation rank** $d_2 G$.
G is briefly called the p-**tower group** of F.

• My principal goal is to draw the attention of the audience to the striking novelty of **three-stage towers** of p-class fields over quadratic, cubic and quartic number fields F, discovered by myself in the past four years, partially in cooperation with M. R. Bush (WLU, VA) and M. F. Newman (ANU, ACT).

• This lecture can be downloaded from
`http://www.algebra.at/ANCI2016DCM.pdf`

• It is an updated and compact version of my article

[9] D. C. Mayer,
Recent progress in determining p-class field towers,
Gulf J. Math. (Dubai, UAE),
arXiv: 1605.09617v1 [math.NT] 31 May 2016.

1. The Hilbert p-class field tower

Assumptions: p . . . a prime number,
F . . . an algebraic number field,
$\mathrm{Cl}_pF := \mathrm{Syl}_p\mathrm{Cl}_F$ . . . the p-class group of F.

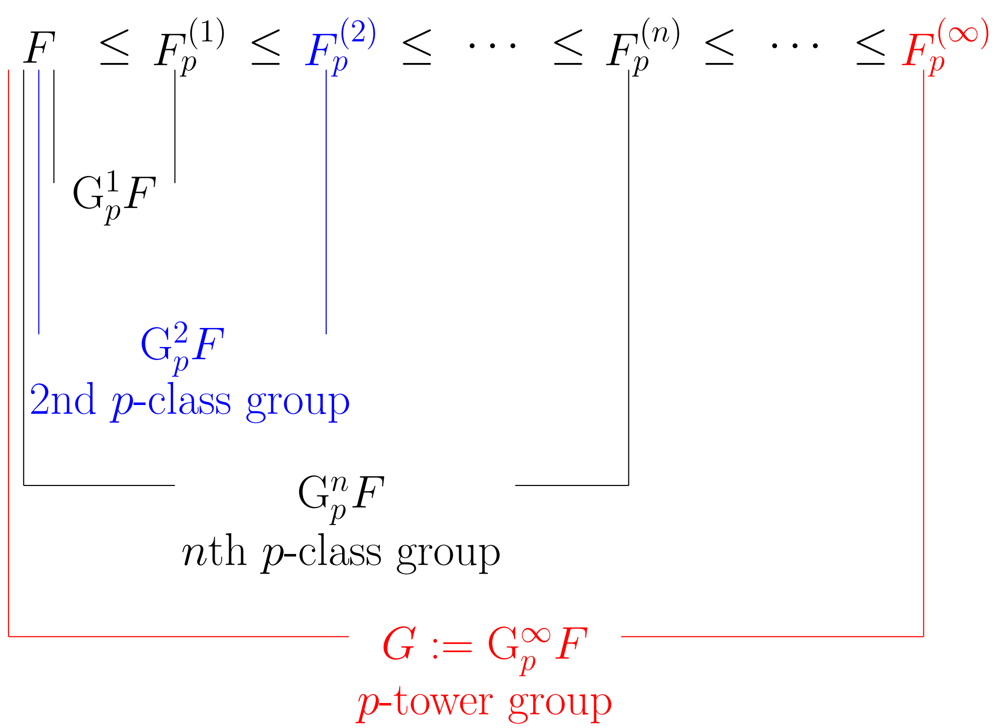
Definition. For $n \geq 0$, the *nth Hilbert p-class field* $F_p^{(n)}$ is the maximal unramified Galois extension of F with group

$$\mathrm{G}_p^n F := \mathrm{Gal}\left(F_p^{(n)}/F\right)$$

having derived length $\mathrm{dl}(\mathrm{G}_p^n F) \leq n$ and order a power of p.

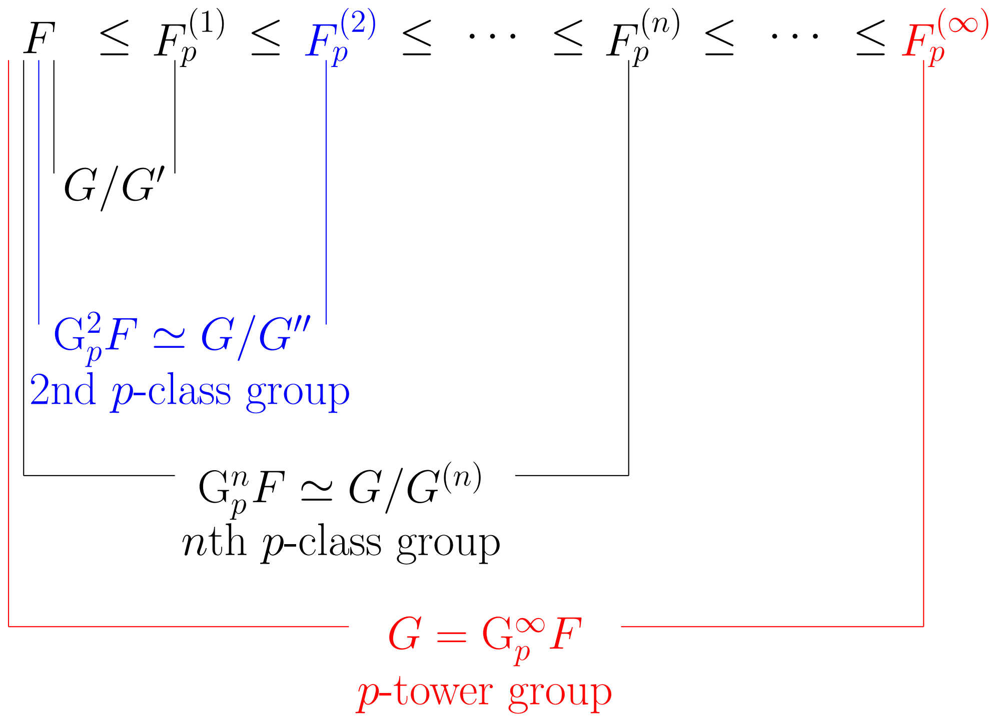
The *Hilbert p-class field tower* $F_p^{(\infty)}$ over F is the maximal unramified pro-p extension of F and has the Galois group

$$\mathrm{G}_p^\infty F := \mathrm{Gal}\left(F_p^{(\infty)}/F\right) \simeq \varprojlim_{n\geq 0} \mathrm{G}_p^n F.$$



The nth *stage* $\mathrm{G}_p^n F$ of the p-class tower arises as the nth derived quotient $G/G^{(n)}$ of the p-tower group G, for $n \geq 1$.

In particular, for $n = 2$, the second p-class group
$\mathfrak{M} := \mathrm{G}_p^2 F \simeq G/G^{(2)}$
can be viewed as a *two-stage approximation* of G.



Definition. The derived length $\mathrm{dl}(G)$ of the p-tower group $G = \mathrm{G}_p^\infty F$ is called the *length* $\ell_p F$ of the p-class tower of F.

When do we consider the p-class tower $F_p^{(\infty)}$ as "known"?

Is it sufficient to determine its length $\ell_p F$?

No, not at all! The length alone is a poor amount of information. According to my conviction, we are not done before we can give an *explicit pro-p presentation* of the p-tower group

$$G = \mathrm{G}_p^\infty F = \mathrm{Gal}\left(F_p^{(\infty)}/F\right).$$

An equivalent inductive definition of the p-class tower:
The tower $F_p^{(\infty)} := \bigcup_{i\ge 0} F_p^{(i)}$ arises recursively from the base field $F_p^{(0)} := F$ by the successive construction of maximal *abelian* unramified p-extensions $F_p^{(n)} := \left(F_p^{(n-1)}\right)_p^{(1)}$, $n \ge 1$.

By the **Artin reciprocity law** of class field theory [1], the relative Galois group $\mathrm{Gal}\left(F_p^{(n)}/F_p^{(n-1)}\right)$ is isomorphic to the (*abelian!*) p-class group $\mathrm{Cl}_pF_p^{(n-1)}$, for each $n \ge 1$.

In particular, for $n = 1$, we have the well-known fact that the first p-class group is isomorphic to the ordinary p-class group

$$\mathrm{G}_p^1 F = \mathrm{Gal}\left(F_p^{(1)}/F\right) \simeq \mathrm{Cl}_pF.$$

In particular, for $n = 2$, the second p-class group
$\mathfrak{M} = \mathrm{Gal}\left(F_p^{(2)}/F\right)$
has an abelian commutator subgroup
$\mathfrak{M}' = \mathrm{Gal}\left(F_p^{(2)}/F_p^{(1)}\right) \simeq \mathrm{Cl}_pF_p^{(1)}$
and is therefore *metabelian*.
The abelianization is

$$\mathfrak{M}/\mathfrak{M}' \simeq \mathrm{Gal}\left(F_p^{(1)}/F\right) \simeq \mathrm{Cl}_pF.$$

Similarly, the abelianization of the p-tower group G is
$G/G' = \mathrm{Gal}\left(F_p^{(\infty)}/F\right)/\mathrm{Gal}\left(F_p^{(\infty)}/F_p^{(1)}\right) \simeq \mathrm{Gal}\left(F_p^{(1)}/F\right) \simeq \mathrm{Cl}_pF$
and consequently G is a *finitely generated pro-p group*.

2. Infinite p-class towers

Let $\mathfrak{G}$ be a finitely generated pro-p group with *generator rank* $d_1 := \dim_{\mathbb{F}_p} \mathrm{H}^1(\mathfrak{G}, \mathbb{F}_p)$ and *relation rank* $d_2 := \dim_{\mathbb{F}_p} \mathrm{H}^2(\mathfrak{G}, \mathbb{F}_p)$.

Theorem (Golod, Shafarevich, 1964 [2]).
If $\mathfrak{G}$ is finite, then $d_2 > \left(\frac{d_1-1}{2}\right)^2$.

Theorem (Refinement by Vinberg, Gaschütz, 1965 [3]).
$\#\mathfrak{G} < \infty \Rightarrow d_2 > \frac{(d_1)^2}{4}$, resp. $d_2 \le \frac{(d_1)^2}{4} \Rightarrow \#\mathfrak{G} = \infty$.

Let $1 \to R \to F \to \mathfrak{G} \to 1$ be a presentation of $\mathfrak{G}$ with a free pro-p group F such that $d_1 F = d_1 \mathfrak{G}$, and suppose that $R \subset F_n$ for a term of the Zassenhaus filtration $(F_n)_{n\ge1}$ of F.

Theorem (Refinement by Koch, Venkov, 1975 [4]).
If $\mathfrak{G}$ is finite, then $d_2 > \frac{(d_1)^n}{n^n} \cdot (n-1)^{n-1}$.

Corollary (Koch, Venkov, 1975 [4]).
Let $p \ge 3$ be odd, then
a complex quadratic field $F = \mathbb{Q}(\sqrt{d})$ with p-class rank $\varrho_p \ge 3$ has an infinite p-class tower with $\ell_p F = \infty$.

