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New Number Fields with known p-Class Tower

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Towers of p-Class Fields over Algebraic Number Fields

ACKNOWLEDGEMENTS

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I am indebted to M.F. Newman (ANU, Canberra, ACT) for pointing out that the capable groups G in Theorems 3 and 7 have p-multiplicator rank $\mu(G) = 4$ and thus satisfy the inequality $d_2(G) \geq 2 + d_1(G)$ between their relation rank $d_2(G)$ and generator rank $d_1(G) = 2$. This disqualifies them as candidates for p-class tower groups of real quadratic fields, according to the Shafarevich Theorem in the Appendix.

Sincere thanks are given to Michael R. Bush (WLU, Lexington, VA) for making available brand-new (July 2015) numerical results on IPADs of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ and the distribution of discriminants $d > 0$ over these IPADs.

Finally, I thank Yasuhiro Kishi (AUE, Nagoya, JP) for our joint investigation of 5-class towers over certain cyclic quartic fields.

PREFACE

In this presentation, I use IPADs of 2^{nd} order to show that real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with one of the 3-capitulation types c.18, (0122), and c.21, (2034), have a 3-class field tower of length 3.

These types are strange because they are unique with the following properties: Every finite metabelian 3-group $\mathfrak{G}$ with one of these types $\varkappa_1(\mathfrak{G})$ has coclass $\mathrm{cc}(\mathfrak{G}) = 2$ and is **infinitely capable** with nuclear rank $\nu(\mathfrak{G}) \geq 1$. So $\mathfrak{G}$ cannot be leaf of a tree. The second 3-class group $\mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ is such a group.

Section c (containing types 18 and 21) was found by Nebelung in 1989, 55 years after the Scholz/Taussky sections D,...,H. However, for 20 years nobody knew examples of fields with these types, which certainly cannot occur for complex quadratic fields, until I discovered suitable real quadratic fields,
$d = 540\,365$ with type c.21 on January 01, 2008,
$d = 534\,824$ with type c.18 on August 20, 2009.

After my return from the ICGA in Shanghai at the end of July 2015, I suddenly had the rewarding idea and the courage to study their 3-class tower.

INTRODUCTION. SUCCINCT SURVEY

• Given a prime p, the Hilbert p-class field tower $\mathrm{F}_p^\infty(K)$ is the maximal unramified pro-p extension of an algebraic number field K.

• The key for determining its Galois group

$$G := \mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K),$$

which is briefly called the p-**class tower group** of K, is the structure of p-class groups $\mathrm{Cl}_p(L)$ of unramified (abelian or non-abelian) extensions $L|K$.

• Our main intention is to present **new criteria** for the occurrence of assigned p-class tower groups G and proofs of their actual **realization** by suitable base fields K.

• This presentation can be downloaded from http://www.algebra.at/22CSICNT2015.pdf

CHAPTER I.
THE GROUP THEORY
OF p-CLASS TOWER GROUPS

§ 0. The Artin Pattern of G

Let $p \geq 2$ be a prime number, G a pro-p group with commutator subgroup G' and finite abelianization G/G' of order p^v, $v \geq 1$.

Definition 0.1. (TTT, TKT and AP)

$\mathrm{Lyr}_n(G) := \{G' \leq H \trianglelefteq G \mid (G:H) = p^n\}$, $0 \leq n \leq v$, the $v+1$ *layers* of intermediate normal subgroups between G and G'.

$T_{G,H} : G/G' \to H/H'$ the *Artin transfer* [12] from G to H,

$\tau_n(G) := (H/H')_{H \in \mathrm{Lyr}_n(G)}$, $0 \leq n \leq v$, the components of the multiple-layered *Transfer Target Type* (TTT) $\tau(G) := [\tau_0(G); \ldots; \tau_v(G)]$,

$\varkappa_n(G) := (\ker(T_{G,H}))_{H \in \mathrm{Lyr}_n(G)}$, $0 \leq n \leq v$, the components of the multiple-layered *Transfer Kernel Type* (TKT) $\varkappa(G) := [\varkappa_0(G); \ldots; \varkappa_v(G)]$.

The pair $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ is called the **Artin pattern** of G.

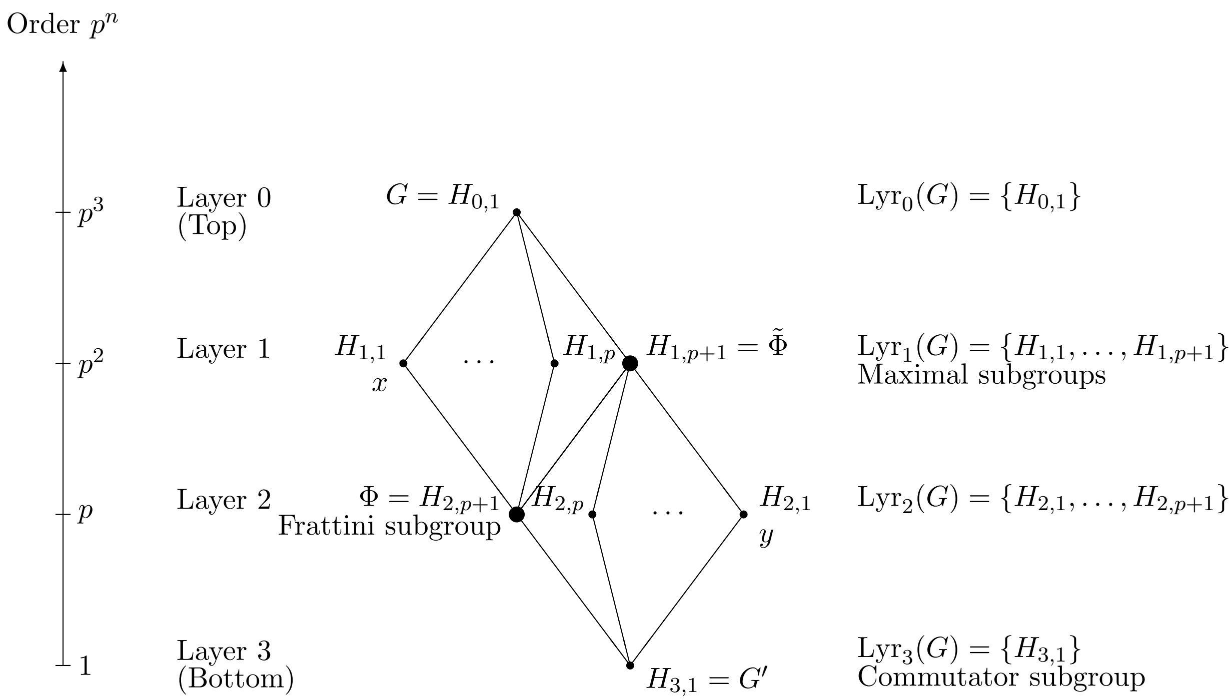
[12] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

Definition 0.2. (IPAD)
(N. Boston, M.R. Bush, F. Hajir, 2011 [5])
The *Index-p Abelianization Data* (IPAD) of G,

$$\tau^{(1)}(G) := [\tau_0(G); \tau_1(G)],$$

arises by restriction of $\tau(G)$ to the 0^{th} and 1^{st} layer. It is a first order approximation of the TTT $\tau(G)$. Figure 1 shows a small non-trivial example of a multi-layered abelianization G/G'.

FIGURE 1. Layers of subgroups $G' \le H_{i,j} \le G$ for $G/G' = \langle x, y, G' \rangle \simeq (p^2, p) \hat{=} 21$



Example: In the situation of Figure 1, the IPAD of G is given by

$$\tau^{(1)}(G) = [G/G'; (H_{1,1}/H'_{1,1}, \dots, H_{1,p+1}/H'_{1,p+1})].$$

CHAPTER II. NUMBER FIELDS WITH 3-CLASS TOWER OF LENGTH 3

§ 1. The Algorithm for Determining G

1^{st} Step. (elementary abelian step)

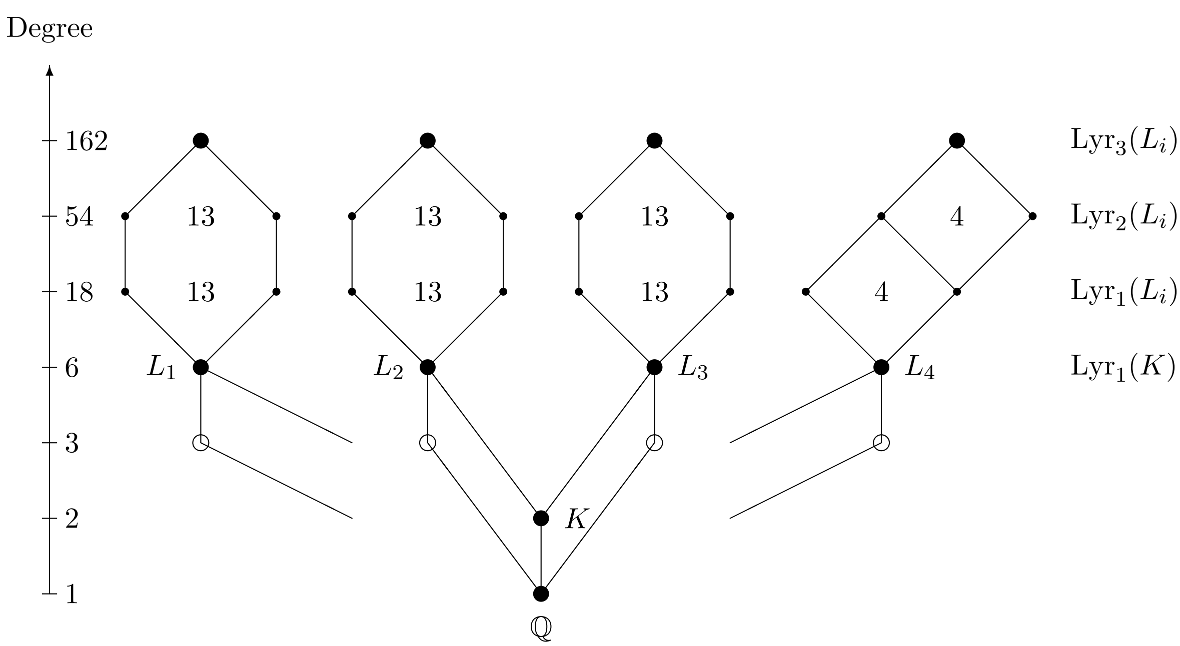
We try to identify the metabelianization $\mathfrak{G} := G/G''$ of the p-class tower group G by means of the p-class groups $\mathrm{Cl}_p(L)$ of unramified cyclic extensions $L|K$ of degree p.

2^{nd} Step. (non-abelian step)

By computing the p-class groups $\mathrm{Cl}_p(M)$ of unramified abelian extensions $M|L$ of increasing degrees $p, p^2, \ldots$, we are occasionally able to determine the p-class tower group G [1, 2]. (The $M|K$ are also unramified but may be non-abelian.)

Figure 2, where $p = 3$, shows a simplified picture of the first few layers of unramified extensions of a quadratic field K with $\mathrm{Cl}_3(K) \simeq (3,3) \hat{=} 1^2$ and $\mathrm{Cl}_3(L_i) \simeq (3,3,3) \hat{=} 1^3$ for $i = 1,2,3$, $\mathrm{Cl}_3(L_4) \simeq (9,3) \hat{=} 21$. It is known that such a field is of type H.4 [9].

FIGURE 2. Layers of unramified extensions of a quadratic field K of type H.4



§ 2. Notation

- G a pro-p group,

$d_1(G) := \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$ the *generator rank* of G,
$d_2(G) := \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ the *relation rank* of G.