Proof. Since F has Dirichlet unit rank $r = 0$, the p-tower group $G = \mathrm{G}_p^\infty F$ has a balanced presentation with $d_2 = d_1$ and $R \subset F_3$. Thus, a sufficient condition for $\ell_p F = \infty$ is $d_1 \le \frac{(d_1)^3}{3^3} \cdot 2^2$, i.e. $\varrho_p = d_1 \ge \sqrt{\frac{27}{4}} = \frac{3}{2}\sqrt{3} \approx 2.598$. □

Example.
$d = -4\,447\,704 \Rightarrow \mathrm{Cl}_3 F \simeq (3,3,3) \Rightarrow \ell_3 F = \infty$. However, we are far from knowing a pro-3 presentation of $G = \mathrm{G}_3^\infty F$. We only have the lower bound $\#\mathfrak{M} \ge 3^{17}$ for $\mathfrak{M} = G/G''$.

3. The relation rank of the p-tower group

Theorem (I. R. Shafarevich, 1964 [5]).
Let $p \geq 2$ be a prime number,
and K be an algebraic number field with signature (r_1, r_2)
and torsion free Dirichlet unit rank $r = r_1 + r_2 - 1$,
ζ a primitive pth root of unity,
$G = \mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$ the Galois group of the maximal unramified pro-p extension $\mathrm{F}_p^\infty(K)$ of K,
$d_1 = \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$ the *generator rank* of G,
$d_2 = \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ the *relation rank* of G. Then

$$d_1 \leq d_2 \leq d_1 + r + \theta, \text{ where } \theta = \begin{cases} 0 & \text{if } \zeta \notin K, \\ 1 & \text{if } \zeta \in K. \end{cases}$$

Corollary. $K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant d,
$G = \mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$
the *p-class tower group* of K. Then

$$\begin{cases} d_2 = d_1 & \text{if } (d < 0 \text{ and } p \geq 3), \\ d_1 \leq d_2 \leq d_1 + 1 & \text{if either } (d < 0 \text{ and } p = 2) \text{ or } (d > 0 \text{ and } p \geq 3), \\ d_1 \leq d_2 \leq d_1 + 2 & \text{if } (d > 0 \text{ and } p = 2). \end{cases}$$

[5] I. R. Shafarevich, Extensions with prescribed ramification points, *Publ. Math., Inst. Hautes Études Sci.* **18** (1964), 71–95 (Russian). English transl. by J. W. S. Cassels: *Am. Math. Soc. Transl.*, II. Ser., **59** (1966), 128–149.

4. Leaving the realm of class field theory

p . . . a prime number,
F . . . an algebraic number field,
E/F . . . an unramified abelian p-extension,
Cl_pE . . . the p-class group of E,
$T_{F,E} : \mathrm{Cl}_pF \to \mathrm{Cl}_pE$, $\mathfrak{a}\cdot\mathcal{P}_F \mapsto (\mathfrak{a}\mathcal{O}_E)\cdot\mathcal{P}_E$
. . . the transfer of p-classes from F to E.

We translate $T_{F,E}$ from number theory to group theory by entering the system of *metabelian* unramified extensions [6]:

System of abelian unramified extensions: $F \le E \le F_p^{(1)}$

System of metabelian unramified extensions: $F_p^{(1)} \le E_p^{(1)} \le F_p^{(2)}$

$$F \le E \le F_p^{(1)} \le E_p^{(1)} \le F_p^{(2)}$$

Galois correspondence

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\mathfrak{M} \trianglerighteq H \trianglerighteq \mathfrak{M}' \trianglerighteq H' \trianglerighteq 1$$

$$\mathfrak{M} = \mathrm{G}\left(F_p^{(2)}/F\right), \quad H = \mathrm{G}\left(F_p^{(2)}/E\right), \quad \mathfrak{M}' = \mathrm{G}\left(F_p^{(2)}/F_p^{(1)}\right), \quad H' = \mathrm{G}\left(F_p^{(2)}/E_p^{(1)}\right)$$

abelian quotient

$$\mathrm{G}\left(E_p^{(1)}/E\right) \simeq \mathrm{Cl}_pE \simeq H/H'$$

Let $H := \mathrm{Gal}(F_p^{(2)}/E)$, then $H' := \mathrm{Gal}(F_p^{(2)}/E_p^{(1)})$, and the p-class transfer $T_{F,E}$ is connected with the Artin transfer $T_{\mathfrak{M},H}$ by two applications of the Artin reciprocity map:

$$\begin{array}{ccccc} & & T_{F,E} & & \\ & \mathrm{Cl}_pF & \longrightarrow & \mathrm{Cl}_pE & \\ \text{Artin isomorphism} & \updownarrow & /// & \updownarrow & \text{Artin isomorphism} \\ & \mathfrak{M}/\mathfrak{M}' & \longrightarrow & H/H' & \\ & & T_{\mathfrak{M},H} & & \end{array}$$

In particular, we have
isomorphic domains, $\mathrm{Cl}_pF \simeq \mathfrak{M}/\mathfrak{M}'$,
isomorphic kernels, $\ker T_{F,E} \simeq \ker T_{\mathfrak{M},H}$,
isomorphic targets, $\mathrm{Cl}_pE \simeq H/H'$.

Definition. Recall that the *Artin transfer* [6] from a group $\mathfrak{G}$ to a subgroup $H \le \mathfrak{G}$ with finite index $n := (\mathfrak{G} : H)$ is defined with the aid of the permutation $\pi \in S_n$ of a transversal $\mathfrak{G} = \dot{\bigcup}_{i=1}^{n} \ell_i \cdot H$ induced by the action of an element $x \in \mathfrak{G}$:

$$T_{\mathfrak{G},H} : \mathfrak{G}/\mathfrak{G}' \to H/H', \ x \cdot \mathfrak{G}' \mapsto \prod_{i=1}^{n} \ell_{\pi(i)}^{-1} x \ell_i \cdot H',$$

where $x\ell_i \cdot H = \ell_{\pi(i)} \cdot H$, for each $1 \le i \le n$.

[6] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

5. The Artin pattern

Assumptions: p ... a prime number,
F ... an algebraic number field,
$\mathfrak{G}$ ... a pro-p group or a finite p-group,
X ... a placeholder, either $X = F$ or $X = \mathfrak{G}$.

Definition. We call

$$\tau_0 X := \begin{cases} \mathrm{Cl}_p F \text{ if } X = F, \\ \mathfrak{G}/\mathfrak{G}' \text{ if } X = \mathfrak{G}, \end{cases}$$

bottom, resp. *top*, *layer* of abelian quotient invariants of X.

Suppose that $\#\tau_0 X = p^t$ for some integer $t \geq 0$,
and let $0 \leq n \leq t$ be an integer.

Definition. The finite set

$$\mathrm{Lyr}_n X := \begin{cases} \{F \leq E \leq F_p^{(1)} \mid [E : F] = p^n\} \text{ if } X = F, \\ \{\mathfrak{G}' \leq H \trianglelefteq \mathfrak{G} \mid (\mathfrak{G} : H) = p^n\} \text{ if } X = \mathfrak{G}, \end{cases}$$

is called the nth *layer* of abelian unramified p-extensions, resp. of intermediate normal subgroups, of X.

Definition. For any $0 \leq n \leq t$ and $Y \in \mathrm{Lyr}_n X$, the mapping $T_{X,Y} : \tau_0 X \to \tau_0 Y$,

$$\begin{cases} x \cdot \mathcal{P}_X \mapsto (x\mathcal{O}_Y) \cdot \mathcal{P}_Y \text{ if } X = F, \\ x \cdot X' \mapsto \prod_{i=1}^{n} \ell_{\pi(i)}^{-1} x \ell_i \cdot Y' \text{ if } X = \mathfrak{G}, \end{cases}$$

is called the *p-class transfer*, resp. the *Artin transfer* [6], from X to Y.

Definition. We call

$\tau_n(X) := (\tau_0 Y)_{Y\in\mathrm{Lyr}_n X}$, for $0 \le n \le t$,
the components of the multi-layered
Transfer Target Type (TTT) $\tau(X) := [\tau_0 X;\ldots;\tau_t X]$,

$\varkappa_n(X) := (\ker T_{X,Y})_{Y\in\mathrm{Lyr}_n X}$, for $0 \le n \le t$,
the components of the multi-layered
Transfer Kernel Type (TKT) $\varkappa(X) := [\varkappa_0 X;\ldots;\varkappa_t X]$.

The pair $\mathrm{AP}(X) := (\tau(X),\varkappa(X))$
is called the *abelian Artin pattern* of X.

Definition. The *index-p abelianization data* (IPAD) of X is a first order approximation of the multi-layered TTT,

$$\tau^{(1)}X := [\tau_0 X;\tau_1 X].$$

A generalization to non-abelian unramified extensions is given recursively by the *iterated* IPAD *of order* $n \ge 2$,

$$\tau^{(n)}X := [\tau_0 X;(\tau^{(n-1)}Y)_{Y\in\mathrm{Lyr}_1 X}],$$

formally supplemented by $\tau^{(0)}X := \tau_0 X$.

Definition. The *cover* of a finite *metabelian* p-group $\mathfrak{M}$ is defined as the set

$$\mathrm{cov}(\mathfrak{M}) := \{\mathfrak{G} \mid \#\mathfrak{G} < \infty,\ \mathfrak{G}/\mathfrak{G}'' \simeq \mathfrak{M}\},$$

and the *total cover* of $\mathfrak{M}$ arises by dropping the finiteness condition:

$$\mathrm{cov}_{tot}\mathfrak{M} := \{\mathfrak{G} \mid \mathfrak{G}/\mathfrak{G}'' \simeq \mathfrak{M}\}.$$

Theorem.
(Uniformity of the Artin pattern on the total cover)
If $\mathfrak{M}$ is a finite metabelian p-group, then

$$(\forall \mathfrak{G} \in \mathrm{cov}_{tot}\mathfrak{M}) \quad \mathrm{AP}(\mathfrak{G}) = \mathrm{AP}(\mathfrak{M}).$$

That is, any pro-p group $\mathfrak{G}$ shares a common abelian Artin pattern with its metabelianization $\mathfrak{M} = \mathfrak{G}/\mathfrak{G}''$.