- $\mathfrak{G}$ a finite p-group,

$\nu(\mathfrak{G})$ the *nuclear rank* of $\mathfrak{G}$,
$\mu(\mathfrak{G})$ the *p-multiplicator rank* of $\mathfrak{G}$.

If $\mathfrak{G}$ is metabelian (i.e. $\mathrm{dl}(\mathfrak{G}) = 2$), then the set

$$\mathrm{cov}(\mathfrak{G}) := \{\text{finite } G \neq \mathfrak{G} \mid G/G'' \simeq \mathfrak{G}\}$$

is called the *cover* of $\mathfrak{G}$.

- K a number field,

$\ell_p(K)$ the *length* of the p-class tower of K.
If $G = \mathrm{G}_p^\infty(K)$, then $\ell_p(K) = \mathrm{dl}(G)$.

In particular, if $\mathrm{Cl}_3(K) \simeq (3,3) \hat{=} 11$ is the 3-class group of K, then $L_1, \ldots, L_4$ denote the unramified cyclic cubic extensions of K, and $\tau^{(1)}(K) = [\mathrm{Cl}_3(K); \mathrm{Cl}_3(L_1), \ldots, \mathrm{Cl}_3(L_4)]$ is the 1^{st} IPAD of K, according to the Artin Theorem in the Appendix.

§ 3. Capitulation Type c.18, Ground State

Assumptions:
K a number field with $\mathrm{Cl}_3(K) \simeq 11$,
$L_1, \ldots, L_4$ unramified cyclic cubic extensions of K,
$\varkappa_1(K) = (0122)$ 3-capitulation type of K in the L_i,
$\tau^{(1)}(K) = [11; \mathbf{22}, 111, 21, 21]$ the 1st IPAD of K,
$\tau^{(2)}(K) =$
$[11; (\mathbf{22}; \tau_1(L_1)), (111; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))]$
the 2nd IPAD of K, with $\tau_1(L_1) = ((21^2)^4)$.

Proposition. (D.C. Mayer, 2010 [9, 11] and 2015)

$\mathfrak{G} = G/G'' \simeq \langle 3^6, \mathbf{49}\rangle$ with $d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G})$, and thus $\ell_3(K) \geq 3$.

Theorem 1. (D.C. Mayer, Aug. 2015 [15])

$\mathrm{cov}(\langle 3^6, \mathbf{49}\rangle) = \{\langle 3^7, \mathbf{284}\rangle, \langle 3^7, \mathbf{291}\rangle\}$ (2 groups) and

1.
$\tau_1(L_2) = ((\mathbf{21^2})^{\mathbf{4}}, (1^2)^9)$, $\tau_1(L_3) = \tau_1(L_4) = (21^2, (21)^3)$
$\Longleftrightarrow G \simeq \langle 3^7, \mathbf{284}\rangle$.

2.
$\tau_1(L_2) = (21^2, (1^3)^3, (1^2)^9)$, $\tau_1(L_3) = (21^2, (21)^3)$,
$\tau_1(L_4) = (21^2, (\mathbf{31})^{\mathbf{3}})$
$\Longleftrightarrow G \simeq \langle 3^7, \mathbf{291}\rangle$.

In both cases, $d_2(G) = 1 + d_1(G)$, $\mathrm{dl}(G) = 3$.

§ 4. Real Quadratic Fields of Type c.18

Proposition. (D.C. Mayer, Feb. 2010 [9, 11])

In the range $0 < d < 10^7$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely **28** cases with 3-capitulation type $\varkappa_1(K) = (0122)$ and $\tau^{(1)}(K) = [11; \mathbf{22}, 111, (21)^2]$.

Theorem 2. (D.C. Mayer, Aug. 2015 [15])

1.

The **10** real quadratic fields (**36**%) with the following discriminants d,

1 030 117, 3 259 597, 3 928 632, 4 593 673, 5 327 080,
5 909 813, 7 102 277, 7 738 629, 7 758 589, 9 583 736,

have 3-class tower group $G \simeq \langle 3^7, \mathbf{284} \rangle$, $\ell_3(K) = 3$.

2.

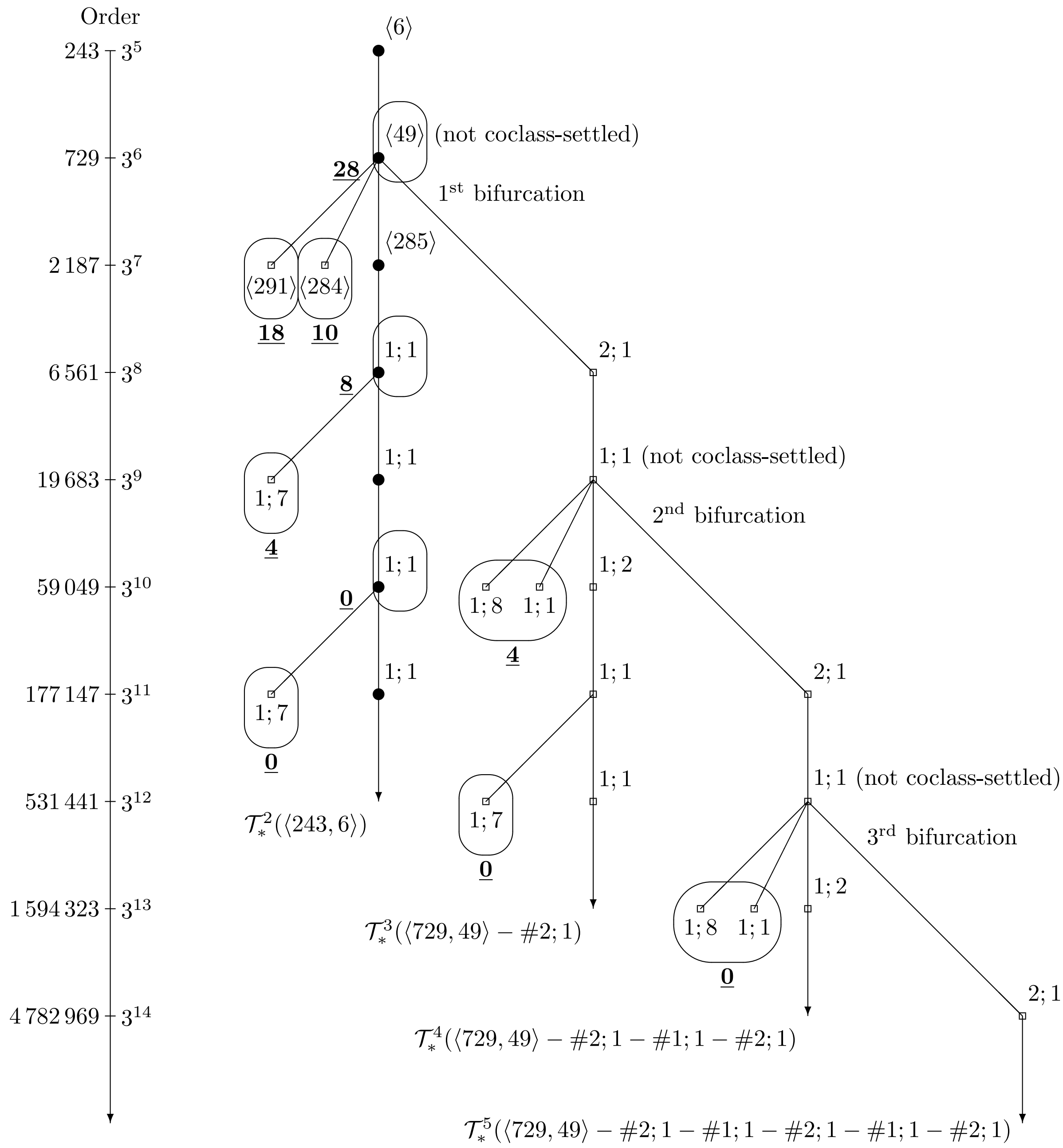
The **18** real quadratic fields (**64**%) with the following discriminants d,

534 824, 2 661 365, 2 733 965, 3 194 013, 3 268 781,
4 006 033, 5 180 081, 5 250 941, 5 489 661, 6 115 852,
6 290 549, 7 712 184, 7 857 048, 7 943 761, 8 243 113,
8 747 997, 8 899 661, 9 907 837,

have 3-class tower group $G \simeq \langle 3^7, \mathbf{291} \rangle$, $\ell_3(K) = 3$.

Figure 3 visualizes the groups in Theorems 1 and 3 and their population in Theorems 2 and 4.

FIGURE 3. Non-metabelian 3-tower groups G on the pruned tree $\mathcal{T}_*(\langle 243,6\rangle)$ [3]



§ 5. **Capitulation Type c.18**, Excited State

Assumptions:
$\varkappa_1(K) = (0122)$ 3-capitulation type of K in the L_i,
$\tau^{(1)}(K) = [11; \mathbf{33}, 111, 21, 21]$ the 1st IPAD of K,
$\tau^{(2)}(K) =$
$[11; (\mathbf{33}; \tau_1(L_1)), (111; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))]$
the 2nd IPAD of K, with $\tau_1(L_1) = ((321)^4)$.

The elementary abelian step of the algorithm yields:

Proposition. (D.C. Mayer, 2010 [9, 11] and 2015)
$\mathfrak{G} = G/G'' \simeq \langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1}$
with $d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G})$, and thus $\ell_3(K) \geq 3$.

Next, the non-abelian step of the algorithm:

Theorem 3. (D.C. Mayer, Aug. 2015 [15])
$\mathrm{cov}(\langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1}) = \{$
$\langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$,
$\langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$,
$\langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$,
$\langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{8}\}$ (4 groups) and
1. $\tau_1(L_2) = (321, (\mathbf{1^3})^\mathbf{3}, (1^2)^9)$, $\tau_1(L_i) = (321, (\mathbf{21})^\mathbf{3})$,
$i = 3, 4 \Longleftrightarrow$
$G \simeq \langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1}$, $|G| = 3^8$, or
$G \simeq \langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$, $|G| = 3^9$.
2. $\tau_1(L_2) = (321, (\mathbf{21^2})^\mathbf{3}, (1^2)^9)$, $\tau_1(L_i) = (321, (\mathbf{31})^\mathbf{3})$,
$i = 3, 4 \Longleftrightarrow$
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$, $|G| = 3^9$, or
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$, $|G| = 3^{10}$,
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{8}$, $|G| = 3^{10}$.

Corollary. (D.C. Mayer, Aug. 2015)
If K has torsion free Dirichlet unit rank 1, then
1. $\tau_1(L_2) = (321, (\mathbf{1^3})^\mathbf{3}, (1^2)^9)$, $\tau_1(L_i) = (321, (\mathbf{21})^\mathbf{3})$,
$i = 3, 4 \Longleftrightarrow G \simeq \langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$.
2. $\tau_1(L_2) = (321, (\mathbf{21^2})^\mathbf{3}, (1^2)^9)$, $\tau_1(L_i) = (321, (\mathbf{31})^\mathbf{3})$,
$i = 3, 4 \Longleftrightarrow$
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$ or
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{8}$.

§ 6. Real Quadratic Fields of Type c.18↑

Proposition. (M.R. Bush, Jul. 2015 [6, 7])
In the range $0 < d < 10^8$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely **8** cases with 1st IPAD $\tau^{(1)}(K) = [11; \mathbf{33}, 111, (21)^2]$.

Corollary. (D.C. Mayer, 2010 [9, 11])
A quadratic field K with $\tau^{(1)}(K) = [11; \mathbf{33}, 111, (21)^2]$ must be a real quadratic field with 3-capitulation type $\varkappa_1(K) = (0122)$.