Corollary.
(Uniformity of the Artin pattern of all higher p-class groups)
All higher p-class groups $\mathrm{G}_p^n F$ of a number field F share a common abelian Artin pattern with $\mathfrak{M} := \mathrm{G}_p^2 F$,

$$(\forall n \geq 2) \quad \mathrm{AP}(\mathrm{G}_p^n F) = \mathrm{AP}(\mathfrak{M}).$$

Proof. $\mathrm{G}_p^n F/\left(\mathrm{G}_p^n F\right)'' \simeq \mathrm{G}_p^2 F$, and thus $\mathrm{G}_p^n F \in \mathrm{cov}_{tot}\mathrm{G}_p^2 F$, for any $n \geq 2$. □

Theorem.
(Coincidence of the Artin pattern of a field F and of $\mathrm{G}_p^2 F$)
Each number field F shares a common abelian Artin pattern with its second p-class group $\mathfrak{M} := \mathrm{G}_p^2 F$,

$$\mathrm{AP}(F) = \mathrm{AP}(\mathfrak{M}).$$

Corollary.
(Coincidence of the IPAD of a field F and of $\mathrm{G}_p^2 F$)
Each number field F shares a common IPAD with its second p-class group $\mathrm{G}_p^2 F$,

$$\tau^{(1)} F = \tau^{(1)} \mathrm{G}_p^2 F.$$

Proof. $\mathrm{AP}(F) = (\tau(F), \varkappa(F))$ and $\tau(F) = [\tau_0 F; \ldots; \tau_t F]$ contains $\tau^{(1)} F = [\tau_0 F; \tau_1 F]$. □

Iterated IPADs of increasing order approximate the complete information on the p-tower group $G = \mathrm{G}_p^\infty F$.

Successive Approximation Theorem.
(Coincidence of the IPAD of order m of a field F and of all groups $\mathrm{G}_p^n F$ with $n \ge m+1$)
For any order $m \ge 0$, the number field F and all higher p-class groups $\mathrm{G}_p^n F$ with $n \ge m+1$ share a common iterated IPAD of order m,

$$(\forall n \ge m+1) \quad \tau^{(m)} F = \tau^{(m)} \mathrm{G}_p^n F.$$

Corollary.
(Coincidence of all iterated IPADs of a field F and of $\mathrm{G}_p^\infty F$)
The number field F shares all iterated IPADs with its p-tower group $G = \mathrm{G}_p^\infty F$,

$$(\forall m \ge 0) \quad \tau^{(m)} F = \tau^{(m)} \mathrm{G}_p^\infty F.$$

The length of the p-class tower can be determined by repeated applications of the following criterion.

Stage Separation Conjecture.
(Proving length bigger than m by means of IPAD of order m)

$$(\forall m \ge 0) \quad \ell_p F > m \Longleftrightarrow \tau^{(m)} F > \tau^{(m)} \mathrm{G}_p^m F.$$

That is, the iterated IPAD of order m indicates when the p-class tower has more than m stages.

Proof. of "$\Longleftarrow$" by contraposition: $\ell_p F \le m \Longrightarrow$
$F_p^{(m)} = F_p^{(m+1)} \Longrightarrow \tau^{(m)} F = \tau^{(m)} \mathrm{G}_p^{m+1} F = \tau^{(m)} \mathrm{G}_p^m F.$ □

Our aim is to give an explicit pro-p presentation of the p-tower group of F,

$$G = \mathrm{G}_p^{\infty} F = \mathrm{Gal}\left(F_p^{(\infty)}/F\right).$$

For this purpose, we proceed in several steps:

(1) We compute the p-class groups $\mathrm{Cl}_p E$ and the p-capitulation $\ker T_{F,E}$ of all unramified cyclic extensions E/F of relative degree p.

(2) With the aid of Artin's reciprocity law, we interpret this number theoretic information $(\tau_1 F, \varkappa_1 F)$ as group theoretic invariants $(\tau_1 \mathfrak{M}, \varkappa_1 \mathfrak{M})$ of the metabelian second p-class group of F,

$$\mathfrak{M} = \mathrm{G}_p^2 F = \mathrm{Gal}\left(F_p^{(2)}/F\right).$$

(3) By a search for the assigned invariants in the descendant tree $\mathcal{T} R$ with root $R \simeq \mathrm{Cl}_p F$ we find a finite batch of contestants for $\mathfrak{M}$, using Thm. M as break-off condition.

(4) We construct small members of the cover $\mathrm{cov}(\mathfrak{M})$ of $\mathfrak{M}$ which satisfy the Shafarevich inequality $d_2 \le d_1 + r + \theta$ for the relation rank $d_2 = d_2(G)$ in dependence on the generator rank $d_1 = d_1(G)$, the Dirichlet unit rank r, and the invariant θ.

Theorem M. (Monotony of the Artin pattern on trees)
Let $\mathcal{T} R$ be the descendant tree with a finite non-trivial p-group as its root $R > 1$ and let $G \to \pi G$ be a directed edge of the tree. Then the abelian Artin pattern $\mathrm{AP} = (\tau, \varkappa)$ satisfies the following monotonicity relations (in componentwise sense)

$$\tau(G) \ge \tau(\pi G),$$
$$\varkappa(G) \le \varkappa(\pi G).$$

6. Contestants for the second p-class group

τ_0 ... finite abelian p-group with generator rank $d \geq 2$,
τ_1 ... family $(\tau_1(i))_{1\leq i\leq n}$ of $n := \frac{p^d-1}{p-1}$ abelian type invariants.

Definition. By $\mathrm{Cnt}_p^2(\tau_0,\tau_1)$ we denote the set of all (isomorphism classes of) finite metabelian p-groups $\mathfrak{M}$ such that $\tau_0\mathfrak{M} = \mathfrak{M}/\mathfrak{M}' \simeq \tau_0$ and $\tau_1\mathfrak{M} = (H/H')_{H\in\mathrm{Lyr}_1\mathfrak{M}} \simeq \tau_1$.

Conjecture. (Finiteness of the batch of contestants) $\mathrm{Cnt}_p^2(\tau_0,\tau_1)$ is a finite set.

Remark. Note that $\mathrm{Cnt}_p^2(\tau_0,\tau_1) = \emptyset$, when τ_1 is *malformed*.

Theorem 1. $p = 3$, $\tau_0 \simeq (3,3) \Longrightarrow \#\mathrm{Cnt}_3^2(\tau_0,\tau_1) < \infty$.

Theorem 2. $p = 2$, $\tau_0 \simeq (2,2,2) \Longrightarrow \#\mathrm{Cnt}_2^2(\tau_0,\tau_1) < \infty$.

The general Conjecture holds if the following Principle is true.

Polarization Principle. There exist a few components of a non-malformed family τ_1 which determine the nilpotency class $c := \mathrm{cl}(\mathfrak{M})$ and the coclass $r := \mathrm{cc}(\mathfrak{M})$ of a finite metabelian p-group $\mathfrak{M}$ with $\tau_1\mathfrak{M} = (H/H')_{H\in\mathrm{Lyr}_1\mathfrak{M}} \simeq \tau_1$.

Proof of Theorem 1, resp. 2, is based on Theorem 3, resp. 4.

Theorem 3. (Bipolarization) $p = 3$, and $\tau_0 \simeq (3,3) \Longrightarrow$ $\#\tau_1(1) = 3^{c-k}$ with $0 \leq k \leq 1$, and $\#\tau_1(2) = 3^{r+1}$.

Theorem 4. (Unipolarization with independent factors) $p = 2$, and $\tau_0 \simeq (2,2,2) \Longrightarrow \tau_1(1) \simeq (3^c, 3^{r-1})$.

7. The first 3-class towers of length 3

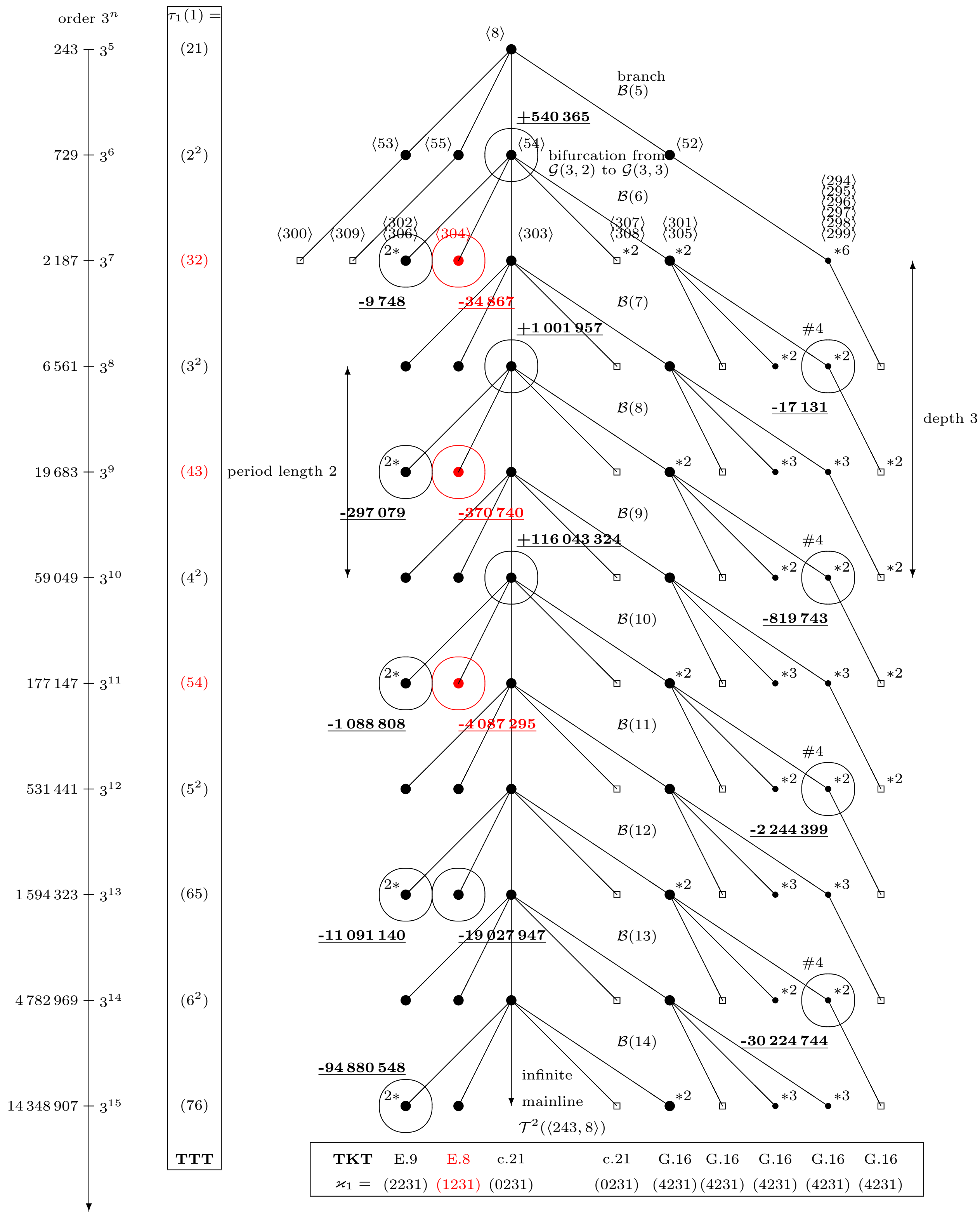
Notation. *Logarithmic type invariants* of abelian 3-groups, for instance $(21)\hat{=}(9,3)$, $(32)\hat{=}(27,9)$, and $(43)\hat{=}(81,27)$.