Theorem 4. (D.C. Mayer, Aug. 2015 [15])
1. The **4** real quadratic fields (**50**%) with the following discriminants d,

13 714 789, 24 037 912, 54 683 977, 94 272 565,

have 3-class tower group $G \simeq \langle 3^7, 285\rangle - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$.
2. The **4** real quadratic fields (**50**%) with the following discriminants d,

14 252 156, 46 748 181, 67 209 369, 78 200 897,

have 3-class tower group either
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{1}$ or
$G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{1} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{8}$.
In each case, the length of the 3-class tower of K is given by $\ell_3(K) = 3$.

§ 7. **Capitulation Type c.21**, Ground State

Assumptions:
$\varkappa_1(K) = (2034)$ 3-capitulation type of K in the L_i,
$\tau^{(1)}(K) = [11; 21, \mathbf{22}, 21, 21]$ the 1st IPAD of K,
$\tau^{(2)}(K) =$
$[11; (21; \tau_1(L_1)), (\mathbf{22}; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))]$
the 2nd IPAD of K, with $\tau_1(L_2) = ((21^2)^4)$.

Proposition. (D.C. Mayer, 2010 [9, 11] and 2015)
$\mathfrak{G} = G/G'' \simeq \langle 3^6, \mathbf{54}\rangle$ with $d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G})$, and thus $\ell_3(K) \geq 3$.

Theorem 5. (D.C. Mayer, Aug. 2015 [15])
$\mathrm{cov}(\langle 3^6, \mathbf{54}\rangle) = \{\langle 3^7, \mathbf{307}\rangle, \langle 3^7, \mathbf{308}\rangle\}$ (2 groups) and

1. $\tau_1(L_1) = \tau_1(L_3) = (21^2, (21)^3)$, $\tau_1(L_4) = (21^2, (\mathbf{31})^{\mathbf{3}})$
$\iff G \simeq \langle 3^7, \mathbf{307}\rangle$.

2. $\tau_1(L_1) = \tau_1(L_4) = (21^2, (21)^3)$, $\tau_1(L_3) = (21^2, (\mathbf{31})^{\mathbf{3}})$
$\iff G \simeq \langle 3^7, \mathbf{308}\rangle$.

In both cases, $d_2(G) = 1 + d_1(G)$, $\mathrm{dl}(G) = 3$.

Remark. Since there is no natural ordering on the four maximal subgroups of the groups G in Theorem 5, the conditions of the two statements in this theorem are indistinguishable.

The two groups $\langle 2187, 307\rangle$ and $\langle 2187, 308\rangle$ have very similar properties and it is a challenge to find properties of a field K which allow its 3-class tower group G to be identified as an abstract group.

§ 8. Real Quadratic Fields of Type c.21

Proposition. (D.C. Mayer, Feb. 2010 [9, 11])

In the range $0 < d < 10^7$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely **27** cases with 3-capitulation type $\varkappa_1(K) = (2034)$. The 1st IPAD of **25** among them is $\tau^{(1)}(K) = [11; 21, \mathbf{22}, (21)^2]$, the remaining **2** have $\tau^{(1)}(K) = [11; 21, \mathbf{33}, (21)^2]$.

Theorem 6. (D.C. Mayer, Aug. 2015 [15])

The **25** real quadratic fields with the following discriminants d,

540 365, 945 813, 1 202 680, 1 695 260, 1 958 629,
3 018 569, 3 236 657, 3 687 441, 4 441 560, 5 512 252,
5 571 377, 5 701 693, 6 027 557, 6 049 356, 6 054 060,
6 274 609, 6 366 029, 6 501 608, 6 773 557, 7 573 868,
8 243 464, 8 251 521, 9 054 177, 9 162 577, 9 967 837,

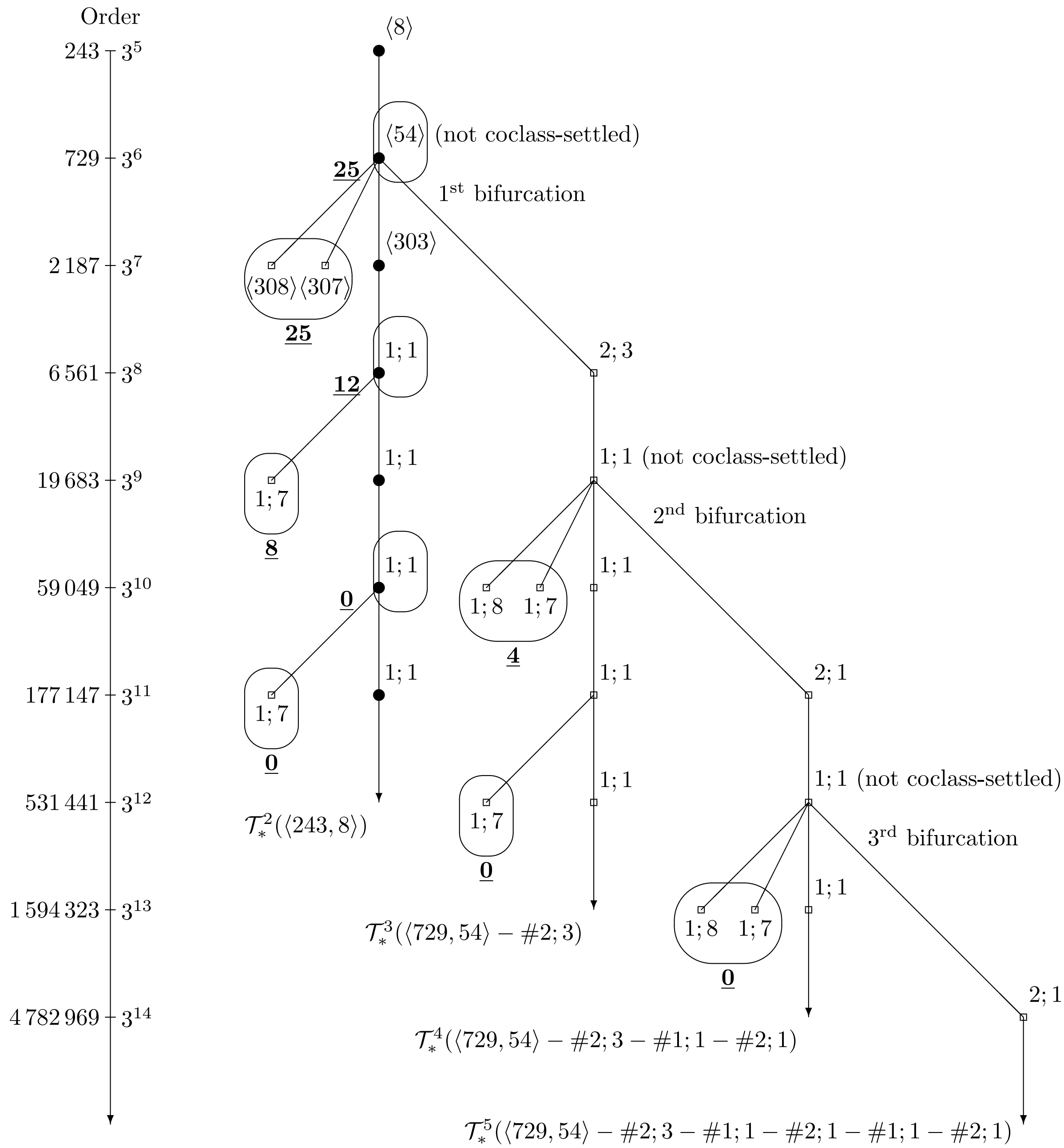
have 3-class tower group
either $G \simeq \langle 3^7, \mathbf{307} \rangle$ or $G \simeq \langle 3^7, \mathbf{308} \rangle$.