Let $F=\mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field, with 3-class group $\mathrm{Cl}_3F\simeq(3,3)$, and $E_1,\ldots,E_4$ be the unramified cyclic cubic extensions of F.

Theorem. Suppose the capitulation of 3-classes of F in $E_1,\ldots,E_4$ is of type $\varkappa_1F\sim(\mathbf{1},\mathbf{2},\mathbf{3},1)$ (called type E.8). Assume further that the 3-class groups of $E_1,\ldots,E_4$ are of type $\tau_1F\sim[T_1,21,21,21]$, where $T_1\in\{32,43,54\}$.

Then the length of the 3-class tower of F is $\ell_3F=3$.

Proof. We employ the p-group generation algorithm for searching the Artin pattern $\mathrm{AP}(F)=(\tau_1F,\varkappa_1F)$ among the descendants of the root $R:=C_3\times C_3=\langle 9,2\rangle$ in the tree $\mathcal{T}R$.
After two steps, $\langle 9,2\rangle\leftarrow\langle 27,3\rangle\leftarrow\langle 243,8\rangle$, we find the next root $U_5:=\langle 243,8\rangle$ of the unique relevant coclass tree $\mathcal{T}^2U_5$, using the assigned TKT E.8, $\varkappa_3=(1231)$, and its scaffold TKT c.21, $\varkappa_0=(0231)$.
Finally, the first component $T_1=\tau_1(1)\in\{32,43,54\}$ of the TTT provides the break-off condition, according to Theorem M, and we get $\mathfrak{M}\simeq\langle 2187,304\rangle=\langle 729,54\rangle-\#1;4$ for the ground state $T_1=(32)$, $\mathfrak{M}\simeq\langle 729,54\rangle-\#1;3-\#1;1-\#1;2$ for the 1st excited state $T_1=(43)$, and $\mathfrak{M}\simeq\langle 729,54\rangle-\#1;3(-\#1;1)^3-\#1;2$ for the 2nd excited state $T_1=(54)$, where $\langle 729,54\rangle-\#1;3=\langle 2187,303\rangle$.
The situation is visualized by the next figure. □



TKT	E.9	E.8	c.21	c.21	G.16	G.16	G.16	G.16	G.16
$\varkappa_1 =$	(2231)	(1231)	(0231)	(0231)	(4231)	(4231)	(4231)	(4231)	(4231)

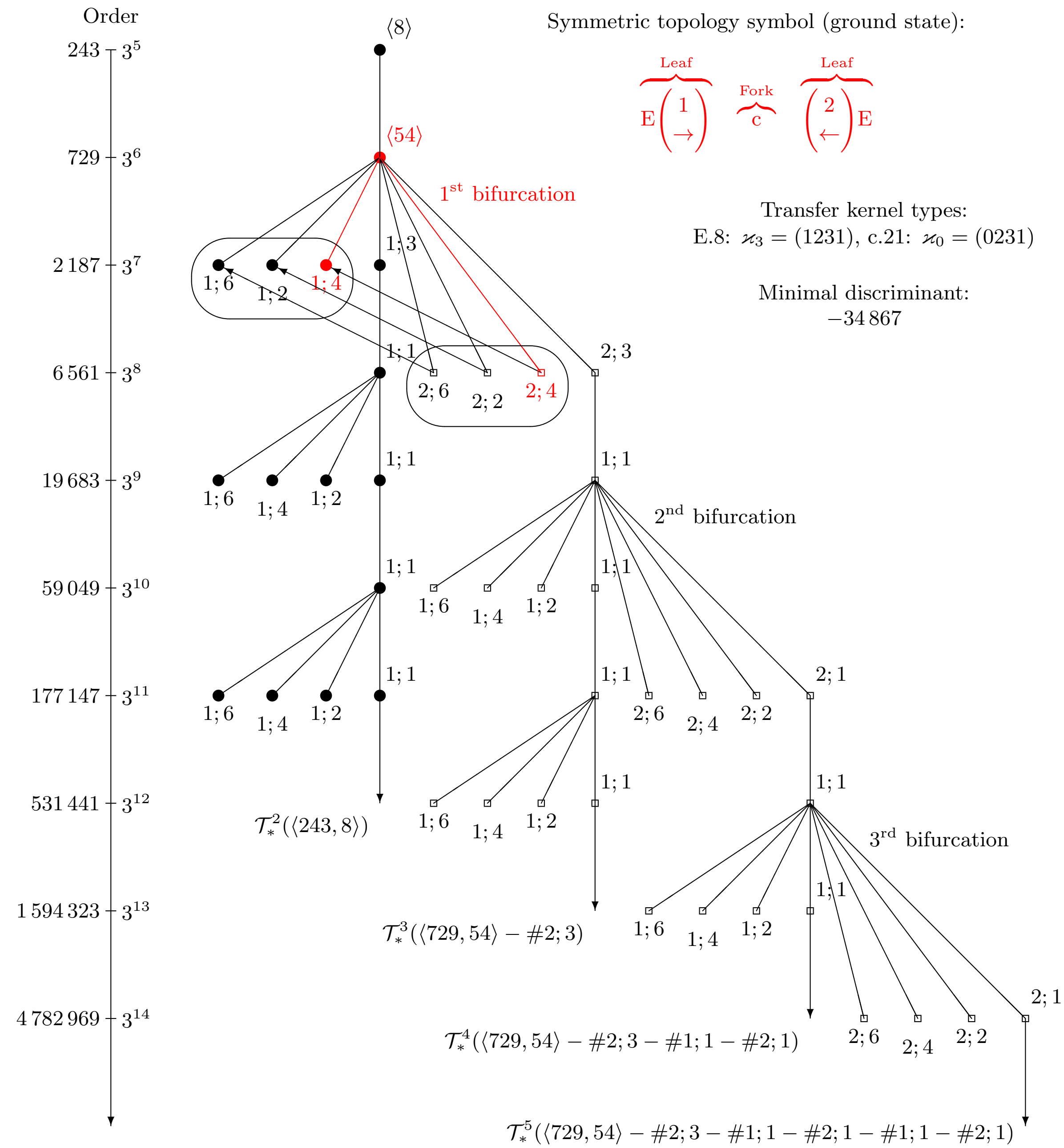
Proof. (continued)
The last figure, showing the second 3-class groups $\mathfrak{M}$, was essentially known to J. A. Ascione in 1979, and to B. Nebelung in 1989.

In the next three figures, which were unknown until 2012, we present the decisive break-through establishing the first rigorous proof for three-stage towers of 3-class fields. The key ingredient is the discovery of periodic bifurcations in the complete descendant tree $\mathcal{T}U_5$ which is of considerably higher complexity than the coclass tree $\mathcal{T}^2U_5$.

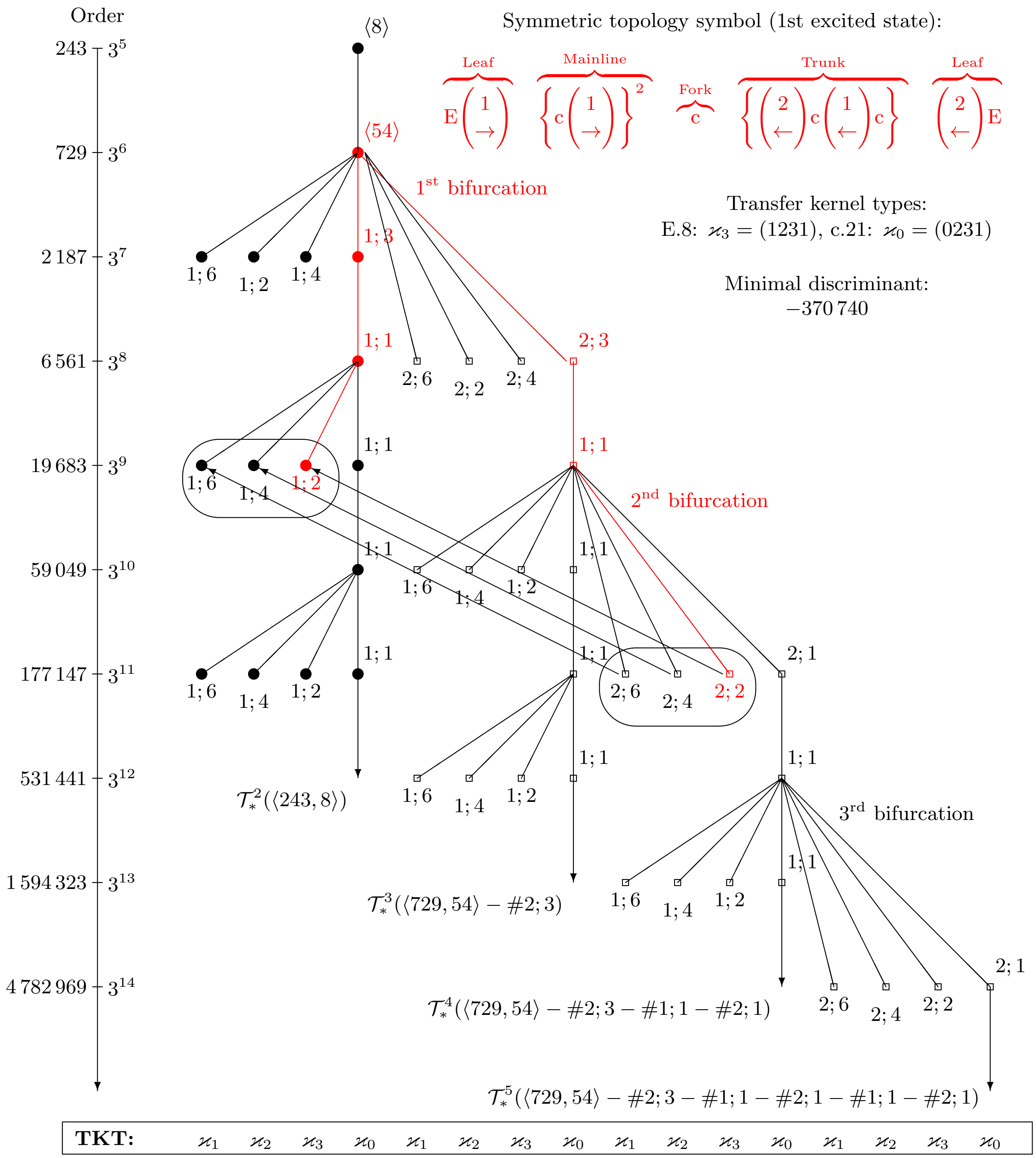
For the ground state $T_1 = (32)$, the first bifurcation yields the cover $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}, \langle 729, 54\rangle - \#2; 4\}$ of $\mathfrak{M} \simeq \langle 2187, 304\rangle = \langle 729, 54\rangle - \#1; 4$. The relation rank $d_2\mathfrak{M} = 3$ eliminates $\mathfrak{M}$ as a candidate for the 3-tower group G, according to the Corollary of the Shafarevich Theorem, and we end up getting $G \simeq \langle 729, 54\rangle - \#2; 4 = \langle 6561, 622\rangle$ with a siblings topology $\mathrm{E}\binom{1}{\rightarrow}$ c $\binom{2}{\leftarrow}\mathrm{E}$ describing the relative location of $\mathfrak{M}$ and G.

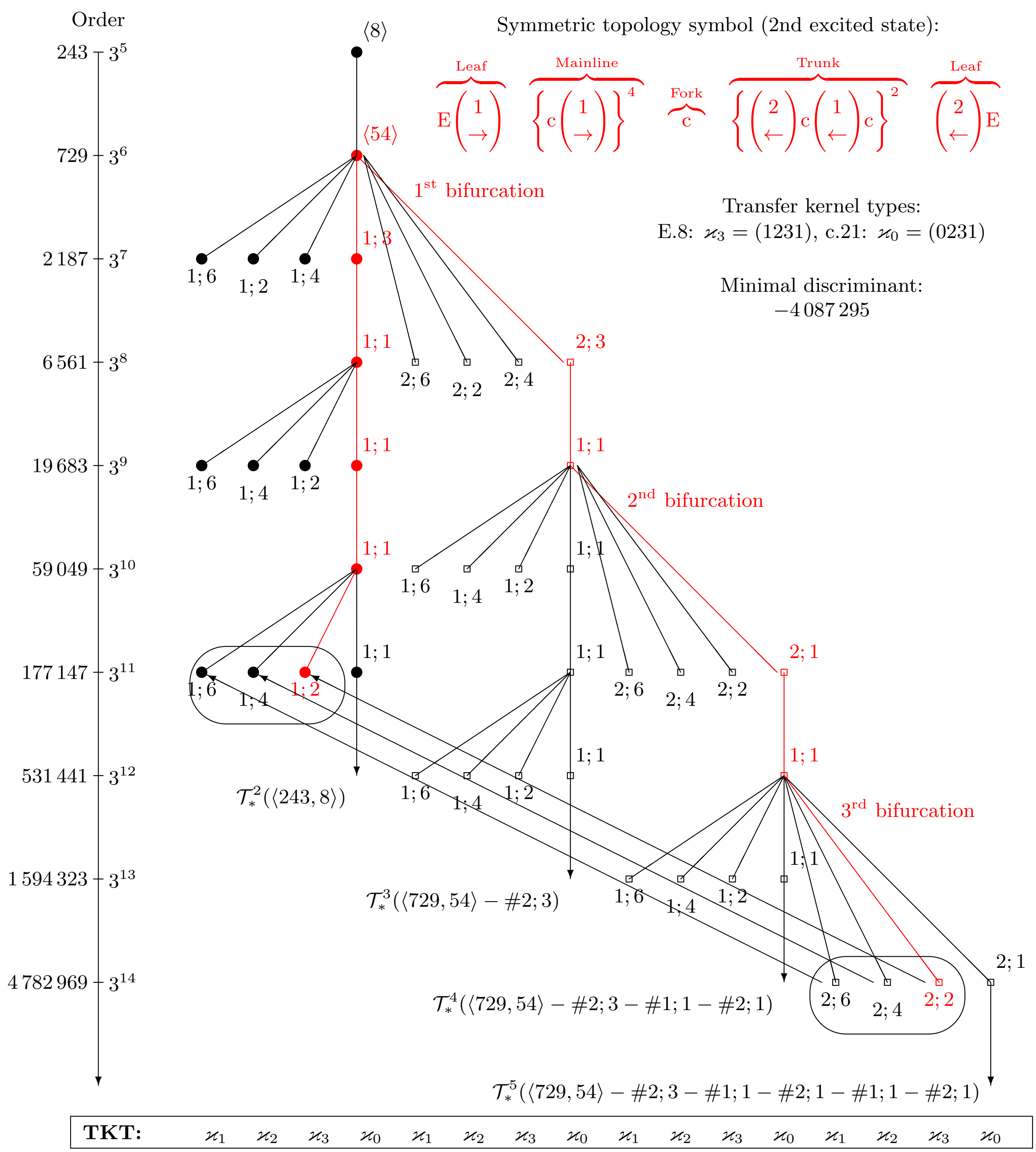
For the 1st excited state $T_1 = (43)$, the second bifurcation yields the cover $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}, \langle 729, 54\rangle - \#2; 3 - \#1; 1 - \#1; 2, \langle 729, 54\rangle - \#2; 3 - \#1; 1 - \#2; 2\}$ of $\mathfrak{M} \simeq \langle 729, 54\rangle - \#1; 3 - \#1; 1 - \#1; 2$. The relation rank $d_2 = 3$ eliminates $\mathfrak{M}$ and $\langle 729, 54\rangle - \#2; 3 - \#1; 1 - \#1; 2$ as candidates for the 3-tower group G, according to Shafarevich, and we get the unique $G \simeq \langle 729, 54\rangle - \#2; 3 - \#1; 1 - \#2; 2$ with an advanced fork topology $\mathrm{E}\binom{1}{\rightarrow}$ $\{\mathrm{c}\binom{1}{\rightarrow}\}^2$ c $\{\binom{2}{\leftarrow}\mathrm{c}\binom{1}{\leftarrow}\mathrm{c}\}$ $\binom{2}{\leftarrow}\mathrm{E}$ describing the relative location of $\mathfrak{M}$ and G.

Similarly, the 2nd excited state $T_1 = (54)$ yields an advanced fork topology $\mathrm{E}\binom{1}{\rightarrow}$ $\{\mathrm{c}\binom{1}{\rightarrow}\}^4$ c $\{\binom{2}{\leftarrow}\mathrm{c}\binom{1}{\leftarrow}\mathrm{c}\}^2$ $\binom{2}{\leftarrow}\mathrm{E}$. □



TKT:	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_0$


Order
243 − 3^5
729 − 3^6
2 187 − 3^7
6 561 − 3^8
19 683 − 3^9
59 049 − 3^10
177 147 − 3^11
531 441 − 3^12
1 594 323 − 3^13
4 782 969 − 3^14
⟨8⟩
⟨54⟩
1st bifurcation
2nd bifurcation
3rd bifurcation
Symmetric topology symbol (1st excited state):
Leaf
Mainline
Fork
Trunk
Leaf
Transfer kernel types:
E.8: ϰ3 = (1231), c.21: ϰ0 = (0231)
Minimal discriminant:
−370 740
T*2(⟨243, 8⟩)
T*3(⟨729, 54⟩ − #2; 3)
T*4(⟨729, 54⟩ − #2; 3 − #1; 1 − #2; 1)
T*5(⟨729, 54⟩ − #2; 3 − #1; 1 − #2; 1 − #1; 1 − #2; 1)
TKT: ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0 ϰ1 ϰ2 ϰ3 ϰ0


Order
243 – 3^5
729 – 3^6
2 187 – 3^7
6 561 – 3^8
19 683 – 3^9
59 049 – 3^{10}
177 147 – 3^{11}
531 441 – 3^{12}
1 594 323 – 3^{13}
4 782 969 – 3^{14}
⟨8⟩
⟨54⟩
1^{st} bifurcation
2^{nd} bifurcation
3^{rd} bifurcation
Symmetric topology symbol (2nd excited state):
Leaf
Mainline
Fork
Trunk
Leaf
Transfer kernel types:
E.8: $\varkappa_3 = (1231)$, c.21: $\varkappa_0 = (0231)$
Minimal discriminant:
−4 087 295
$\mathcal{T}_*^2(\langle 243, 8\rangle)$
$\mathcal{T}_*^3(\langle 729, 54\rangle - \#2;3)$
$\mathcal{T}_*^4(\langle 729, 54\rangle - \#2;3 - \#1;1 - \#2;1)$
$\mathcal{T}_*^5(\langle 729, 54\rangle - \#2;3 - \#1;1 - \#2;1 - \#1;1 - \#2;1)$
TKT: $\varkappa_1$ $\varkappa_2$ $\varkappa_3$ $\varkappa_0$ $\varkappa_1$ $\varkappa_2$ $\varkappa_3$ $\varkappa_0$ $\varkappa_1$ $\varkappa_2$ $\varkappa_3$ $\varkappa_0$ $\varkappa_1$ $\varkappa_2$ $\varkappa_3$ $\varkappa_0$

General theorems on three-stage towers of 3-class fields over quadratic fields $F = \mathbb{Q}(\sqrt{d})$ whose second 3-class group $\mathfrak{M} = \mathrm{G}_3^2 F$ belongs to a periodic sequence on a coclass tree.

Theorem E. A parametrized infinite family of **fork topologies** is given uniformly for the states $\uparrow^n$, $n \geq 0$, of any TKT in section **E** by the symmetric topology symbol

$$\overbrace{\mathrm{E}\begin{pmatrix}1\\ \rightarrow\end{pmatrix}}^{\text{Leaf}} \quad \overbrace{\left\{\mathrm{c}\begin{pmatrix}1\\ \rightarrow\end{pmatrix}\right\}^{2n}}^{\text{Mainline}} \quad \overbrace{\mathrm{c}}^{\text{Fork}} \quad \overbrace{\left\{\begin{pmatrix}2\\ \leftarrow\end{pmatrix}\mathrm{c}\begin{pmatrix}1\\ \leftarrow\end{pmatrix}\mathrm{c}\right\}^{n}}^{\text{Trunk}} \quad \overbrace{\begin{pmatrix}2\\ \leftarrow\end{pmatrix}\mathrm{E}}^{\text{Leaf}}$$

with scaffold type **c** and the following invariants:
distance $d = 4n + 2$, weighted distance $w = 5n + 3$,
class increment $\Delta\mathrm{cl} = (2n + 5) - (2n + 5) = 0$,
coclass increment $\Delta\mathrm{cc} = (n + 3) - 2 = n + 1$,
logarithmic order increment $\Delta\mathrm{lo} = (3n+8)-(2n+7) = n+1$.