In both cases, the length of the 3-class tower of K is given by $\ell_3(K) = 3$.

Remark. The remaining **2** discriminants are treated in Theorem 8.

Figure 4 visualizes the groups in Theorems 5 and 7 and their population in Theorems 6 and 8.

FIGURE 4. Non-metabelian 3-tower groups G on the pruned tree $\mathcal{T}_*(\langle 243, 8\rangle)$ [3]



§ 9. **Capitulation Type c.21**, Excited State

Assumptions:
$\varkappa_1(K) = (2034)$ 3-capitulation type of K in the L_i,
$\tau^{(1)}(K) = [11; 21, \mathbf{33}, 21, 21]$ the 1st IPAD of K,
$\tau^{(2)}(K) =$
$[11; (21; \tau_1(L_1)), (\mathbf{33}; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))]$
the 2nd IPAD of K, with $\tau_1(L_2) = ((321)^4)$.

The elementary abelian step of the algorithm yields:

Proposition. (D.C. Mayer, 2010 [9, 11] and 2015)

$\mathfrak{G} = G/G'' \simeq \langle 3^7, 303\rangle - \#\mathbf{1}; \mathbf{1}$
with $d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G})$, and thus $\ell_3(K) \geq 3$.

Next, the non-abelian step of the algorithm:

Theorem 7. (D.C. Mayer, Aug. 2015 [15])
$\mathrm{cov}(\langle 3^7, 303\rangle - \#\mathbf{1};\mathbf{1}) = \{$
$\langle 3^7, 303\rangle - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{7}$,
$\langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1}$,
$\langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{1}$,
$\langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{8}\}$ (4 groups) and

1. $\tau_1(L_i) = (321, (\mathbf{21})^{\mathbf{3}})$, $i = 1, 3, 4$
$\Longleftrightarrow$
$G \simeq \langle 3^7, 303\rangle - \#\mathbf{1};\mathbf{1}$, $|G| = 3^8$, or
$G \simeq \langle 3^7, 303\rangle - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{7}$, $|G| = 3^9$.

2. $\tau_1(L_i) = (321, (\mathbf{31})^{\mathbf{3}})$, $i = 1, 3, 4$
$\Longleftrightarrow$
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1}$, $|G| = 3^9$, or
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{7}$, $|G| = 3^{10}$,
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{8}$, $|G| = 3^{10}$.

Corollary. (D.C. Mayer, Aug. 2015)
If K has torsion free Dirichlet unit rank 1, then

1. $\tau_1(L_i) = (321, (\mathbf{21})^{\mathbf{3}})$, $i = 1, 3, 4$
$\Longleftrightarrow G \simeq \langle 3^7, 303\rangle - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{7}$.

2. $\tau_1(L_i) = (321, (\mathbf{31})^{\mathbf{3}})$, $i = 1, 3, 4$
$\Longleftrightarrow$
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{7}$ or
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2};\mathbf{3} - \#\mathbf{1};\mathbf{1} - \#\mathbf{1};\mathbf{8}$.

§ 10. Real Quadratic Fields of Type c.21↑

Proposition. (M.R. Bush, Jul. 2015 [6, 7])
In the range $0 < d < 10^8$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely **12** cases with 1st IPAD $\tau^{(1)}(K) = [11; 21, \mathbf{33}, (21)^2]$.

Corollary. (D.C. Mayer, 2010 [9, 11])
A quadratic field K with $\tau^{(1)}(K) = [11; 21, \mathbf{33}, (21)^2]$ must be a real quadratic field with 3-capitulation type $\varkappa_1(K) = (2034)$.

Theorem 8. (D.C. Mayer, Aug. 2015 [15])
1. The **8** real quadratic fields (**67**%) with the following discriminants d,

$$1\,001\,957, \quad 9\,923\,685, \; 20\,633\,209, \; 58\,650\,717,$$
$$63\,404\,792, \; 72\,410\,413, \; 84\,736\,636, \; 92\,578\,472,$$

have 3-class tower group $G \simeq \langle 3^7, 303\rangle - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$.