Theorem c. A parametrized infinite family of **fork topologies** is given uniformly for the states $\uparrow^n$, $n \geq 0$, of any TKT in section **c** by the symmetric topology symbol

$$\overbrace{\left\{\mathrm{c}\begin{pmatrix}1\\ \rightarrow\end{pmatrix}\right\}^{2n}}^{\text{Mainline}} \quad \overbrace{\mathrm{c}}^{\text{Fork}} \quad \overbrace{\left\{\begin{pmatrix}2\\ \leftarrow\end{pmatrix}\mathrm{c}\begin{pmatrix}1\\ \leftarrow\end{pmatrix}\mathrm{c}\right\}^{n}}^{\text{Path}} \quad \overbrace{\begin{pmatrix}1\\ \leftarrow\end{pmatrix}\mathrm{c}}^{\text{Leaf}}$$

(with identical scaffold type **c**) and the following invariants:
distance $d = 4n + 1$, weighted distance $w = 5n + 1$,
class increment $\Delta\mathrm{cl} = (2n + 5) - (2n + 4) = 1$,
coclass increment $\Delta\mathrm{cc} = (n + 2) - 2 = n$,
logarithmic order increment $\Delta\mathrm{lo} = (3n+7)-(2n+6) = n+1$.
The ground state $\uparrow^0$ degenerates to a **child topology**.

8. Illuminating 3-class towers of length 3

Notation. *Logarithmic type invariants* of abelian 3-groups, for instance $(321)\hat{=}(27,9,3)$, $(2^21)\hat{=}(9,9,3)$, and $(41^2)\hat{=}(81,3,3)$.

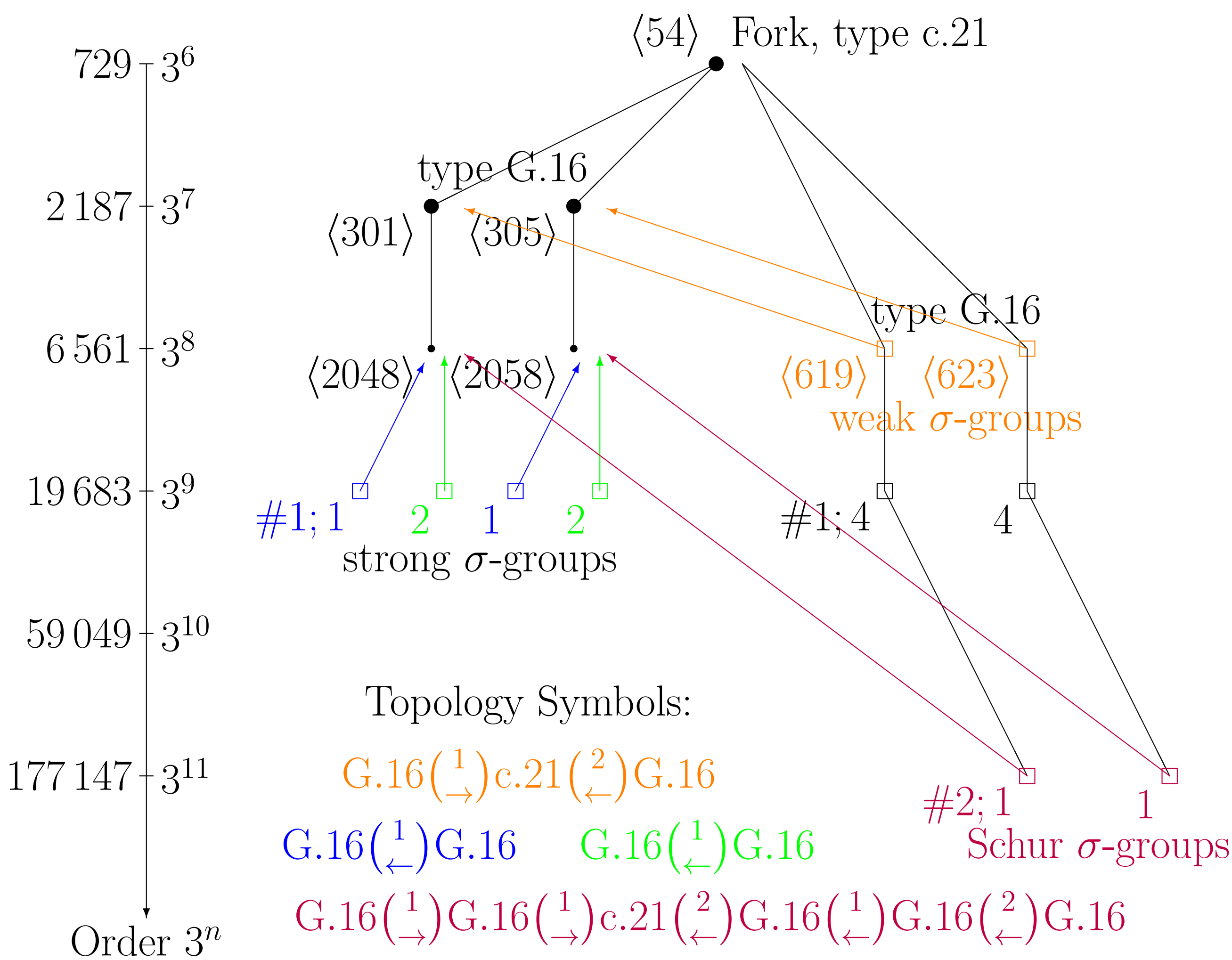
Let F be a number field with $\mathrm{Cl}_3 \simeq (3,3)$
and Artin pattern $\mathrm{AP}(F) = (\tau_1 F, \varkappa_1 F)$,
where $\varkappa_1 F \sim (4,2,3,1)$ is of type G.16
and $\tau_1 F \sim [32,21,21,21]$ indicates the ground state.

Theorem 1. (Imaginary quadratic with fork topology)
If $F = \mathbb{Q}(\sqrt{d})$, $d < 0$, is imaginary quadratic,
and $\tau^{(2)} F = [(32; 321, (\mathbf{4}1^2)^3), (21; 321, (\mathbf{3}1)^3)^3]$,
then $\mathfrak{M} \simeq \langle 3^8, 2048|2058\rangle$,
and $G \simeq \langle 3^8, 619|623\rangle - \#1; 4 - \#2; 1$ is a Schur σ-group.

Theorem 2. (Real quadratic with child topology)
If $F = \mathbb{Q}(\sqrt{d})$, $d > 0$, is real quadratic,
and $\tau^{(2)} F = [(32; 321, (\mathbf{4}1^2)^3), (21; 321, (\mathbf{2}1)^3)^3]$,
resp. $\tau^{(2)} F = [(32; 321, (\mathbf{3}1^2)^3), (21; 321, (\mathbf{2}1)^3)^3]$,
then $\mathfrak{M} \simeq \langle 3^8, 2048|2058\rangle$,
and $G \simeq \mathfrak{M} - \#1; 1$, resp. $\mathfrak{M} - \#1; 2$ is a strong σ-group.

Theorem 3. (Cyclic cubic with fork topology)
If F is cyclic cubic,
and $\tau^{(2)} F = [(32; \mathbf{2^2}1, (\mathbf{3}1^2)^3), (21; \mathbf{2^2}1, (\mathbf{3}1)^3)^3]$,
then $\mathfrak{M} \simeq \langle 3^7, 301|305\rangle$,
and $G \simeq \langle 3^8, 619|623\rangle$ is a weak σ-group.

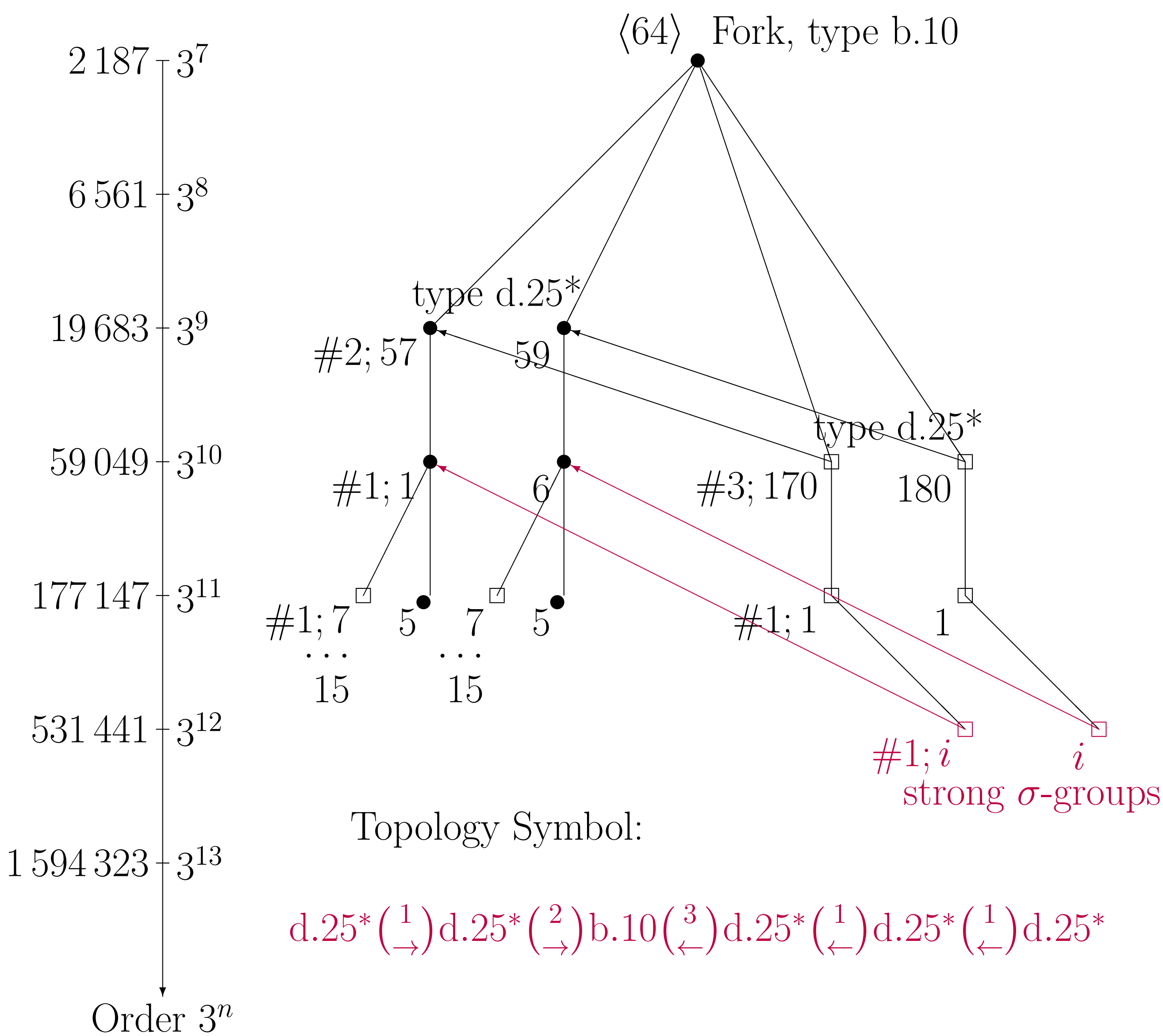
By arrows we denote projections $G \to \mathfrak{M} = G/G''$ from the 3-tower group G onto its metabelianization $\mathfrak{M}$.



Smallest concrete realizations:

$F = \mathbb{Q}(\sqrt{d})$ with $d = -17\,131$,
$F = \mathbb{Q}(\sqrt{d})$ with $d = +8\,711\,453$,
$F = \mathbb{Q}(\sqrt{d})$ with $d = +9\,448\,265$,
F cyclic cubic field with conductor $c = 48\,393$.

Theorem 4. (Real quadratic with fork topology)
If $F = \mathbb{Q}(\sqrt{d})$, $d > 0$, is real quadratic,
and $\tau^{(2)}F = [(3^2; 32^21, (321^2)^3), (32; 32^21, (31^2)^3), (1^3; 32^21, (21^2)^3, (1^3)^9)^2]$,
then $\mathfrak{M} \simeq \langle 3^7, 64\rangle - \#2; 57|59 - \#1; 1|6$,
and $G \simeq \langle 3^7, 64\rangle - \#3; 170|180 - \#1; 1 - \#1; i$
with $i \in \{7, 8, 10, 12, 14, 15\}$ is a strong σ-group.



Topology Symbol:

$$\mathrm{d.25^*}\binom{1}{\rightarrow}\mathrm{d.25^*}\binom{2}{\rightarrow}\mathrm{b.10}\binom{3}{\leftarrow}\mathrm{d.25^*}\binom{1}{\leftarrow}\mathrm{d.25^*}\binom{1}{\leftarrow}\mathrm{d.25^*}$$

Smallest concrete realization:
$F = \mathbb{Q}(\sqrt{d})$ with $d = +8\,491\,713$.

9. The first 5-class tower of length 3

Notation. *Logarithmic type invariants* of abelian 5-groups, for instance $(1^2)\hat{=}(5,5)$, $(1^3)\hat{=}(5,5,5)$, and $(21^3)\hat{=}(25,5,5,5)$.

Let $F=\mathbb{Q}(\sqrt{d})$ be a real quadratic field
with 5-class group $\mathrm{Cl}_5F\simeq(5,5)$, and
$E_1,\ldots,E_6$ be the unramified cyclic quintic extensions of F.

Theorem 1. Suppose the 5-capitulation of F in $E_1,\ldots,E_6$ is of type $\varkappa_1(F)\sim(1,0^5)$.

Then the length of the 5-class tower of F is
(1) $\ell_5F=2$, if $\tau_1(F)\sim[1^3,(1^2)^5]$,
(2) $\ell_5F=3$, if $\tau_1(F)\sim[21^3,(1^2)^5]$.

(For the given TKT $\varkappa_1$, the TTT τ_1 determines the length.)

Examples. Among the 377 quadratic fields $F=\mathbb{Q}(\sqrt{d})$ with 5-class group $\mathrm{Cl}_5F\simeq(1^2)$ and $0<d<26\,695\,193$, there are 57 with 5-capitulation of type $\varkappa_1(F)\sim(1,0^5)$.

(1) 55 of them have $\tau_1(F)\sim[1^3,(1^2)^5]$ and thus $\ell_5F=2$,
(2) only 2 have $\tau_1(F)\sim[21^3,(1^2)^5]$ and thus $\ell_5F=3$.

The discriminants of the former start with 1 167 541,
the latter have $d\in\{3\,812\,377, 19\,621\,905\}$.

Corollary 1. Under the assumptions of Theorem 1,

(1) $\mathrm{G}_5^\infty F = \mathrm{G}_5^2 F \simeq \langle 625, 8\rangle$ with order 5^4, class 3, coclass 1 and relation rank 3, if $\tau_1(F) \sim [1^3, (1^2)^5]$,

(2) $\mathrm{G}_5^2 F \simeq \langle 15625, 635\rangle$ with order 5^6, class 5, coclass 1 and relation rank 4, if $\tau_1(F) \sim [21^3, (1^2)^5]$,

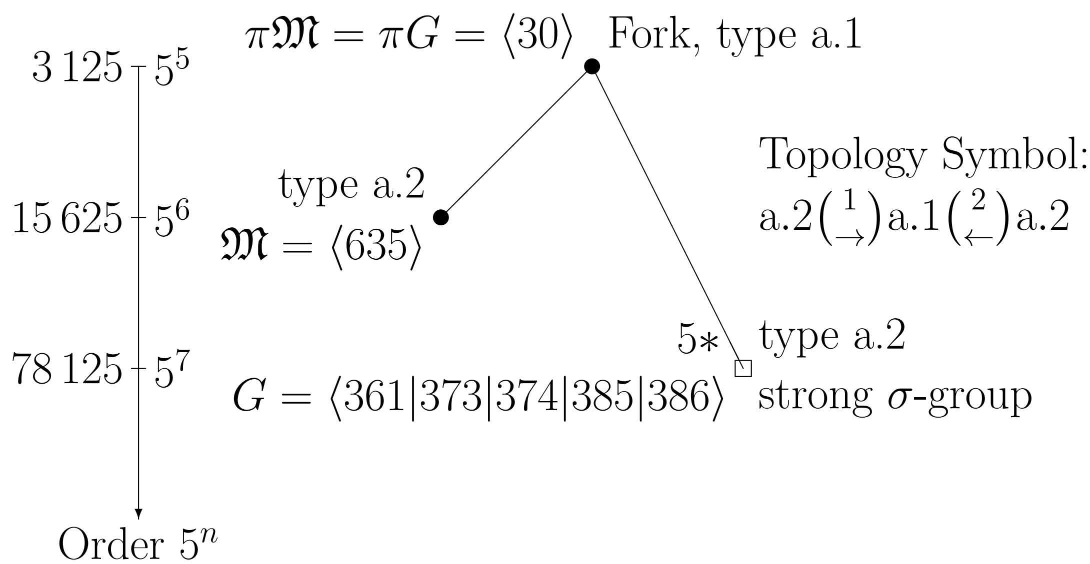
(3) $\mathrm{G}_5^\infty F = \mathrm{G}_5^3 F \simeq \langle 78125, n\rangle$, $n \in \{361, 373, 374, 385, 386\}$, each with order 5^7, class 5, coclass 2 and relation rank 3, if $\tau_1(F) \sim [21^3, (1^2)^5]$.

Let
$\bigl([21^3; 1^4, (21^2)^{155}], [1^2; 1^4, (1^3)^5]^a, [1^2; 1^4, (21)^5]^b\bigr)$ with $a+b = 5$ denote the IPAD $\tau^{(2)}F$ of second order of F, when $\ell_5 F = 3$. (The common component (1^4) belongs to $F_5^{(1)}$.)

Corollary 2. Under the assumptions of Corollary 1, $\tau^{(2)}F$ admits a distinction among the identifiers n.

(1) $n \in \{361, 373\}$, if $(a, b) = (2, 3)$,
(2) $n \in \{374, 385\}$, if $(a, b) = (0, 5)$,
(3) $n = 386$, if $(a, b) = (1, 4)$.

The following figure shows the *siblings topology* which describes the relative position of the second 5-class group $\mathfrak{M} = \mathrm{G}_5^2 F$ and the 5-tower group $G = \mathrm{G}_5^\infty F$ of a real quadratic field $F = \mathbb{Q}(\sqrt{d})$ satisfying the assumptions of Theorem 1. It is a special instance of a fork topology, characterized by a minimal vertex distance 2.



10. Action of $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathrm{G}_p^n F$

When $F/\mathbb{Q}$ is a normal extension,
then its Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on the class group Cl_F
and thus also on all commutator quotients $\mathrm{G}_p^n F/\left(\mathrm{G}_p^n F\right)' \simeq \mathrm{Cl}_p F$ of higher p-class groups with $p \in \mathbb{P}$ and $n \in \mathbb{N} \cup \{\infty\}$.

Let p be a prime number and $d \geq 2$ be an integer.

Definition. (σ-Groups of various degrees)
A pro-p group $\mathfrak{G}$ is called a *σ-group of degree d*, if it has an automophism $\sigma \in \mathrm{Aut}(\mathfrak{G})$ of order $\mathrm{ord}(\sigma) = d$ such that the trace $T_\sigma := \sum_{i=0}^{d-1} \sigma^i \in \mathbb{Z}[\mathrm{Aut}(\mathfrak{G})]$ of σ annihilates the commutator quotient $\mathfrak{G}/\mathfrak{G}'$, that is,

$$x^{T_\sigma} = \prod_{i=0}^{d-1} \sigma^i(x) \in \mathfrak{G}', \text{ for all } x \in \mathfrak{G}.$$

If $d = 2$, then $\mathfrak{G}$ is simply called a *σ-group*.

Definition. (Strong σ-group)
A pro-p group $\mathfrak{G}$ is called a *strong σ-group*, if it possesses an automophism $\sigma \in \mathrm{Aut}(\mathfrak{G})$ of order $\mathrm{ord}(\sigma) = 2$ such that σ acts as the inversion $x \mapsto x^{-1}$ on the cohomology groups $\mathrm{H}^1(\mathfrak{G}, \mathbb{F}_p)$ and $\mathrm{H}^2(\mathfrak{G}, \mathbb{F}_p)$.

Theorem. (Absolutely cyclic number fields)
If $F/\mathbb{Q}$ is a cyclic extension of degree d, then all higher p-class groups $\mathrm{G}_p^n F$ with $n \in \mathbb{N} \cup \{\infty\}$ are σ-groups of degree d.
If $d = 2$, that is, when $F/\mathbb{Q}$ is a quadratic field, then the p-tower group $G = \mathrm{G}_p^\infty F$ is a strong σ-group.

10.1. Cyclic cubic fields F with $\mathrm{Cl}_5 F \simeq (5,5)$

Proposition 1. (σ-groups of degree 3 with type $(5,5)$)
The transfer kernel type of a pro-5 group $\mathfrak{G}$ with abelianization $\mathfrak{G}/\mathfrak{G}' \simeq (5,5)$ which is a σ-group of degree 3 is restricted to the following admissible types

- $(0,0,0,0,0,0)$,
- two 3-cycles,
- a 6-cycle,
- the identity,
- three 2-cycles.

Corollary 1. (Cyclic cubic fields of type $(5,5)$)
The second 5-class group $\mathrm{G}_5^2 F$ of a cyclic cubic field F with 5-class group $\mathrm{Cl}_5 F \simeq (5,5)$ is restricted to the isomorphism classes of the following groups

- $\langle 25, 2\rangle$,
- $\langle 125, 3\rangle$,
- a descendant of $\langle 3125, 3\rangle$,
- $\langle 3125, 9\rangle$,
- $\langle 3125, 12\rangle$,
- $\langle 3125, 14\rangle$,
- a descendant of $\langle 3125, 10\rangle$

All these finite 5-groups are σ-groups of degree 3.