2. The **4** real quadratic fields (**33**%) with the following discriminants d,

$$25\,283\,701, \; 36\,100\,840, \; 42\,531\,528, \; 81\,398\,865,$$

have 3-class tower group either
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2}; \mathbf{3} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{7}$ or
$G \simeq \langle 3^6, 54\rangle - \#\mathbf{2}; \mathbf{3} - \#\mathbf{1}; \mathbf{1} - \#\mathbf{1}; \mathbf{8}$.

In each case, the length of the 3-class tower of K is given by $\ell_3(K) = 3$.

CHAPTER III. NUMBER FIELDS WITH 5-CLASS TOWER OF LENGTH 2

§ 11. Cyclic Quartic Fields

Let $M = \mathbb{Q}((\zeta - \zeta^{-1})\sqrt{d})$ be a cyclic quartic field where $\zeta = \exp(\frac{1}{5}2\pi i)$ is a primitive fifth root of unity and $d > 0$ with $\gcd(d, 5) = 1$ is a real quadratic fundamental discriminant.

Proposition. (D.C. Mayer, Jun. 2012 [8, 13])
In the range $0 < d < 5\,000$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with $\gcd(5, d) = 1$ there exist precisely **37** cases such that the 5-dual field $M = \mathbb{Q}((\zeta - \zeta^{-1})\sqrt{d})$ of K has a 5-class group $\mathrm{Cl}_5(M)$ of type $(5, 5)$.

Theorem 9. (Y. Kishi, D.C. Mayer, Jul. 2015)
1. For the **7** real quadratic fields (**19**%) with the following discriminants d,

$$457,\ 501,\ 1\,996,\ 2\,573,\ 3\,253,\ 4\,189,\ 4\,957,$$

the 5-dual field M of K has 5-capitulation type $\varkappa_1(M) = (124563)$, a 4-cycle with two fixed points, and 5-class tower group $G \simeq \langle 5^5, \mathbf{11} \rangle$.
2. For the **5** real quadratic fields (**14**%) with the following discriminants d,

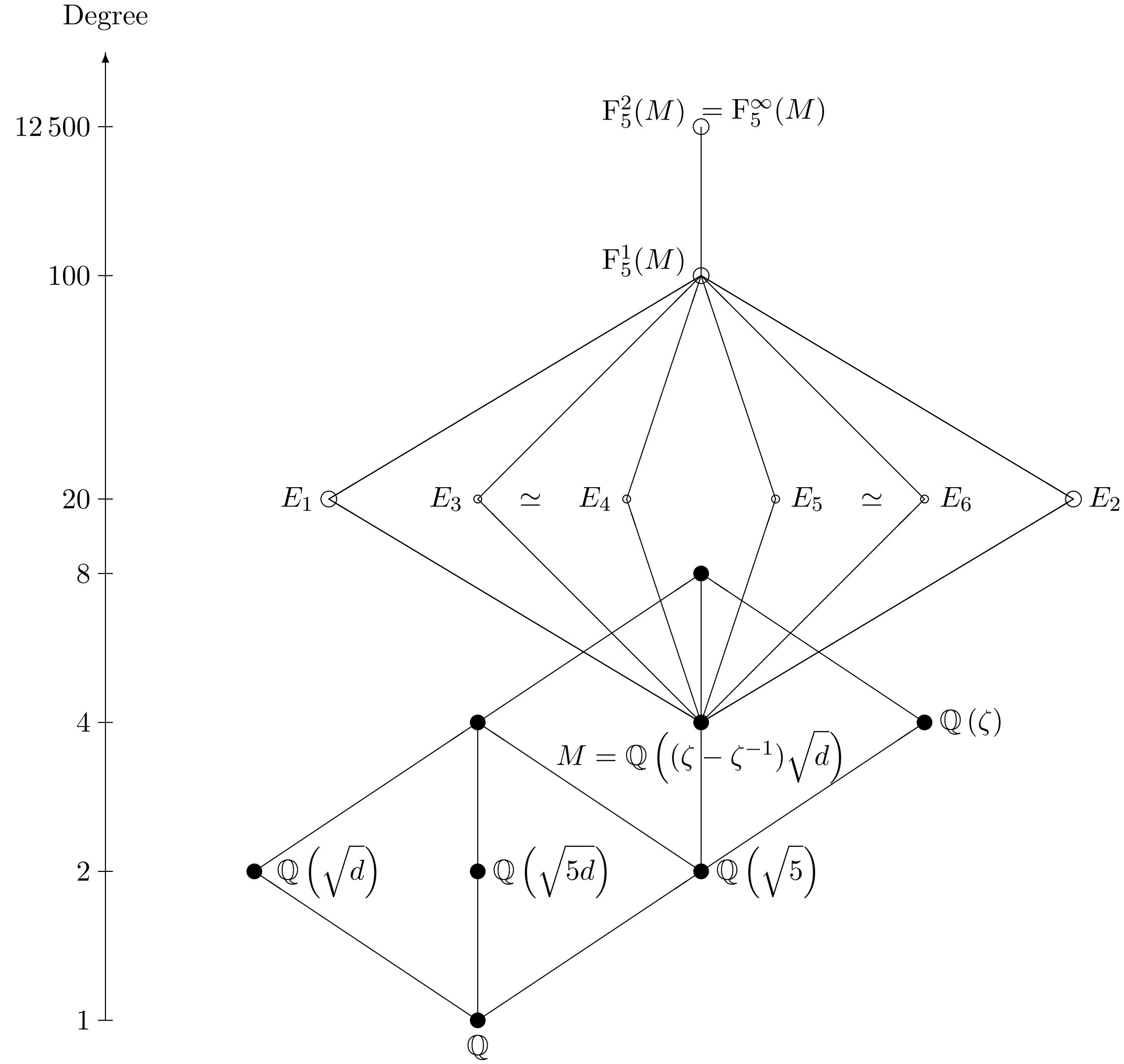
$$581,\ 753,\ 2\,296,\ 2\,829,\ 4\,553,$$

the 5-dual field M of K has 5-capitulation type $\varkappa_1(M) = (123456)$, the identity permutation, and 5-class tower group $G \simeq \langle 5^5, \mathbf{14} \rangle$ (O. Taussky, 1970).

In each case, the length of the 5-class tower of M is given by $\ell_5(M) = 2$, and $d_2(G) = d_1(G)$.

Figure 5 visualizes the situation of a two-stage 5-class tower in Theorem 9.

FIGURE 5. 5-class tower $\mathrm{F}_5^\infty(M)$ of the mirror field $M = \mathbb{Q}\left((\zeta-\zeta^{-1})\sqrt{d}\right)$



Theorem 10. (Y. Kishi, D.C. Mayer, Jul. 2015) For the **2** real quadratic fields (**5**%) with discriminants $d \in \{4\,357,\ 4\,444\}$ the 5-dual field M of K has 5-capitulation type $\varkappa_1(M) = (000000)$, a constant with six total capitulations, and abelian 5-class tower group $G \simeq \langle 5^2, \mathbf{2}\rangle$, $\ell_5(M) = 1$.

$k_1 := \mathbb{Q}(\sqrt{d})$, $d > 0$, $\gcd(5, d) = 1$, $k_2 := \mathbb{Q}(\sqrt{5d})$.

Reflection Theorem for $p = 5$. (Y. Kishi [14])

Relation between the 5-class ranks of k_1, k_2 and M:

$$r_5(M) = r_5(k_1) + r_5(k_2) + 2 - \delta_1 - \delta_2,$$

where $0 \leq \delta_i \leq 1$ (for the precise definition see [14]).

Now let $\mathrm{Cl}_5(M) \simeq (5, 5)$,

$E_1, \ldots, E_6$ unramified cyclic 5-extensions of M,

$$F_5(w) := \langle \rho, \sigma \mid \rho^4 = \sigma^5 = 1, \rho^{-1}\sigma\rho = \sigma^w \rangle$$

two Frobenius groups of order 20 with $2 \leq w \leq 3$.

Theorem 11. (Y. Kishi, D.C. Mayer, Jul. 2015)

The properties of the absolute extensions $E_i|\mathbb{Q}$ and the values of the invariants in the Reflection Theorem for the **37** cases in the Proposition are:

1. For the **2** cases with $\ell_5(M) = 1$ in Theorem 10, we have $r_5(k_1) = 1$, $r_5(k_2) = 0$, $\delta_1 = 0$, $\delta_2 = 1$, and

$$\mathrm{Gal}(E_i|\mathbb{Q}) \simeq F_5(2) \text{ for } 1 \leq i \leq 6.$$

2. For the other **35** cases, including the **12** cases of $\ell_5(M) = 2$ in Theorem 9, we have pairwise conjugate non-Galois extensions $E_3 \simeq E_4$, $E_5 \simeq E_6$,

$$\mathrm{Gal}(E_1|\mathbb{Q}) \simeq F_5(2),\ \mathrm{Gal}(E_2|\mathbb{Q}) \simeq F_5(3),$$

$r_5(k_1) = 1$, $r_5(k_2) = 0$, $\delta_1 = 1$, $\delta_2 = 0$

for $d \in \{1\,996,\ 3\,121,\ 3\,129,\ 3\,253\}$,

$r_5(k_1) = r_5(k_2) = \delta_1 = \delta_2 = 1$ for $d = 4\,504$, and

$r_5(k_1) = r_5(k_2) = \delta_1 = \delta_2 = 0$ otherwise.

APPENDIX.
THE ARITHMETIC
OF p-CLASS TOWER GROUPS

§ 12. Capitulation of p-Classes

Definition 12.1.
K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa_1(K) = (\ker(j_i))_{1 \le i \le p+1}$
is called the p-*capitulation type* of K [10, 16].

The family $\tau_1(K) = (\mathrm{Cl}_p(L_i))_{1 \le i \le p+1}$
is called the p-*class group type* of K [8, 9].

Theorem 12.1. (E. Artin, 1929 [12])
The p-capitulation type $\varkappa_1(K)$, resp. p-class group type $\tau_1(K)$, of K coincides with the first layer TKT $\varkappa_1(G)$, resp. TTT $\tau_1(G)$, of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $2 \le n \le \infty$.

Table 1. Class extension and transfer

$$\begin{array}{ccccc} & & j_{L|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L) & \\ \text{Artin} & \updownarrow & & \updownarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H/H' & \text{isomorphism} \\ & & T_{G,H} & & \end{array}$$

§ 13. Relation Rank of p-Class Tower Groups

Theorem 13.1. (I.R. Shafarevich, 1964 [18])
$p \geq 2$ prime number, K number field with signature (r_1, r_2) and torsion free unit rank $r = r_1 + r_2 - 1$,
S finite set of places of K not divisible by p,
ζ primitive pth root of unity,
$G := \mathrm{G}_{S,p}^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_{S,p}^{\infty}(K)|K)$ the Galois group of the maximal pro-p extension $\mathrm{F}_{S,p}^{\infty}(K)$ of K which is unramified outside of S,
$d_1 := \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$ the *generator rank* of G,
$d_2 := \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ the *relation rank* of G. Then

$$d_1 \leq d_2 \leq \begin{cases} d_1 + r & \text{if } S \neq \emptyset \text{ or } \zeta \notin K, \\ d_1 + 1 & \text{if } S = \emptyset \text{ and } \zeta \in K. \end{cases}$$

Corollary 13.1. $K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant d and $S = \emptyset$,
$G := \mathrm{G}_{p}^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_{p}^{\infty}(K)|K)$ the Galois group of the maximal unramified pro-p extension $\mathrm{F}_{p}^{\infty}(K)$ of K, i.e., the *p-class tower group* of K. Then

$$\begin{cases} d_2 = d_1 & \text{if } (d < 0 \text{ and } p \geq 3 \text{ odd}), \\ d_1 \leq d_2 \leq d_1 + 1 & \text{if either } (d < 0 \text{ and } p = 2) \text{ or } d > 0. \end{cases}$$

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