Examples 1. In the range $1 < c < 1\,000\,000$ of conductors, there are 481 occurrences of $\mathrm{Cl}_5 F \simeq (5,5)$. The dominating part of 463 fields (96%) has $\mathrm{G}_5^2 F \simeq \langle 125, 3\rangle$ with six total transfer kernels $\varkappa_1(F) = (0,0,0,0,0,0)$. Exceptions occur for the following 18 conductors only, confirming Corollary 1:

No.	c	Factors	$\mathrm{G}_5^2 F$	$\varkappa(F)$
1	66 313	13, 5101	$\langle 3125, 12\rangle$	a 6-cycle
2	68 791	prime	$\langle 3125, 14\rangle$	the identity
3	77 971	103, 757	$\langle 3125, 14\rangle$	the identity
4	87 409	7, 12487	$\langle 3125, 12\rangle$	a 6-cycle
5	199 621	prime	$\langle 3125, 9\rangle$	two 3-cycles
6	317 853	9, 35317	$\langle 3125, 12\rangle$	a 6-cycle
7	425 257	7, 79, 769	$\langle 3125, 14\rangle$	the identity
8	464 191	7, 13, 5101	$\langle 3125, 12\rangle$	a 6-cycle
9	481 537	7, 68791	$\langle 3125, 14\rangle$	the identity
10	545 797	7, 103, 757	$\langle 3125, 14\rangle$	the identity
11	596 817	9, 13, 5101	$\langle 3125, 12\rangle$	a 6-cycle
12	619 119	9, 68791	$\langle 3125, 14\rangle$	the identity
13	678 303	9, 75367	$\langle 3125, 12\rangle$	a 6-cycle
14	701 739	9, 103, 757	$\langle 3125, 14\rangle$	the identity
15	767 623	prime	$\langle 3125, 9\rangle$	two 3-cycles
16	786 681	7, 9, 12487	$\langle 3125, 12\rangle$	a 6-cycle
17	894 283	13, 68791	$\langle 3125, 14\rangle$	the identity
18	909 229	487, 1867	$\langle 3125, 14\rangle$	the identity

The first, resp. last, example of the dominating part is

$$c = 6\,901 = 67 \cdot 103, \text{ resp. } c = 96\,733 = 7 \cdot 13 \cdot 1063.$$

10.2. Cyclic quartic fields F with $\mathrm{Cl}_3F \simeq (3,3)$

Proposition 2. (σ-groups of degree 4 with type $(3,3)$)
The transfer kernel type of a pro-3 group G with abelianization $G/G' \simeq (3,3)$ which is a σ-group of degree 4 is restricted to the 6 admissible types a.1, b.10, D.5, F.7, G.16, G.19 among the 23 possible types. The remaining 17 types a.2, a.3, c.18, c.21, d.19, d.23, d.25, A.1, D.10, E.6, E.8, E.9, E.14, F.11, F.12, F.13, H.4 are forbidden.

Corollary 2. (Cyclic quartic fields of type $(3,3)$)
The second 3-class group G_3^2F of a cyclic quartic field F with 3-class group $\mathrm{Cl}_3F \simeq (3,3)$ is restricted to the isomorphism classes of $\langle 9,2\rangle$, $\langle 27,3\rangle$, $\langle 243,7\rangle$, a descendant of $\langle 243,9\rangle$ or a descendant of $\langle 243,3\rangle$.
All these finite 3-groups are σ-groups of degree 4.

Examples 2. A cyclic quartic field has a unique representation $F = \mathbb{Q}\left(\sqrt{a(d+b\sqrt{d})}\right)$ with $d = b^2 + c^2$, $a \in \mathbb{Z}$, $b, c \in \mathbb{N}$, a and d squarefree and coprime. We have $\mathrm{G}_3^2F \simeq$
$\langle 27,3\rangle$, $\varkappa(F) \sim (0,0,0,0)$, for $(a,b,c,d) = (3,9,5,106)$,
$\langle 243,7\rangle$, $\varkappa(F) \sim (4,2,2,4)$, for $(a,b,c,d) = (-1,10,7,149)$,
$\langle 729,57\rangle$, $\varkappa(F) \sim (2,1,4,3)$, for $(a,b,c,d) = (-3,10,1,101)$,
$\langle 729,37\rangle$, $\varkappa(F) \sim (0,0,4,3)$, for $(a,b,c,d) = (-5,1,6,37)$,
$Y - \#2; 48$, $\varkappa(F) \sim (1,2,4,3)$, for $(a,b,c,d) = (-7,5,4,41)$,
where $\langle 729,57\rangle$ is an immediate descendant of $\langle 243,9\rangle$,
$\langle 729,37\rangle$ is an immediate descendant of $\langle 243,3\rangle$, and
$Y := \langle 2187,64\rangle$ is a descendant of step size 2 of $\langle 243,3\rangle$.

10.3. Cyclic quartic fields F with $\mathrm{Cl}_5 F \simeq (5,5)$

Proposition 3. (σ-groups of degree 4 with type $(5,5)$)
The transfer kernel type of a pro-5 group G with abelianization $G/G' \simeq (5,5)$ which is a σ-group of degree 4 is restricted to the admissible types with two 2-cycles, a 4-cycle, identity, and descendant types of $(0,0,0,0,0,0)$, $(0,2,2,2,2,2)$, $(0,1,1,1,1,1)$.

Corollary 3. (Cyclic quartic fields of type $(5,5)$)
The second 5-class group $\mathrm{G}_5^2 F$ of a cyclic quartic field F with 5-class group $\mathrm{Cl}_5 F \simeq (5,5)$ is restricted to the isomorphism classes of $\langle 25,2\rangle$, $\langle 3125,11|14\rangle$, a descendant of $\langle 125,3\rangle$, a descendant of $\langle 3125,7\rangle$ or a descendant of $\langle 3125,3\rangle$, $\langle 3125,4\rangle$, $\langle 3125,5\rangle$.
All these finite 5-groups are σ-groups of degree 4.

Examples 3. We have $\mathrm{G}_5^2 F \simeq$
$\langle 3125,11\rangle$, $\varkappa(F)$ a 4-cycle, for $(a,b,c,d) = (-457,2,1,5)$,
$\langle 3125,14\rangle$, $\varkappa(F)$ the identity, for $(a,b,c,d) = (-581,2,1,5)$.

11. The first 3-class tower of length 2

Notation. *Logarithmic type invariants* of abelian 3-groups, for instance $(1^3) = (111) \hat{=} (3,3,3)$ and $(21) \hat{=} (9,3)$.

Let $F/\mathbb{Q}$ be an algebraic number field
with 3-class group $\mathrm{Cl}_3 F \simeq (3,3)$, and
$E_1, \ldots, E_4$ be the unramified cyclic cubic extensions of F.

Theorem. Suppose the capitulation of 3-classes of F in $E_1, \ldots, E_4$ is of type $\varkappa_1 F \sim (2,2,4,1)$ (called type D.10).

Then the length of the 3-class tower of F is $\ell_3 F = 2$
and $\tau_1 F \sim [21, 21, 1^3, 21]$.

(The given TKT $\varkappa_1$ determines the length and the TTT τ_1.)

Examples. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field.

(1) Among the 2 020 imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 F \simeq (3,3)$ and $-10^6 < d < 0$, there are 667 with 3-capitulation of type $\varkappa_1 F \sim (2,2,4,1)$. With a relative frequency of 33.0%, this type is definitely dominating.

(2) Among the 2 576 real quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3 F \simeq (3,3)$ and $0 < d < 10^7$, there are 263 with a second 3-class group of even coclass. Among the latter, there are 93 with 3-capitulation of type $\varkappa_1 F \sim (2,2,4,1)$. With a relative frequency of 35.4%, this type is dominating when it is taken with respect to $\mathrm{cc}(\mathrm{G}_3^2 F) \equiv 0 \pmod 2$.

The discriminants of the former start with $d = -4\,027$,
those of the latter with $d = 422\,573$.

Proof. A search for type D.10, $\varkappa_1 \sim (2,2,4,1)$, in the descendant tree $\mathcal{T}R$ of the root $R := C_3 \times C_3$ leads to the unique candidate for $\mathfrak{M} = \mathrm{G}_3^2 F$ after two steps with path $R = \langle 9,2\rangle \leftarrow \langle 27,3\rangle \leftarrow \langle 243,5\rangle = \mathfrak{M}$. The group $\mathfrak{M}$ is a metabelian Schur σ-group and any epimorphism onto $\mathfrak{M}$ must be an isomorphism [7]. Thus, we have $\ell_3 F = \mathrm{dl}(\mathfrak{M}) = 2$. $\square$

Theorem 1. A metabelian p-group $\mathfrak{M}$ of nilpotency class $\mathrm{cl}(\mathfrak{M}) = 3$ has a trivial cover $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}\}$.

Theorem 2. A capable p-group G of odd nilpotency class $\mathrm{cl}(G) \equiv 1 \pmod 2$ cannot be a strong σ-group.

Corollary. A capable metabelian p-group $\mathfrak{M}$ of nilpotency class $\mathrm{cl}(\mathfrak{M}) = 3$ is forbidden as the second p-class group $\mathrm{G}_p^2 F$ of any quadratic field $F = \mathbb{Q}(\sqrt{d})$.

Proof. By Theorem 1, we have $\mathrm{cov}(\mathfrak{M}) = \{\mathfrak{M}\}$. If $\mathrm{G}_p^2 F$ were isomorphic to $\mathfrak{M}$, then $\mathrm{G}_p^\infty F \simeq \mathrm{G}_p^2 F \simeq \mathfrak{M}$. However, then Theorem 2 yields the contradiction that $\mathrm{G}_p^\infty F$ were not a strong σ-group. $\square$

Examples. 1. The parent $\pi\mathfrak{M} = \langle 243,4\rangle$ of the metabelian 3-group $\mathfrak{M} = \langle 729,45\rangle$, both with TKT H.4, $\varkappa_1 \sim (4443)$, is capable and has $\mathrm{cl}(\pi\mathfrak{M}) = 3$.
Thus, a quadratic field F cannot have $\mathrm{G}_3^2 F \simeq \pi\mathfrak{M}$.
2. The parent $\pi\mathfrak{M} = \langle 2187,301\rangle$ of the metabelian 3-group $\mathfrak{M} = \langle 6561,2048\rangle$, both with TKT G.16, $\varkappa_1 \sim (4231)$, has a finite cover $\mathrm{cov}(\pi\mathfrak{M}) = \{\pi\mathfrak{M}, \langle 6561,619\rangle\}$ and relation rank $d_2(\pi\mathfrak{M}) = 4$. Since $G = \langle 6561,619\rangle$ is capable with odd class $\mathrm{cl}(G) = 5$, a quadratic field F cannot have $\mathrm{G}_3^2 F \simeq \pi\mathfrak{M}$.

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